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A Note on Test Factors

By

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Summary.—The calculation of test factors is reviewed. The distribution of the population from which the test sample is taken is assumed to be Gaussian. Three cases are discussed, in which

(i) there is no prior knowledge of the mean or standard deviation

(ii) there is no prior knowledge of the mean but the standard deviation is a given fraction of the mean (*i.e.*, coefficient of variation known)

(iii) there is no prior knowledge of the mean but the standard deviation is known.

In each case an estimate is made of the average proportion of items under strength which go into service as a result of the continued application of a given test factor. The distributions of the statistics used in the solution of cases (i) and (iii) can be found from published Tables. The corresponding distributions for case (ii) for the appropriate ranges are given in this paper.

1. *Introduction*.—When many components are made to the same nominal specification, the strengths, or the values of any other property under consideration, are in general scattered about the mean value. If this distribution is known it is a simple matter to state below what strength any given fraction of the total number of specimens (*i.e.*, of the 'population') will fall. In practice, although the general form of the distribution can usually be inferred, with fair accuracy, from previous investigations, the parameters of the distribution, such as the mean and the variance, are unknown. In order to obtain further information about the population it is usual to test a small number of specimens, perhaps only a single specimen, assumed to be selected at random. The mean strength of the sample specimens, however, does not necessarily coincide with the mean of the whole population, and merely provides an estimate of it which may be high or low. This point has been discussed by Starkey and Cox[1] and by Atkinson[2] who also calculate test factors necessary to fulfil certain requirements, a test factor being defined as the factor by which the sample mean is required to exceed the design strength. The form in which their results are given, however, does not indicate the proportion of items under strength likely to result from the continued use of a given test factor. Such information is important when considering how the additional cost caused by an increase in the required test factor compares with the saving brought about by having fewer defects in service. In the present paper the results are presented in such a form that this information is readily available and methods are given for finding the appropriate test factor for any permissible proportion of items under strength.

2. *Discussion of the Problem*.—From a large population, a small number of specimens, n, is taken at random and tested for strength (or other property) and found to give values $x_1, x_2, \ldots x_n$.

* R.A.E. Report Structures 215, received 26th October, 956.

Out of the further specimens going into service we shall fix our attention on a single specimen selected at random and of undetermined strength x. Then any conclusion drawn about this single specimen will apply equally to all others going into service.

Now the sample values $x_1, x_2, \ldots x_n$ provide only estimates of the population parameters, but it is often possible to find the distribution of a statistic which is a function of $x_1, x_2, \ldots x_n$ and x only, and such that it is independent of the unknown population parameters. The sample values $x_1, x_2, \ldots x_n$ are known, and hence the distribution of x can be inferred. The statistical principles involved are discussed fully by Fisher[3] who considers as an example the case discussed in Section 3 (i) below, the solution of which depends on 'Student's' t distribution.

Before any progress can be made in the analysis of the problem, some assumptions must be made about the distribution of the parent population. It is assumed that this is normal (*i.e.*, Gaussian). Three cases are considered here, depending on the existing state of knowledge regarding the population parameters. They are the cases in which

(i) there is no prior knowledge of the mean or standard deviation

(ii) there is no prior knowledge of the mean but the standard deviation is a given fraction of the mean (*i.e.*, coefficient of variation known)

(iii) there is no prior knowledge of the mean but the standard deviation is known.

3. *Consideration of Particular Cases.*—

(i) *Case* (*i*).—This example is discussed by Fisher[3] and is, in fact, 'Student's' t test applied to two small samples, the first sample giving the values $x_1, x_2, \ldots x_n$ and the second sample being the further single item giving the value x. Since no assumptions are made regarding the scatter of the population, the sample itself is used to provide an estimate of this, in addition to providing an estimate of the mean. From the sample, the statistics $\bar{x}$ and s are estimated, where

$$\bar{x} = \sum_{r=1}^{n} \frac{x_r}{n},$$

$$s^2 = \sum_{r=1}^{n} \frac{(x_r - \bar{x})^2}{(n-1)}.$$

It can be shown that the statistic

$$\frac{\bar{x} - x}{s} \sqrt{\left(\frac{n}{n+1}\right)}$$

is distributed as 'Student's' t with $n - 1$ degrees of freedom. It will be noted that the statistic estimated is a function of $x_1, x_2, \ldots x_n$ and x only. Tables of the t distribution are given by Fisher and Yates[4].

As an example, a numerical case is considered. A random sample of five specimens from a normal population gives values of

10·33

9·76

10·53

9·58

10·35

(These five values are taken at random from Tables giving normal variates of mean 10 and standard deviation 0·5). Calculating $\bar{x}$ and s :

$$\bar{x} = 10{\cdot}11,$$

$$s = 0{\cdot}414.$$

If we can tolerate, for example, a proportion of 1 in 100 below strength we obtain from Tables of the t distribution for 4 degrees of freedom, the information that 1 per cent of the distribution is cut off at a value of $t = 3{\cdot}747$. The value of x corresponding to 1 per cent is hence given by :

$$3{\cdot}747 = \frac{10{\cdot}11 - x}{0{\cdot}414} \sqrt{\left(\frac{5}{5+1}\right)}$$

and $x = 8{\cdot}41$.

Care should be used in the interpretation of this result. It does not imply that for this particular case 1 per cent of further specimens will fall below $8{\cdot}41$. It merely implies that if we continue to apply the same procedure to a series of samples, then the overall proportion obtained from summing all the cases, will tend to 1 per cent. In this sense, the value calculated above corresponds to an expected proportion of 1 per cent.

(ii) *Case (ii)*.—In the example considered in Section 3 (i), allowance is made not only for the fact that the mean as estimated from the sample may be high, but also for the fact that the variance as estimated from the sample may be low. This results from assuming a complete lack of prior information about the variance. It may happen, however, that we have a considerable amount of information in this respect. For a given type of structure the standard deviation is often found to be a constant fraction of the strength. When previous experience points to such a relationship it is justifiable to take advantage of the fact, and when only one specimen is available for test it is necessary to make some such assumption involving the standard deviation. In case (ii) it is assumed that there is no prior knowledge of the population mean, but that its standard deviation is some known fraction v of that unknown mean, where v is termed the coefficient of variation. Atkinson[2] gives values of v for typical cases :

Built up light alloy structure (*i.e.*, typical metal wing) ..	0·03
Built up wooden structure (*i.e.*, typical wooden wing) ..	0·07
Light alloy castings	0·10
Glass in sheet form	0·20

Making the assumption that v is known suggests considering the distribution of the ratio $\bar{x}/x$ or its reciprocal. The result is obtained from work by Geary[5] who considers the distribution of the ratio of two normal variates. In the notation used here, his result shows that the statistic z, where

$$z = \frac{\left(1 - \frac{x}{\bar{x}}\right)}{v \sqrt{\left(1 + \frac{x^2}{n\bar{x}^2}\right)}}$$

is distributed normally about zero mean with unit standard deviation. For a given value of z the value of $x/\bar{x}$ is quickly found by iteration from the relation

$$\frac{x}{\bar{x}} = 1 - vz \sqrt{\left(1 + \frac{x^2}{n\bar{x}^2}\right)}$$

by substituting the previous approximation to $x/\bar{x}$ in the right-hand side of the equation. The test factor is then given by $\bar{x}/x$.

In order to show the importance of additional information about the population variance, the sample data of Section 3 (i) is taken and the 1 per cent level calculated on the assumption that $v = 0{\cdot}05$. Substituting $z = 2{\cdot}326$, corresponding to the 1 per cent point for the normal distribution, and $n = 5$ gives $\bar{x}/x = 1{\cdot}143$. Finally substituting for $\bar{x}$, we have $x = 8{\cdot}85$, which may be compared with the previous value of $8{\cdot}41$.

Tables 1 to 5 give values of $\bar{x}/x$ for values of v ranging from $0{\cdot}03$ to $0{\cdot}20$, the expected proportion below strength, P, ranging from 1 in 10 to 1 in 10,000 and n = 1, 2, 3, 5 and ∞.

The results are shown graphically in Figs. 1 to 5.

(iii) *Case (iii)*.—It is assumed that the mean is unknown but the standard deviation is known. This is seldom a practical case but it is discussed for completeness. It is easily shown that $\bar{x} - x$ is distributed normally with standard deviation $\sigma\sqrt{\{1 + (1/n)\}}$, where σ is the standard deviation of the parent population (*see*, for example, Kendall[6], Vol. 1, p. 234). Thus the statistic z, where

$$z = \frac{\bar{x} - x}{\sigma\left(\sqrt{1 + \frac{1}{n}}\right)},$$

is distributed normally about zero mean with unit standard deviation.

4. *Applications*.—The numerical example already considered shows the method of calculation of the test factor and serves as a comparison between cases (i) and (ii). In practice, however, each method is used in its own particular situation. Case (i) is applicable to a small component where a reasonable number of specimens can be tested. The fraction defective to be expected in service, on the basis of the test factor achieved is then determined directly. Case (ii) may be used to meet existing airworthiness requirements in the case of a large component such as a wing, when very few specimens (perhaps only a single specimen) are available for test. In this case the requirements are intended to ensure that a large majority, say 90 per cent, of items reach the design strength, which itself includes a factor of safety. As an over-riding requirement when the coefficient of variation is large it is specified that only a negligible proportion fall short of 90 per cent of the design strength. It is conventional to take this negligible proportion as 0·1 per cent of the distribution, although as the form of the distribution curve at its ' tails ' is usually uncertain, this proportion is not likely to be exact. Fig. 6 gives values of the test factor plotted against the coefficient of variation when the requirement is for 90 per cent greater than the design strength or 99·9 per cent greater than 90 per cent of the design strength, whichever is the more severe. Fig. 7 is a similar Figure for test factors which are required to give 90 per cent greater than design strength or 99·9 per cent greater than 80 per cent of the design strength.

In the practical application of these test factors there is an additional safeguard. The assumption that no further action is taken if the test specimens achieve a satisfactory standard, is not in general justified. When the test values are unduly high and this leads to a much higher proportion of weak specimens in service than is acceptable, a further investigation is soon made. On the other hand when the proportion below strength is less than would be acceptable, no action is taken. This results in a slight decrease of the overall proportion of weak specimens.

5. *Conclusions*.—The determination of test factors can be based on finding a statistic which is a function of the sample values $x_1, x_2 \ldots x_n$ and the value x of a further random item from the population ; the sampling distribution of which is independent of unknown population parameters. In the case where such a statistic can be found, its sampling distribution leads immediately to the determination of the required test factor.

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6	M. G. Kendall	*The Advanced Theory of Statistics*. Griffin. 1943.

TABLE 1

Test Factors for $v = 0 \cdot 03$

P	$n = 1$	$n = 2$	$n = 3$	$n = 5$	$n = \infty$
1 in 10	1·056	1·049	1·046	1·044	1·040
1 in 30	1·081	1·071	1·067	1·063	1·058
1 in 100	1·104	1·091	1·086	1·082	1·075
1 in 300	1·122	1·107	1·101	1·096	1·089
1 in 1,000	1·141	1·123	1·117	1·111	1·102
1 in 3,000	1·156	1·137	1·130	1·123	1·114
1 in 10,000	1·172	1·151	1·143	1·136	1·126

TABLE 2

Test Factors for $v = 0 \cdot 05$

P	$n = 1$	$n = 2$	$n = 3$	$n = 5$	$n = \infty$
1 in 10	1·095	1·083	1·078	1·075	1·068
1 in 30	1·139	1·122	1·115	1·110	1·101
1 in 100	1·180	1·158	1·150	1·143	1·132
1 in 300	1·213	1·187	1·178	1·170	1·157
1 in 1,000	1·247	1·218	1·207	1·198	1·183
1 in 3,000	1·276	1·243	1·231	1·221	1·205
1 in 10,000	1·306	1·270	1·257	1·246	1·228

TABLE 3

Test Factors for $v = 0 \cdot 07$

P	$n = 1$	$n = 2$	$n = 3$	$n = 5$	$n = \infty$
1 in 10	1·136	1·119	1·112	1·107	1·099
1 in 30	1·201	1·176	1·167	1·160	1·147
1 in 100	1·262	1·231	1·220	1·210	1·195
1 in 300	1·314	1·277	1·264	1·253	1·234
1 in 1,000	1·366	1·325	1·310	1·297	1·276
1 in 3,000	1·412	1·367	1·350	1·335	1·313
1 in 10,000	1·461	1·411	1·392	1·377	1·352

TABLE 4

Test Factors for $v = 0 \cdot 10$

P	$n = 1$	$n = 2$	$n = 3$	$n = 5$	$n = \infty$
1 in 10	1·200	1·176	1·167	1·159	1·147
1 in 30	1·301	1·266	1·253	1·242	1·225
1 in 100	1·400	1·356	1·339	1·325	1·303
1 in 300	1·486	1·434	1·415	1·398	1·372
1 in 1,000	1·577	1·517	1·495	1·477	1·447
1 in 3,000	1·659	1·593	1·569	1·548	1·516
1 in 10,000	1·749	1·677	1·650	1·628	1·592

TABLE 5

Test Factors for $v = 0 \cdot 20$

P	$n = 1$	$n = 2$	$n = 3$	$n = 5$	$n = \infty$
1 in 10	1·452	1·403	1·385	1·369	1·345
1 in 30	1·734	1·663	1·637	1·614	1·579
1 in 100	2·069	1·977	1·943	1·915	1·870
1 in 300	2·422	2·312	2·272	2·239	2·186
1 in 1,000	2·890	2·762	2·716	2·678	2·618
1 in 3,000	3·435	3·291	3·240	3·197	3·131
1 in 10,000	4·241	4·080	4·023	3·976	3·903

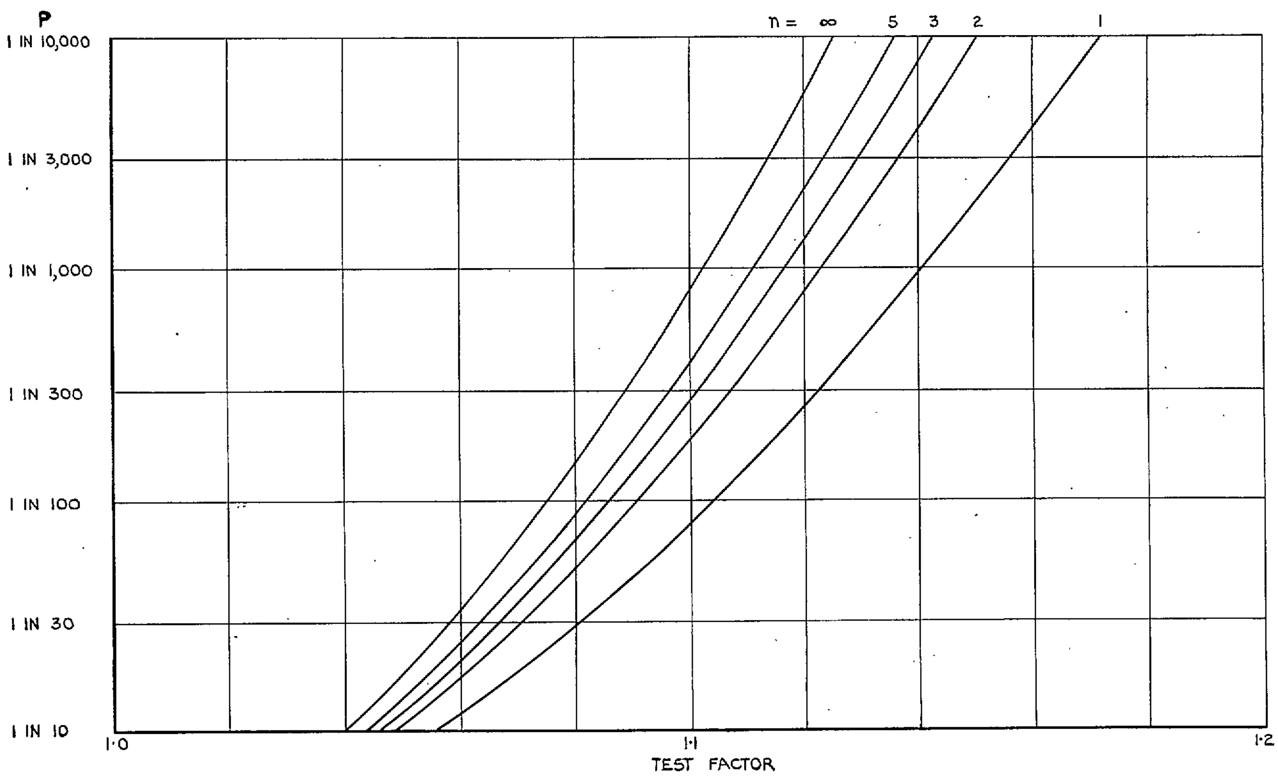

FIG. 1. Test factors for $v = 0 \cdot 03$.

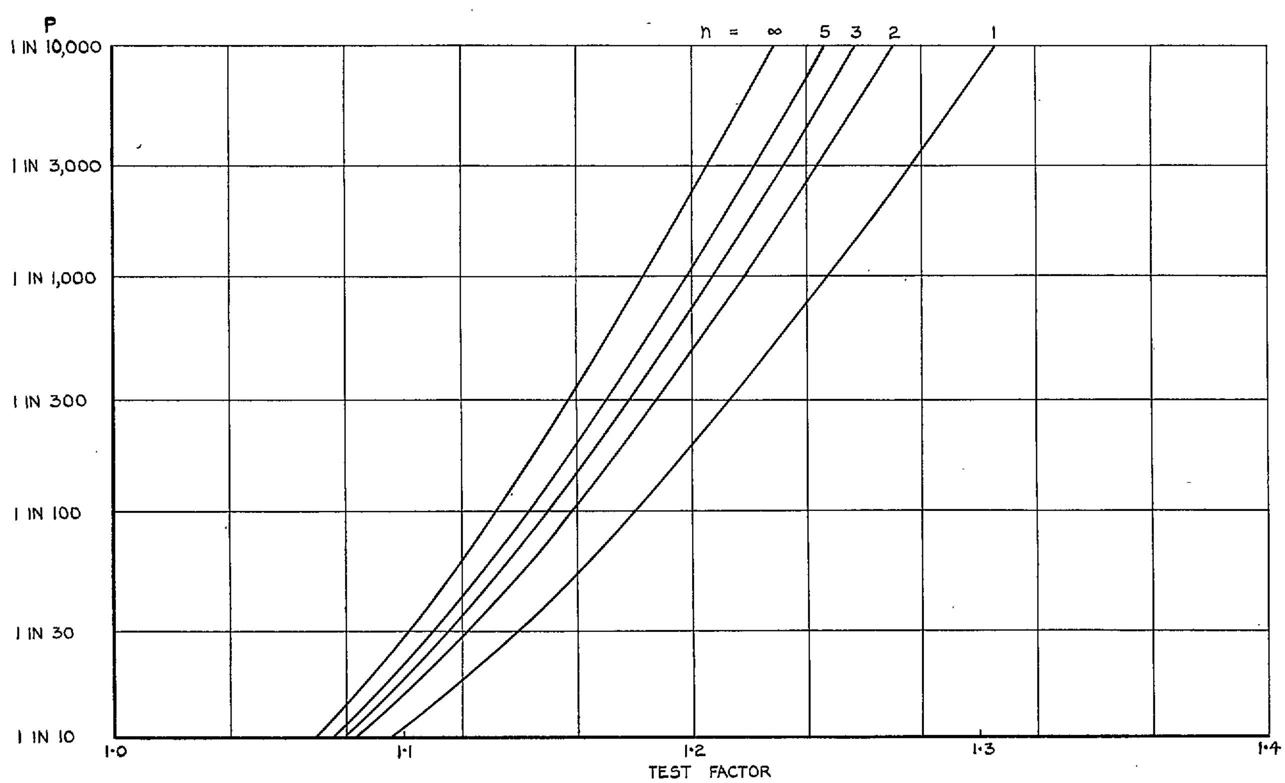

FIG. 2. Test factors for $v = 0 \cdot 05$.

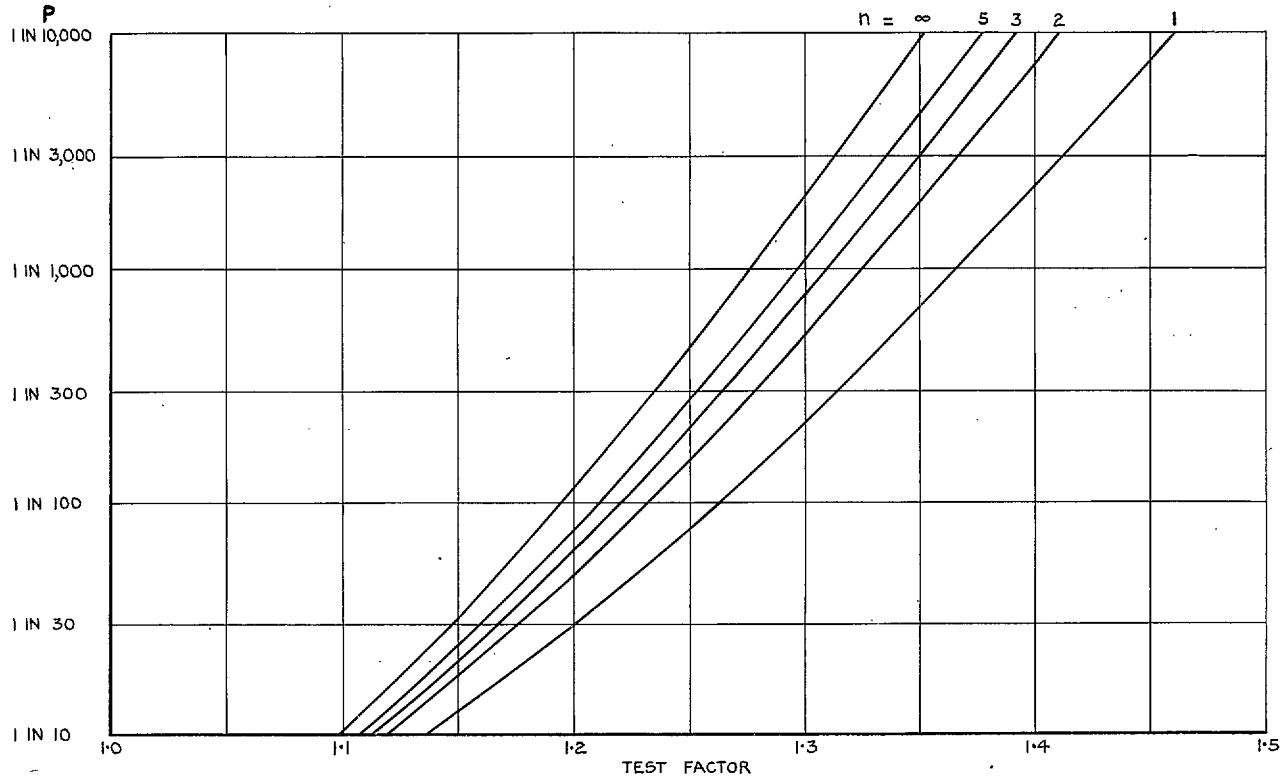

FIG. 3. Test factors for $v = 0 \cdot 07$.

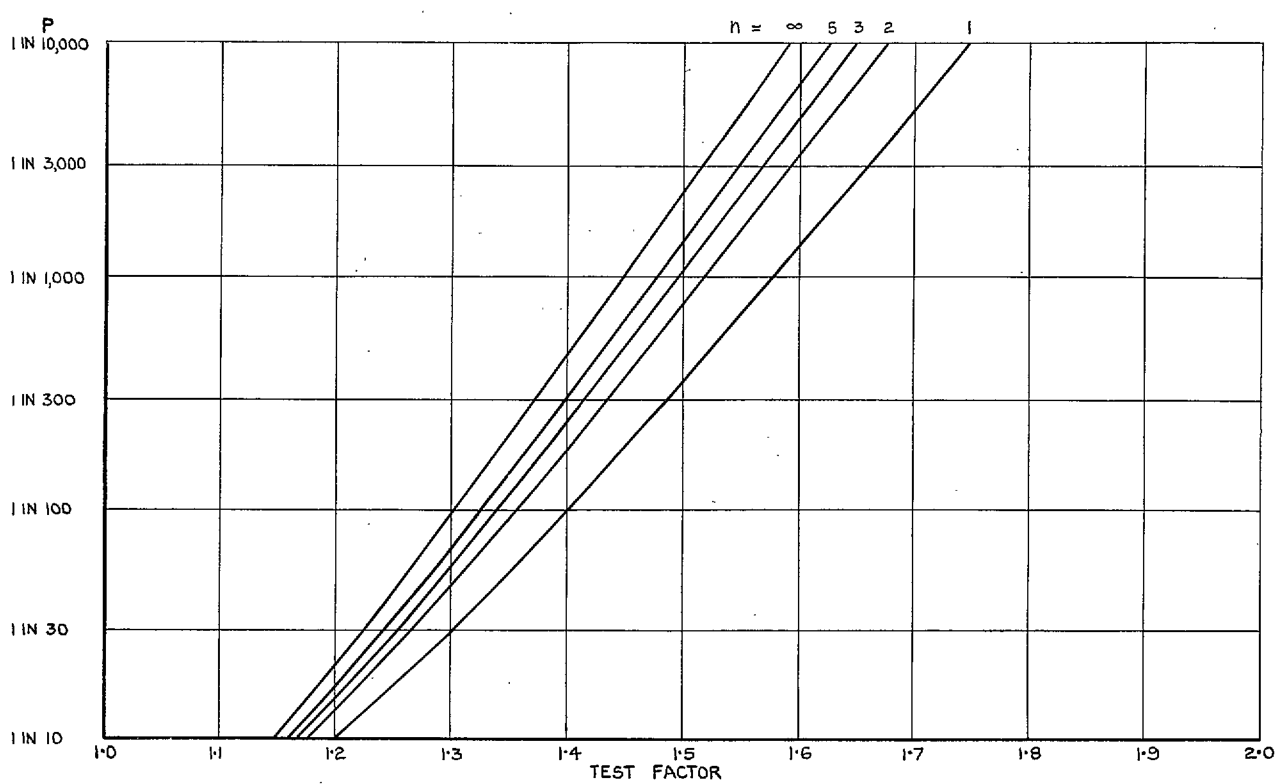

FIG. 4. Test factors for $v = 0 \cdot 10$.

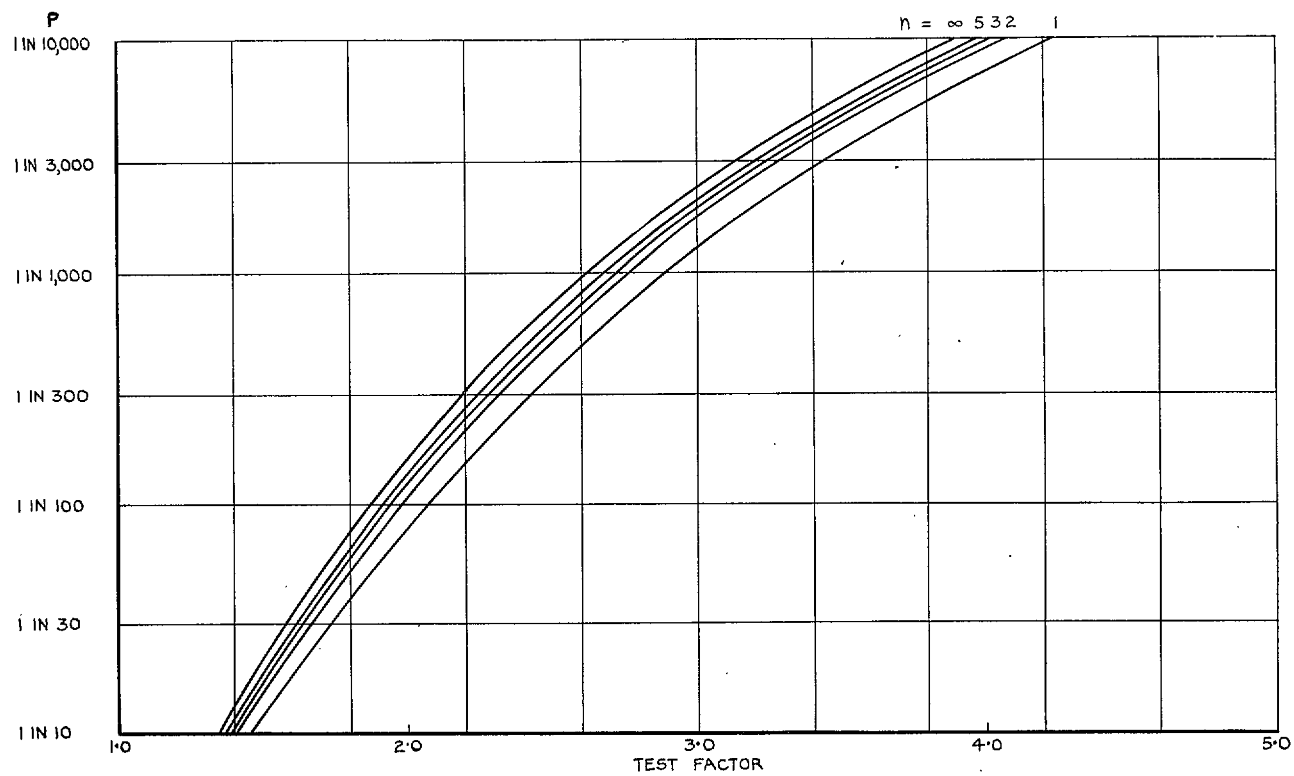

FIG. 5. Test factors for $\dot{v} = 0 \cdot 20$.

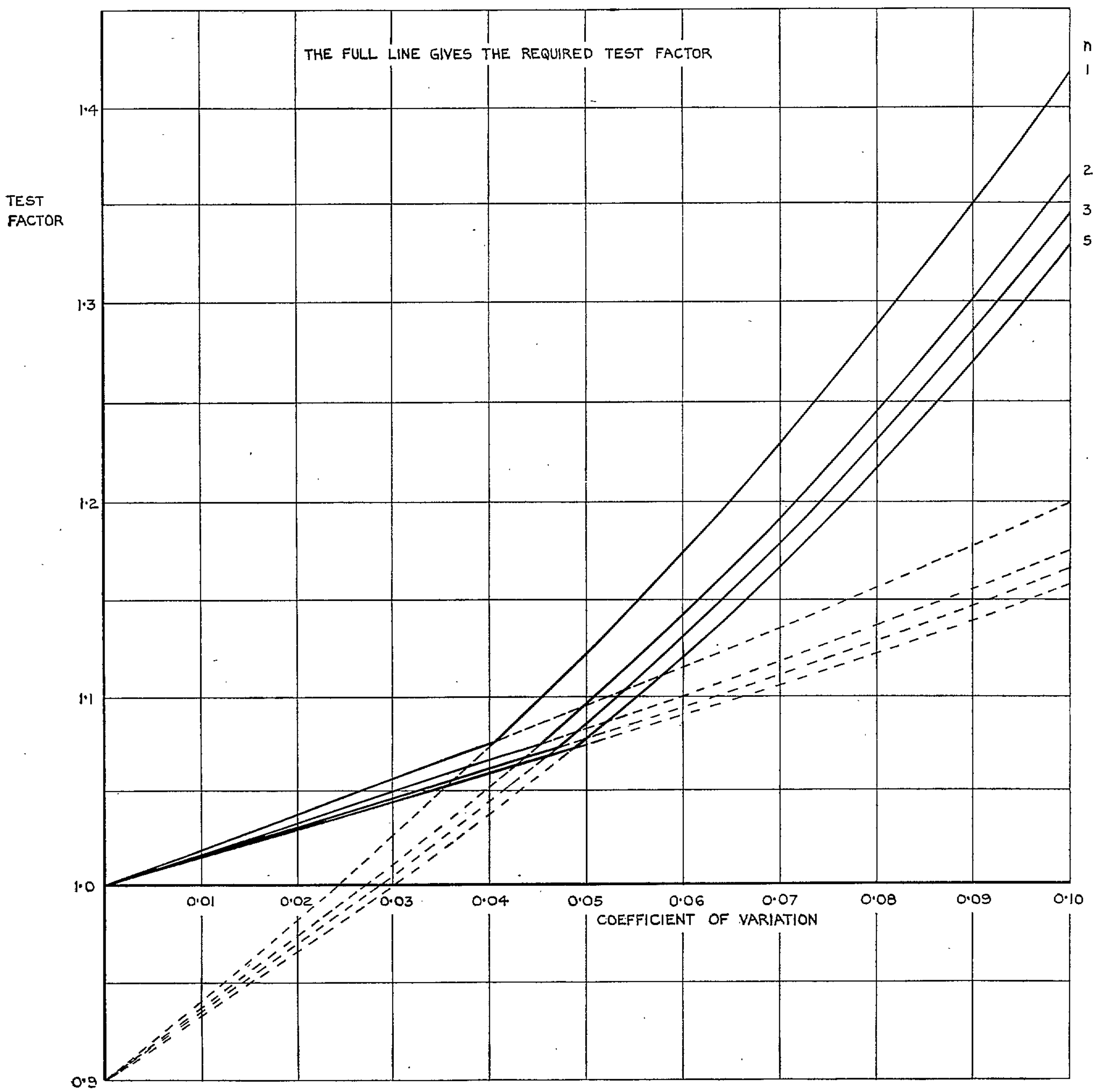

FIG. 6. Test factor required to give:

(i) 90 per cent items greater than design strength ;

(ii) 99·9 per cent items greater than 90 per cent of design strength.

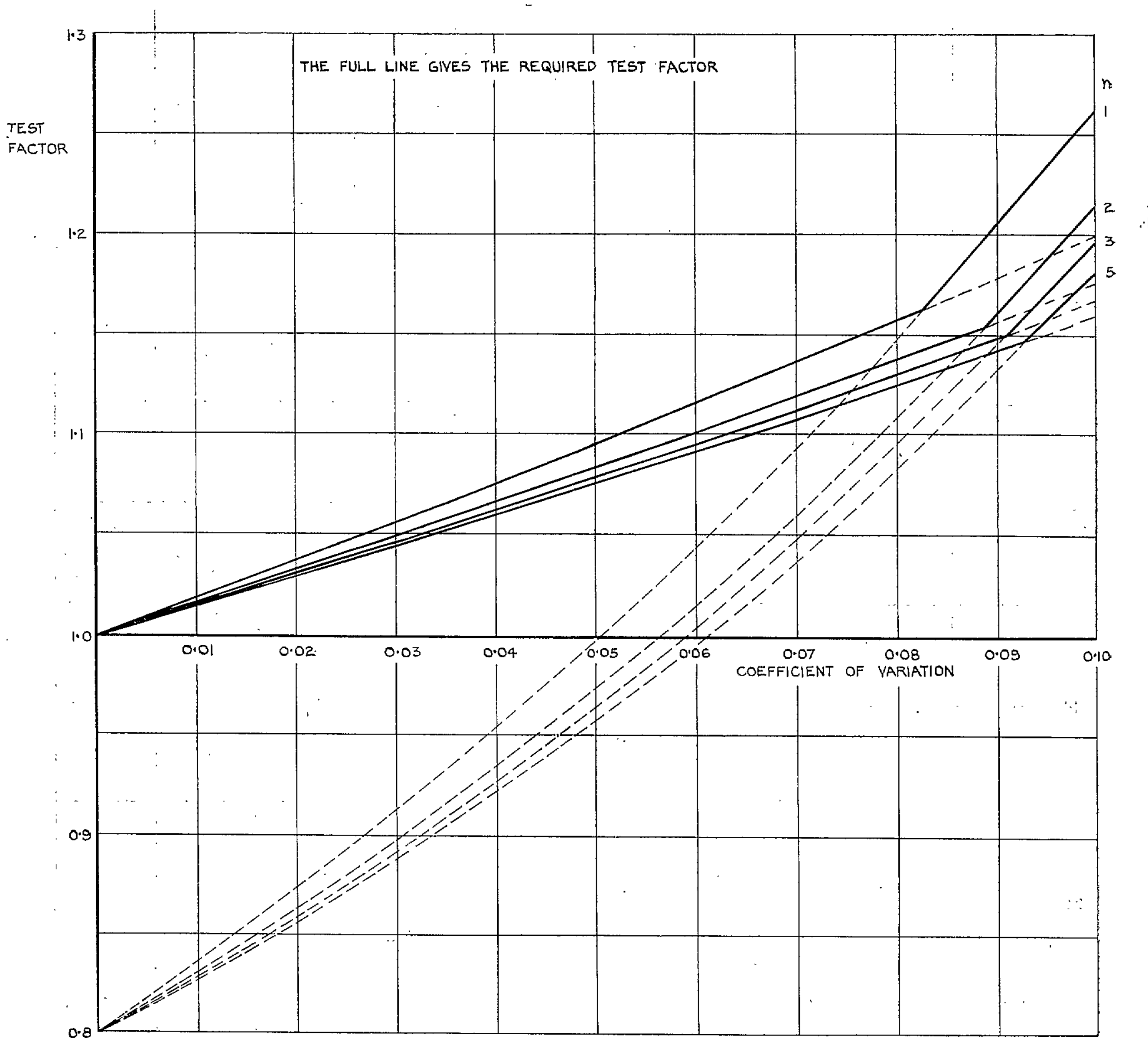

FIG. 7. Test factor required to give :
(i) 90 per cent items greater than design strength ;
(ii) 99·9 per cent items greater than 80 per cent. of design strength.

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