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Note on the Convergence of Numerical Solutions of the Navier-Stokes Equations

By

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Summary.—A criterion is given for the convergence of numerical solutions of the Navier-Stokes equations in two dimensions under steady conditions. The criterion applies to all cases of steady viscous flow in two dimensions and shows that if the local 'mesh Reynolds number', based on the size of the mesh used in the solution, exceeds a certain fixed value, the numerical solution will not converge.

1. *Introduction.*—Although it is of considerable importance to have solutions to the Navier-Stokes equations in particular cases other than the few which can be dealt with analytically, very little attention has been given to this matter until recently. In 1933 Thom published a numerical solution of the viscous flow past a circular cylinder at Reynolds numbers of 10 and 20[1]. Recently, Kawaguti has published a solution for the same problem at Reynolds number 40[2] and Allen and Southwell have published solutions at Reynolds numbers 1, 10, 100 and 1000[3].

It is important in such numerical solutions of the Navier-Stokes equations to consider the convergence of the process. Apart from the statement of Thom in Ref. 4 that the process may not be convergent if the distance between points of the mesh on which the solution is calculated exceeds some limiting figure, the authors are not aware of any published criterion for convergence.

2. *Navier-Stokes Equations.*—The Navier-Stokes equations for the two-dimensional flow of a viscous fluid under conditions invariant with time can be written as

$$\nabla^2\psi = \zeta \quad \ldots \quad (1)$$

$$\nabla^2\zeta = \frac{1}{\nu}\left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}\right), \quad \ldots \quad (2)$$

where ν is the kinematic viscosity, ψ is the stream function and ζ is the vorticity. For the numerical solution of these equations the field of flow is replaced by a rectangular mesh at the discrete points of which values of ψ and ζ are calculated by finite difference approximations to the equations (1) and (2).

In this investigation the approximate equations used were

$$\psi_0 = \psi_m - \frac{n^2}{2}\zeta_0 \quad \ldots \quad (3)$$

$$\zeta_0 = \zeta_m - \frac{1}{16\nu}[(a-c)(B-D) + (b-d)(C-A)], \quad \ldots \quad (4)$$

where ψ_0 is the value of ψ at the centre of a square of side $2n$, recalculated from the corner values, and ψ_m is the mean of these corner values. In equation (4) the small letters represent ζ values and capital letters ψ values at the mesh points, as shewn in Fig. 1. The method of solution is one of reiteration. Assumed values of ψ and ζ are placed at the discrete points of the mesh and these values are then progressively improved at each point in turn by recalculation from values at surrounding points by the use of equations (3) and (4).

The convergence of this process was examined for general two-dimensional viscous flow. The method employed was as follows. A finite disturbance was applied at a point in the steady flow such as point O in Fig. 2. At the outer points $E, L, F \ldots K$, the values of ψ and ζ were assumed to remain fixed at the settled values and at points A, B, C, D new values of these functions were calculated using equations (3) and (4). From these disturbed values at A, B, C, D, ψ and ζ were recalculated at O. If the new value of ζ is nearer the correct value than the original disturbed one, the process is convergent. Further details of the analysis are given in Appendix I.

The criterion for convergence of the numerical solutions was found to be:

$$\frac{1}{\nu^2}\left| 2(E-G)^2 + 2(F-H)^2 + (\overline{E-G} + \overline{F-H})(L-N) + (\overline{E-G} - \overline{F-H})(K-M) \right| < 1280$$

where capital letters, as before, represent ψ values at the points. This criterion can be expressed with good approximation (*see* Appendix I) as

$$R_s^2 < 20$$

where R_s, the local 'mesh Reynolds number', is defined by $R_s = (qn)/\nu$, q being the local velocity of flow and $2n$ the distance between adjacent points of the mesh.

The analysis of convergence has been based on a simplified representation of the numerical process, in that a finite disturbance was applied at only one point in the settled field and its effect was allowed to spread over only four squares of the mesh. Nevertheless, in actual solutions the criterion derived above has been found to predict quite closely the areas in which the routine numerical process failed to converge.

These investigations are a development and extension of some considerably earlier work of the senior author, who had examined the convergence of the numerical process when applied to the particular case of Plane Poiseuille Motion[4].

3. *Boundary Conditions*.—At solid boundaries ψ is usually known. On the other hand ζ at the boundary must be calculated from the pattern of the flow in the vicinity of the boundary. The method of solution is to calculate the values of ψ and ζ in the flow for certain assumed values of ζ on the boundary. Boundary values of ζ are then recalculated from these new values of ψ and ζ in the flow and the sequence is continued until the boundary values repeat themselves to the desired degree of accuracy.

Several formulae for recalculation of boundary values of ζ have been suggested. Of these, the ones most frequently used by the authors are:

$$\zeta_0 = \frac{2(\psi_1 - \psi_0)}{m^2} \quad \ldots \quad (5)$$

$$\zeta_0 = \frac{3(\psi_1 - \psi_0)}{m^2} - \frac{\zeta_1}{2}, \quad \ldots \quad (6)$$

where ζ_0 is the boundary value and ψ_1, ζ_1, are values at a point in the flow distant m from the boundary (Fig. 3). (For the derivation of these formulae *see* Refs. 5 and 6).

The convergence of this process was examined by a method similar to that described in the preceding section. It was found that under certain conditions this process, too, will fail to converge. The criterion for convergence depends on which equation, (5) or (6), is used for recalculation of ζ on the boundary. Its magnitude depends also on whether the component, v, of the local velocity normal to the boundary is directed away from or towards the boundary. If equation (5) is used the criterion is

$$\left|\frac{nv}{\nu}\right| < 9 \cdot 9 \text{ when } v \text{ is directed away from}$$

the boundary and

$$\left|\frac{nv}{\nu}\right| < 12 \cdot 7 \text{ when } v \text{ is directed towards}$$

the boundary. If equation (6) is used the corresponding criteria are

$$\left|\frac{nv}{\nu}\right| < 3 \cdot 1 \text{ and}$$

$$\left|\frac{nv}{\nu}\right| < 5 \cdot 9 \text{ respectively.}$$

n is as shown in Fig. 3.

In practice the authors have found that even when these criteria are satisfied it is often necessary to employ a technique similar to that described in Appendix II to achieve a reasonable rate of convergence. This consists in applying to the boundary values of ζ only part of the movement indicated by the cycle of calculations.

4. *Conclusions*.—The investigation described above showed that the convergence of the numerical solution of the Navier-Stokes equations depends on the local 'mesh Reynolds number'. It was found also that convergence of the solution at a solid boundary is governed by a similar criterion. In particular cases experience shows that a mesh size of order one-half that indicated by these criteria will certainly give convergence, but if the mesh is made larger than this while still being less than the critical size, the solution, though it probably converges, approaches the final values only slowly.

The critical size of mesh will depend on the actual finite difference approximations to equations (1) and (2) which are used and also on the sequence of calculation (*see* Appendix II). What is significant is that a critical size of mesh does exist for each problem; provided the mesh is made small enough a solution can be obtained for the Navier-Stokes equations for steady viscous flow at any value of Reynolds number. There is, of course, the practical consideration that if the critical size of mesh in a problem is very small, the labour involved in a solution by desk computation may be prohibitive.

It should be noted that these considerations do not have bearing on the questions of hydrodynamic stability and turbulence; all derivatives with respect to time have been omitted as steady flow was postulated.

LIST OF SYMBOLS

(x, y)	The physical plane
(α, β)	The transformed plane
M	The modulus of transformation from the physical to the $\alpha\beta$-plane
ψ	The stream function in viscous flow
ζ	The vorticity
δ	A finite disturbance to the value of ζ at a point, as defined in Appendix I.
∇^2	The Laplacian operator, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
ν	The kinematic viscosity
$2n$	The distance between adjacent points of the mesh on which a numerical solution of viscous flow is obtained
m	The distance of a point in the field of flow from a solid boundary
R_s	The local 'mesh Reynolds number' defined by $R_s = \frac{qn}{\nu}$
q	The local velocity of flow
u, v	The rectangular components of q, parallel to the axes of the mesh in use for the numerical solution

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APPENDIX I

For the sake of conciseness the method of examining the convergence of numerical solutions of the Navier-Stokes equations was described only in outline in the body of this paper. The method is treated in greater detail below.

In Fig. 2, which represents part of a field of general two-dimensional viscous flow, the values of ψ and ζ at all points were assumed to be, initially, the settled values. At O a finite disturbance was applied to the flow; the value of ζ at O was changed from ζ_0 to $\zeta_0 + \delta$, the value of ψ_0 being unaltered. The values of ψ and ζ at the outer points $E, L, F \ldots K$ were assumed to remain unchanged from the settled values and at points A, B, C, D equations (4) and (3) were used to obtain new (*i.e.*, disturbed) values of ζ and ψ, in that order. From these disturbed values of ζ and ψ, a new value of ζ at O was obtained from equation (4);

$$\zeta_0' = \zeta_0 + \frac{\delta}{4} - \frac{\delta}{1024\nu^2}[2(E-G)^2 + 2(F-H)^2 + (\overline{E-G} + \overline{F-H})(L-N)$$
$$+ (\overline{E-G} - \overline{F-H})(K-M)],$$

where capital letters signify the values of ψ at the points. The first disturbed value of ζ at O was $\zeta_0 + \delta$ and hence for convergence

$$\left|\frac{\delta}{4} - \frac{\delta}{1024\nu^2}[2(E-G)^2 + 2(F-H)^2 + (\overline{E-G} + \overline{F-H})(L-N) + (\overline{E-G} - \overline{F-H})(K-M)]\right| < \delta.$$

The second term of the left-hand side of this inequality is itself always positive and the condition for convergence can be expressed as

$$\frac{1}{\nu^2}[2(E-G)^2 + 2(F-H)^2 + (\overline{E-G} + \overline{F-H})(L-N) + (\overline{E-G} - \overline{F-H})(K-M)] < 1280.$$

Provided the mesh is not too coarse this can be written, with sufficient accuracy, as

$$\frac{4}{\nu^2}[(E-G)^2 + (F-H)^2] < 1280.$$

The accuracy of this approximation was tested by application to a large number of points in the field of flow past a circular cylinder at Reynolds number 40. In every case, even where the mesh was of such size that convergence would not be obtained, the approximation differed from the exact expression by less than 10 per cent. This is considered of sufficient accuracy for defining the limiting size of a mesh because in actual solutions when the field is being sub-divided the mesh size is always reduced by a factor of $1/\sqrt{2}$ or $\frac{1}{2}$.

Now $$\frac{F-H}{4n} = u$$

and $$\frac{E-G}{4n} = v,$$

where u and v are the magnitudes of the average velocity at right angles to FH and EG in Fig. 2, respectively. Hence the criterion for convergence can be expressed as

$$\frac{q^2 n^2}{\nu^2} < 20,$$

i.e.,

$$R_s^2 < 20,$$

where q (given by $q^2 = u^2 + v^2$) is the average velocity of flow through the mesh in Fig. 2.

The analysis above has been described as applying to calculations performed in the physical plane. When the solution is to be obtained in a transformed plane, say the $\alpha\beta$-plane, the Navier-Stokes equations take the form[1]:

$$\nabla_{\alpha\beta}^2 \psi = \frac{\zeta}{M^2} \quad \ldots \tag{1a}$$

$$\nabla_{\alpha\beta}^2 \zeta = \frac{1}{\nu}\left(\frac{\partial\psi}{\partial\alpha}\frac{\partial\zeta}{\partial\beta} - \frac{\partial\psi}{\partial\beta}\frac{\partial\zeta}{\partial\alpha}\right), \quad \ldots \tag{2a}$$

where M is the modulus of transformation. The corresponding finite difference approximations are

$$\psi_0 = \psi_m - \frac{n^2\zeta_0}{2M^2} \quad \ldots \tag{3a}$$

$$\zeta_0 = \zeta_m - \frac{1}{16\nu}\,[(a-c)(B-D) + (b-d)(C-A)]. \quad \ldots \tag{4a}$$

The convergence of numerical solutions of these equations is governed by a similar criterion to that for solutions in the physical plane. In fact, if the criterion is expressed in terms of ψ values it is exactly the same as that for the physical plane, *i.e.*,

$$\frac{4}{\nu^2}\,[(E-G)^2 + (F-H)^2] < 1280.$$

If this is expressed in terms of the ' mesh Reynolds number ' it becomes

$$\frac{q^2 n^2}{\nu^2} < 20M^2$$

where n is one-half the size of the mesh in the $\alpha\beta$-plane.

It should be noted that the criteria for convergence given in this paper are criteria for convergence of ζ. However, since ψ is obtained from ζ by equation (1) or (1a), if ζ is convergent so should be ψ. This has been the experience in actual problems. While it has not been possible to demonstrate the validity of this deduction in a theoretical analysis of general two-dimensional viscous flow, convergence of the numerical process when the finite disturbance is applied as a change of value of ψ_0 has been examined for two particular cases; Plane Poiseuille Motion and Plane Shear Motion. In each case the criterion was found to be less stringent than the one obtained above by considering the convergence of ζ.

APPENDIX II

In this paper the analysis of convergence has been made on the assumption that the re-calculated value of ζ at O obtained by the sequence described in Appendix I is taken as the new value to be used in the next cycle of calculations.

Thus $\qquad \zeta_0 \text{ initial} = \zeta_0 + \delta$

$$\zeta_0 \quad \text{new} = \zeta_0 + \frac{\delta}{4} - \frac{\delta}{256\nu^2}\left[(E - G)^2 + (F - H)^2\right].$$

However, the new value of ζ at O may be taken as intermediate between these two:

$$\zeta_0 \quad \text{new} = \zeta_0 + \frac{\delta}{(a+1)}\left\{a + \frac{1}{4} - \frac{1}{256\nu^2}\left[(E - G)^2 + (F - H)^2\right]\right\},$$

where the two values of ζ have been combined in the proportion of $a : 1$.

If the criterion of convergence is obtained, using this revised value of ζ_0, it reduces to

$$\frac{1}{\nu^2}\left[(E - G)^2 + (F - H)^2\right] < 512a + 320$$

$$\text{or } \frac{n^2q^2}{\nu^2} < 32a + 20.$$

It has been assumed that only positive values of a are admissible. Since a can be chosen to have any positive value it follows that convergence can be obtained in this way for almost any value of R_s. The proviso should be made that if nq/ν is very great the rate of convergence is likely to be too slow for practical purposes. Further, there is an upper limit to the value of n which may be used, because if n is made too great the neglected terms in the finite difference approximations will become significant and serious ' truncation errors ' will be introduced.

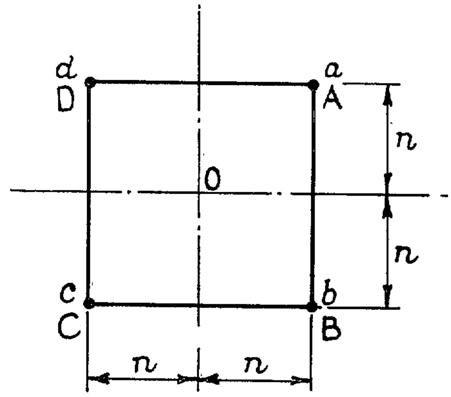

FIG. 1.

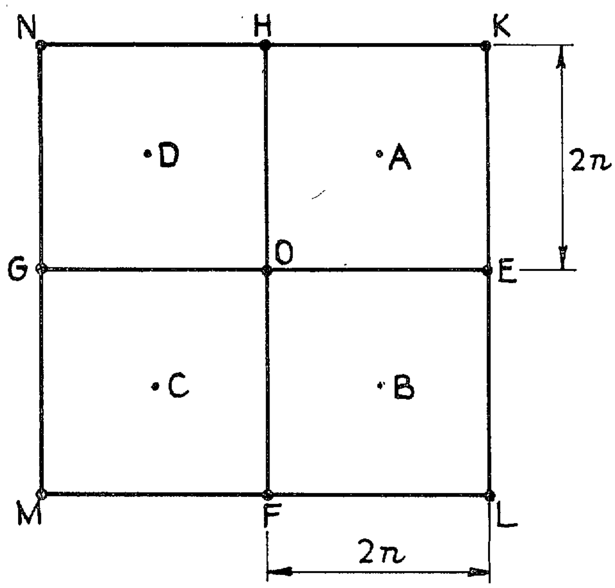

FIG. 2.

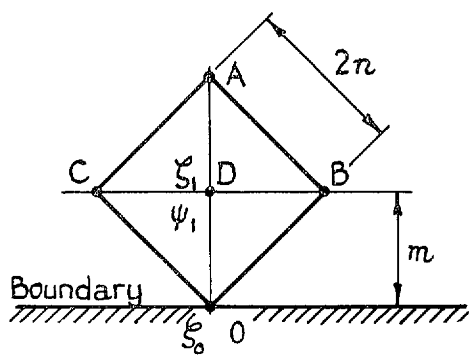

FIG. 3.

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