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By

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The Theory of Torsional Vibrations of a Four-Boom Thin-Walled Cylinder of Rectangular Cross-Section

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COMMUNICATED BY THE PRINCIPAL DIRECTOR OF SCIENTIFIC RESEARCH (AIR), MINISTRY OF SUPPLY

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Summary.—The torsional vibrations of a four-boom cylinder of doubly symmetrical rectangular cross-section are considered and the differential equation of motion is derived on the assumption that the ribs maintain the section shape but do not themselves resist any warping out of their plane and that the walls of the cylinder are effective only in shear.

Frequency equations are derived for a length of cylinder, free at both ends and prevented from rotating at the mid-section. The complete behaviour of the cylinder is determined by two non-dimensional parameters and curves are given from which the frequencies for any such cylinder may be determined. It is shown that the higher frequencies in particular may be underestimated by between 40 to 80 per cent if warping constraint effects are ignored.

An approximate method is given for estimating the torsional frequencies of a cylinder with non-uniform characteristics.

1. *Introduction.*—Under static loading the torsional stiffness at any section along a thin-walled cylinder is given sufficiently accurately by Batho theory unless the section is in the neighbourhood of some constraint. Such constraints may be caused externally, as when one end of the cylinder is built-in and therefore restrained against warping, or internally, as when there is an abrupt change in loading at a section and therefore an interaction at this section between the two parts of the cylinder. In either case the change in torsional stiffness is caused by the tendency of the booms to resist variations in the axial warping of cross-sections, a tendency which causes a redistribution of shear in the walls of the cylinder and a consequent increase in torsional stiffness.

Under dynamic loading, in which the inertia torque varies continuously along the length of the cylinder, it is therefore to be expected that the simple Batho theory will not be sufficiently accurate. That the constraints are not of secondary importance can be seen by considering the limiting case of very high frequencies when adjacent wavelengths effectively oppose each other's tendency to warp; for a typical wing section the stiffness determined on the basis of zero warping is about three times that determined by Batho theory. The increase in torsional stiffness is, however, less marked for the lower frequencies. Furthermore, there are cylinders which do not warp when resisting torsion; for such cylinders the torsional stiffness will not vary.

In order to investigate this stiffening effect in detail, a four-boom, thin-walled cylinder of doubly symmetrical, rectangular cross-section has been considered. By assuming that the ribs maintain

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the section shape and that the walls are effective only in shear, it has been possible to find the torsional frequencies and modes of vibration of a cylinder, free at both ends and prevented from rotating at the mid-section.

An approximate method, based on the vibrations of an infinitely long cylinder, has been developed; a dynamic torsional stiffness has been introduced as a simplifying aid to the calculation of frequencies in a non-uniform cylinder.

2. *Derivation of the Differential Equation of Motion*.—2.1. *Assumptions*.—The following assumptions are made regarding the structures:

(*a*) Stress-strain relations are linear

(*b*) Buckling does not take place

(*c*) Rivet flexibility is negligible

(*d*) Ribs maintain the section shape but do not offer any resistance to warping out of their plane

(*e*) The walls of the cylinder are effective only in shear.

Assumptions (*a*) to (*d*) are standard practice. The direct load-carrying capacity of the walls may be taken into account by adding to each boom 1/6th of the section area of the adjacent walls.

2.2. *Stress-strain Relations*.—The differential equation of motion of a thin-walled cylinder of doubly symmetrical, rectangular section will be formed. First, two equilibrium equations will be found in terms of θ the rotation of a section, and u the warping of a section as measured by the axial displacement of a boom. The structure under consideration and the directions of positive θ and u are shown in Fig. 1. From symmetry only a quarter of the cross-section need be considered.

$d\theta/dx$ and u both contribute to the shear strains in the four shear panels of the cylinder. The shear strain in the side panels is

$$a\frac{d\theta}{dx}+\frac{u}{b}$$

so that the shear per unit length is

$$Gt_1\left\{a\frac{d\theta}{dx}+\frac{u}{b}\right\}. \qquad (1)$$

Similarly the shear per unit length in the top and bottom panels is

$$Gt_2\left\{b\frac{d\theta}{dx}-\frac{u}{a}\right\}. \qquad (2)$$

The total torque at a section is therefore given by

$$T=4G\left\{ab(at_1+bt_2)\frac{d\theta}{dx}+(at_1-bt_2)u\right\}. \qquad (3)$$

2.3. *Equilibrium of Boom Element*.—Differences in the shears per unit length in the panels adjacent to a boom will alter the direct load in the boom, so that

$$\frac{dP}{dx}=EF\frac{d^2u}{dx^2}$$

$$=Gt_1\left\{a\frac{d\theta}{dx}+\frac{u}{b}\right\}-Gt_2\left\{b\frac{d\theta}{dx}-\frac{u}{a}\right\}. \qquad (4)$$

Strictly, there should be an inertia term in this equation, arising from the fact that the boom is vibrating axially, but it is shown in Appendix I that this can be neglected.

2.4. *Torsional Equilibrium.*—By considering the equilibrium of an elemental slice of the cylinder it is found that

$$J\frac{d^2\theta}{dt^2} = \frac{dT}{dx}$$

$$= 4G\left\{ab(at_1 + bt_2)\frac{d^2\theta}{dx^2} + (at_1 - bt_2)\frac{du}{dx}\right\} \quad (5)$$

where J is the polar moment of inertia per unit length.

θ and u are functions of x and t, but if the cylinder is vibrating with a frequency ω we can write

$$\left.\begin{aligned}\theta &= \theta(x)\sin\omega t,\\ u &= u(x)\sin\omega t\end{aligned}\right\} \quad (6)$$

and t may now be eliminated from all the equations by dividing throughout by $\sin\omega t$. In particular equation (5) becomes

$$J\omega^2\theta + \frac{dT}{dx} = 0\,. \quad (7)$$

2.5. *Equation of Motion.*—Either θ or u may be eliminated from equations (4) and (7) to give an equation in u or θ alone:

$$\left[\frac{d^4}{d\xi^4} - (\alpha^2 - \beta^2\Omega^2)\frac{d^2}{d\xi^2} - \alpha^2\Omega^2\right][\theta \text{ or } u] = 0 \quad (8)$$

where the following non-dimensional parameters have been introduced:

$$\alpha = \frac{2L}{\pi}\left\{\frac{4Gt_1t_2}{EF(at_1 + bt_2)}\right\}^{1/2}, \quad (9)$$

$$\beta = \frac{2\sqrt{(abt_1t_2)}}{at_1 + bt_2} \leqslant 1\,, \quad (10)$$

$= 1$ if there is no tendency to warp,

$$\xi = \frac{\pi x}{2L}\,, \quad (11)$$

$$\Omega = \frac{\omega}{\omega_{B,0}} \quad (12)$$

where

$$\omega_{B,0} = \frac{2\pi ab}{L}\left\{\frac{Gt_1t_2}{J(at_1 + bt_2)}\right\}^{1/2}. \quad (13)$$

If there are no warping constraints the torsional stiffness is that given by Batho theory and for the fundamental frequency $\Omega = 1$.

2.6. *General Solution to the Equation.*—Solutions of equation (13) are of the form $e^{\mu\xi}$ where μ is a root of

$$\mu^4 - (\alpha^2 - \beta^2\Omega^2)\mu^2 - \alpha^2\Omega^2 = 0. \quad (14)$$

By introducing

$$\left.\begin{aligned} \mu_1{}^2 &= \tfrac{1}{2}\{[(\alpha^2 - \beta^2\Omega^2)^2 + 4\alpha^2\Omega^2]^{1/2} - \alpha^2 + \beta^2\Omega^2\} \\ \mu_2{}^2 &= \tfrac{1}{2}\{[(\alpha^2 - \beta^2\Omega^2)^2 + 4\alpha^2\Omega^2]^{1/2} + \alpha^2 - \beta^2\Omega^2\} \\ \mu_1 &\equiv \frac{\alpha\Omega}{\mu_2} \end{aligned}\right\} \quad \ldots \quad (15)$$

the general solution may be written

$$\left.\begin{aligned} \theta &= W_\theta \sin \mu_1\xi + X_\theta \cos \mu_1\xi + Y_\theta \sinh \mu_2\xi + Z_\theta \cosh \mu_2\xi \\ \left\{\frac{-2L}{\pi ab\sqrt{(1-\beta^2)}}\right\}u &= W_u \sin \mu_1\xi + X_u \cos \mu_1\xi + Y_u \sinh \mu_2\xi + Z_u \cosh \mu_2\xi \end{aligned}\right\} \quad \ldots \quad (16)$$

The factor in the expression for u has been chosen so that if warping constraints are ignored (i.e., $\alpha = \infty$) and the fundamental mode is considered:

$$X_u = W_\theta\,,$$

the other constants being zero.

From equations (4) or (7) the eight arbitrary constants satisfy the following relations:

$$\left.\begin{aligned} \mu_1(1-\beta^2)W_u &= -(\mu_1{}^2 - \beta^2\Omega^2)X_\theta \\ \mu_1(1-\beta^2)X_u &= +(\mu_1{}^2 - \beta^2\Omega^2)W_\theta \\ \mu_2(1-\beta^2)Y_u &= +(\mu_2{}^2 + \beta^2\Omega^2)Z_\theta \\ \mu_2(1-\beta^2)Z_u &= +(\mu_2{}^2 + \beta^2\Omega^2)Y_\theta \end{aligned}\right\} \quad \ldots \quad (17)$$

Four other relations are needed between these constants and they will come from a consideration of the two boundary conditions at each end of the cylinder.

3. *The Frequency Equations.*—Consider a cylinder of length $2L$, free at its ends but prevented from rotating at its centre-section (at $\xi = 0$). Referring to the behaviour of one half of the cylinder it will be seen that when warping constraints are ignored there is no distinction between symmetrical and anti-symmetrical torsional vibration, the condition at the centre-section being merely that $\theta = 0$ in each case. But when warping constraints are taken into account it will be seen that at the centre, in addition to zero rotation, $u = 0$ for the symmetrical case (from symmetry) and $du/d\xi = 0$ for the anti-symmetrical case (no boom load from symmetry).

It must be pointed out that for the anti-symmetrical case there is no need to prevent rotation at the centre-section as there is a θ-node at that point; in fact the cylinder can be regarded as being completely free.

3.1. *Symmetrical Vibration.*—The four boundary conditions are:

$$\left.\begin{aligned} &\text{at } \xi = 0, \quad \theta = 0 \quad \text{and } u = 0\,, \\ &\text{at } \xi = \frac{\pi}{2}\,, \quad T = 0 \quad \text{and } \frac{du}{d\xi} = 0 \end{aligned}\right\} \quad \ldots \quad (18)$$

Using equations (17) these four conditions may be expressed in terms of W_u, X_u, Y_u, Z_u. For there to be a solution other than

$$W_u = X_u = Y_u = Z_u = 0$$

the determinant of these four equations must vanish. This determinantal equation determines the frequency of vibration Ω.

Thus, for the symmetrical vibration

$$\begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{|c|c|c|c|} \multicolumn{1}{c}{(W_u)} & \multicolumn{1}{c}{(X_u)} & \multicolumn{1}{c}{(Y_u)} & \multicolumn{1}{c}{(Z_u)} \\ \hline (\beta^2\Omega^2 + \mu_2{}^2)\mu_1 & 0 & (\beta^2\Omega^2 - \mu_1{}^2)\mu_2 & 0 \\ 0 & 1 & 0 & 1 \\ (\beta^2\Omega^2 + \mu_2{}^2)\sin\frac{\pi}{2}\mu_1 & (\beta^2\Omega^2 + \mu_2{}^2)\cos\frac{\pi}{2}\mu_1 & (\beta^2\Omega^2 - \mu_1{}^2)\sinh\frac{\pi}{2}\mu_2 & (\beta^2\Omega^2 - \mu_1{}^2)\cosh\frac{\pi}{2}\mu_2 \\ \mu_1\cos\frac{\pi}{2}\mu_1 & -\mu_1\sin\frac{\pi}{2}\mu_1 & \mu_2\cosh\frac{\pi}{2}\mu_2 & \mu_2\sinh\frac{\pi}{2}\mu_2 \\ \hline \end{array} = 0 \qquad (19)$$

which reduces to

$$\alpha\Omega(1-\beta^2)\left\{2\alpha\Omega + (\alpha^2 - \beta^2\Omega^2)\sin\frac{\pi}{2}\mu_1\sinh\frac{\pi}{2}\mu_2\right\}$$
$$+\left\{\alpha^4 + 2\alpha^2\Omega^2 + \beta^4\Omega^4\right\}\cos\frac{\pi}{2}\mu_1\cosh\frac{\pi}{2}\mu_2 = 0 . \quad .. \quad .. \quad .. \qquad (20)$$

3.2. *Anti-symmetrical Vibration.*—The boundary conditions are as before, except that at $\xi = 0$, $du/d\xi = 0$ instead of $u = 0$. Thus, the determinantal equation is the same as (19) except that the second line (0, 1, 0, 1) becomes $(\mu_1, 0, \mu_2, 0)$.

This reduces to

$$(\mu_1{}^2 - \beta^2\Omega^2)\mu_1\tan\frac{\pi}{2}\mu_1 = (\mu_2{}^2 + \beta^2\Omega^2)\mu_2\tanh\frac{\pi}{2}\mu_2 . \quad .. \quad .. \quad .. \qquad (21)$$

4. *Numerical Values.*—When warping constraint effects are ignored Ω_B is a root of the equation

$$\cos\frac{\pi}{2}\Omega_B = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (22)$$

so that

$$\Omega_{B,n} = 2n + 1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (23)$$

where n is the number of the mode, $n = 0$ corresponding to the fundamental mode.

Now the values of Ω which satisfy equations (20) and (21) are functions of α, β as well as n, but if a further non-dimensional frequency parameter λ_n is introduced such that

$$\left.\begin{aligned} \lambda_n &= \frac{\omega_n}{\omega_{B,n}} \\ &= \frac{\Omega_n}{2n+1} \end{aligned}\right\} \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \qquad (24)$$

for any particular value of n, λ_n will be a function only of α and β and will lie between 1 and $1/\beta$, approaching $1/\beta$ as n increases. λ_n therefore affords a useful comparison with the results based on simple Batho theory.

λ_0 and λ_1 have been plotted in Figs. 2, 3, 4 and 5 for various values of α and β covering the practical range. $\lambda_{0,\text{ anti.}}$ differs very slightly from unity and the difference between $\lambda_{0,\text{ symm.}}$ and unity is practically all due to the local increase in stiffness near the root and an approximate expression could be found for it as in Appendix II.

λ_n has been plotted against n in Figs. 6 and 7 for $\alpha = 5$, $\beta = 0{\cdot}6$ and $\alpha = 2{\cdot}5$, $\beta = 0{\cdot}6$, both cases representative of typical wing structures.

The frequencies in the symmetrical vibration are naturally higher than in the anti-symmetrical case, though as higher modes are considered the corresponding frequencies approach each other.

5. *Approximate Analysis for the Higher Modes.*—It has been pointed out that $\lambda_{n,\,\text{symm.}}$ and $\lambda_{n,\,\text{anti.}}$ approach each other as n increases. The reason is that the end effects become relatively less important for the higher modes and this suggests that a simple approximate analysis might be possible.

If the cylinder is regarded as part of an infinitely long one vibrating uniformly equation (16) becomes

$$\theta = \sin \mu_1 \xi \ .$$

Now, in the nth mode there are approximately $(2n + 1)$ half-waves in the actual cylinder, from $\xi = 0$ to $\pi/2$, so that

$$\mu_1 = 2n + 1$$

whence

$$\left.\begin{aligned} \lambda_n &= \left[\frac{1 + \alpha_n^{\,2}}{\beta^2 + \alpha_n^{\,2}}\right]^{1/2} \\ \text{where} \qquad \alpha_n &= \frac{\alpha}{2n + 1} \ . \end{aligned}\right\} \qquad \ldots \qquad (25)$$

This approximate value has been plotted in Figs. 6 and 7 and compared with the true values.

An alternative way of appreciating these results is to introduce a dynamic torsional stiffness. Equation (25) could then be written

$$\frac{\text{dynamic torsional stiffness}}{\text{static torsional stiffness}} = \frac{1 + \alpha_n^{\,2}}{\beta^2 + \alpha_n^{\,2}} \ . \qquad \ldots \qquad (26)$$

6. *The Shape of the Modes.*—Once the frequency of vibration is known the relative magnitudes of the constants W_u, W_θ, etc., may be obtained from equation (17) and the appropriate boundary conditions.

For the cylinder considered in Fig. 7 the fundamental and first harmonic modes have been plotted in Figs. 8 and 9. The symmetrical and anti-symmetrical modes are plotted so that they may be compared with each other and with the simple mode obtained by ignoring warping constraints. The modes are plotted on the basis of a unit amplitude of rotation at the tip. It will be seen that the effect of ignoring warping constraints does not appreciably alter the θ-modes, though the ratio of amplitude of rotation at the tip to the amplitude of rotation near the root is somewhat underestimated. The shape of the u-modes, of little practical importance, are appreciably altered when warping constraints are taken into account. By considering the modes of vibration of an infinitely long cylinder, as in section 5, it is found that

$$\frac{\text{actual amplitude of } u\text{-displacement}}{\text{amplitude of } u\text{-displacement by simple Batho theory}} = \frac{1}{1 + (\beta/\alpha_n)^2} \ . \qquad \ldots \qquad (27)$$

7. *Application to Cylinder with Non-uniform Characteristics along its Length.*—The conclusions reached so far are strictly applicable only to the cylinder of doubly symmetrical rectangular cross-section, and an exact solution for any other type of cross-section, or for a cylinder with non-uniform characteristics along its length, will clearly be very complicated. It has, however, been shown that the θ-modes are not appreciably altered by warping constraints, particularly in the higher modes: the main effect of warping constraints being rather to cause an increase in the apparent torsional stiffness of the cylinder. This fact may be used to obtain an estimate of the torsional frequencies in a non-uniform cylinder.

Consider for example a non-uniform cylinder vibrating in the second harmonic torsional mode—no distinction being drawn between symmetrical and anti-symmetrical vibration—and suppose that the θ-mode, based on simple Batho theory (*i.e.*, constraint effects ignored) is as given below in Fig. i. There are a number of approximate methods for obtaining this mode[2,3].

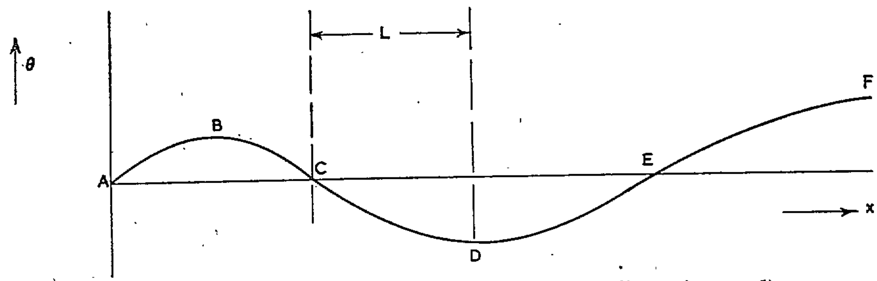

FIG. i. Second harmonic mode (constraint effects ignored).

It will be assumed that the actual θ-mode will not differ significantly from this, so that, in particular, a typical half-wave, such as CD, will not alter significantly. It is possible to obtain an average value for α and β over each such half-wave and then by applying equation (26) a modified value for the torsional stiffness in each half-wave will be obtained. If L is redefined as the length of each half-wave the factor in equation (26) becomes simply

$$\left(\frac{1+\alpha^2}{\beta^2+\alpha^2}\right) . \qquad (26a)$$

The modified stiffness along the length of the cylinder obtained in this manner should give a sufficiently accurate value for the frequency.

To determine the frequency in the fundamental symmetrical vibration the approximate analysis of Appendix II may be used.

8. *Conclusions.*—An exact solution has been obtained for the torsional vibrations of a four-boom, thin-walled cylinder of doubly-symmetrical rectangular cross-section. It is shown that the torsional stiffness based on simple Batho theory should not generally be used, especially if the higher frequencies are being considered. The vibrations of a length of cylinder, free at both ends and prevented from rotating at the mid-section, have been considered in detail and the following conclusions have been drawn.

(*a*) The complete behaviour of the cylinder may be described by the two non-dimensional parameters α and β

(*b*) The simple Batho stiffness is strictly correct only when cross-sections of the cylinder do not tend to warp under torsion, in which case $\beta = 1$

(*c*) The torsional frequencies will be underestimated if warping constraints are not taken into account and the relative amount will be greatest for the highest modes. The frequency will be underestimated by a factor which approaches β as the frequency increases

(*d*) There is a difference between the frequencies in the symmetrical and anti-symmetrical torsional vibrations unless there is no warping

(*e*) The frequency of the fundamental anti-symmetrical mode is practically that given by simple Batho theory; the frequency of the fundamental symmetrical mode may be up to 20 per cent higher than that given by simple Batho theory, but this increase may be estimated fairly accurately by assuming a reduced length for the cylinder to represent the local stiffening at the root[4]

(*f*) The frequencies in the symmetrical and anti-symmetrical modes approach each other as the order of the mode increases

(*g*) The θ-modes are not seriously affected by warping constraints.

A simplified method of approach is given for estimating the torsional vibrations of a cylinder with non-uniform characteristics along its length.

LIST OF SYMBOLS

$2a$		Width of cylinder
$2b$		Depth of cylinder
t_1		Skin thickness of vertical walls
t_2		Skin thickness of horizontal walls
x		Distance along cylinder
θ		Rotation of a section about x-axis
u		Axial warping of a section as measured by the axial displacement of a boom
t		Time
E, G		Elastic moduli
T		Torque
P		Load in a boom
F		Section area of a boom
J		Polar moment of inertia of cylinder per unit length
$2L$		Total length of cylinder
ω		Angular frequency, *i.e.*, cycles per second $\times\ 2\pi$
$\omega_{B,0}$		Fundamental angular frequency assuming Batho stiffness
Ω	=	$\omega/\omega_{B,0}$
α	=	$\dfrac{2L}{\pi}\left(\dfrac{4Gt_1t_2}{EF(at_1+bt_2)}\right)^{1/2}$
β	=	$\dfrac{2\sqrt{(abt_1t_2)}}{at_1+bt_2}$
ξ	=	$\dfrac{\pi x}{2L}$
μ_1, μ_2		Determined from equations (14) and (15)
$W_\theta, X_\theta, Y_\theta, Z_\theta$ W_u, X_u, Y_u, Z_u		Constants occurring in equation (16)
λ_n	=	$\dfrac{\omega_n}{\omega_{B,n}}=\dfrac{\Omega_n}{2n+1}$
α_n	=	$\dfrac{\alpha}{2n+1}$
m		Effective mass per unit length of a boom
γ	=	$4ab\left\{\dfrac{Gmt_1t_2}{EJF(at_1+bt_2)}\right\}^{1/2}$

Suffix $_B$ refers to results predicted by simple Batho theory

Suffix $_n$ refers to the nth mode, $n = 0$ corresponding to the fundamental.

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APPENDIX I

Effect of Boom Inertia in Axial Vibration

It has been pointed out in section 2.1 that there should strictly be an additional inertia term in the general equation of motion, arising from the fact that the booms are vibrating axially (unless $\beta = 1$). This additional inertia effect will reduce the frequencies, but it is shown here that this reduction may be neglected in all cases.

Referring to equation (4) the inertia term $m\omega^2 u$ must be added to $EF\, d^2u/dx^2$, where m is the effective mass per unit length of a boom.

The differential equation of motion may now be written

$$\left[\frac{d^4}{d\xi^4} - (\alpha^2 - \beta^2\Omega^2 - \gamma^2\Omega^2)\frac{d^2}{d\xi^2} - \Omega^2(\alpha^2 - \beta^2\gamma^2\Omega^2)\right]\left[\theta \text{ or } u\right] = 0 \quad \text{(27)}$$

where

$$\gamma = 4ab\left[\frac{Gmt_1t_2}{EJF(at_1 + bt_2)}\right]^{1/2} \quad \text{(28)}$$

a non-dimensional parameter, which, like β, is independent of the length of the cylinder.

To get a clearer idea of the magnitude of γ it is worth noting that for the type of cylinder considered here in which $a > b$, $t_1 > t_2$

$$\gamma \simeq \frac{1 \cdot 2b}{a}\sqrt{\left(\frac{at_2}{F}\right)}.$$

Referring to the case in which $\alpha = 5$, $\beta = 0 \cdot 6$ we may take

$$\gamma^2 \simeq 0 \cdot 08$$

Making use of the simplified analysis of section 5 the relation

$$\theta = \sin(2n + 1)\xi$$

may be substituted in equation (27) to obtain an equation for Ω:

$$(2n + 1)^4 + (\alpha^2 - \beta^2\Omega^2 - \gamma^2\Omega^2)(2n + 1)^2 - \Omega^2(\alpha^2 - \beta^2\gamma^2\Omega^2) = 0 \quad \text{(29)}$$

whence

$$\lambda_n = \left[\frac{2(1 + \alpha_n^2)}{\alpha_n^2 + \beta^2 + \gamma^2 + \{(\alpha_n^2 + \beta^2 + \gamma^2)^2 - 4\beta^2\gamma^2(1 + \alpha_n^2)\}^{1/2}}\right]^{1/2} \quad \text{(30)}$$

where

$$\alpha_n = \frac{\alpha}{2n + 1}.$$

Equation (30) may be compared with the simpler equation (25) in which these effects are ignored. Taking $\beta = 0 \cdot 6$, $\gamma = 0 \cdot 283$, Table 1 below shows the percentage reduction in frequency for a number of values of α_n.

TABLE 1

α_n	∞	2·5	1	0·5	0·1	0
per cent reduction in frequency	0	0.4	1·5	2·0 (a max.)	0·2	0

Effect of Boom Inertia in Axial Vibration upon the Frequency

Equation (29) is a quadratic in Ω^2 and so will have two distinct roots. The second, corresponding to (30) with a minus sign attached to the square root in the denominator, corresponds to a frequency in which axial vibrations, instead of torsional, are predominant. It has been pointed out that if $\beta = 1$ there is no tendency to warp when twisted; conversely there is no tendency to twist if forced to warp. It is worth noting that for this case in which there is no interaction the two roots of equation (29) are

$$\lambda_n = 1$$

$$\lambda_n' = \frac{\sqrt{(1 + \alpha_n^2)}}{\gamma}.$$

It is also interesting to note that, because of the interchangeability of β and γ in equation (29): if $\gamma = 1$

$$\lambda_n = 1\,,$$

whatever the value for β.

APPENDIX II

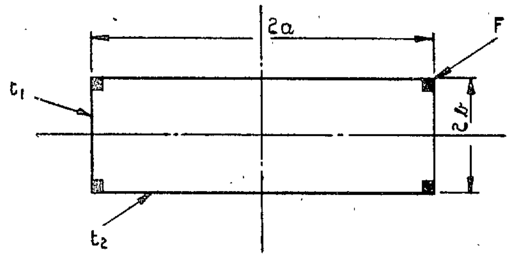
Increase in Stiffness Near the Root Due to Building-In

If concentrated torques are applied to the free ends of the cylinder it can be shown[4] that the symmetrical torque loading shows an increase in stiffness over the anti-symmetrical case given by the factor

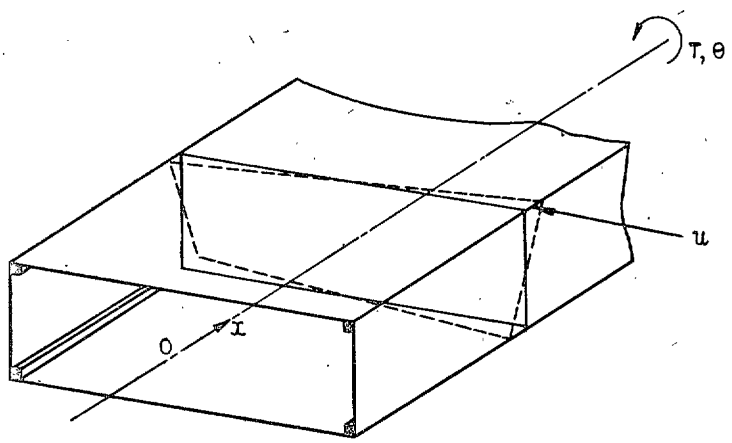
$$\frac{1}{1 - \frac{2(1 - \beta^2)}{\pi\alpha}\tanh\frac{\pi\alpha}{2}}.$$

If this increase is regarded as due to an effective shortening of the cylinder, as opposed to a uniformly distributed increase in stiffness, it follows from the arguments of section 4, that

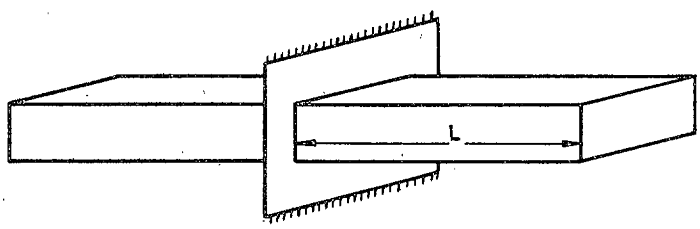
$$\Omega_{0,\,\mathrm{symm.}} \simeq 1 + \frac{2(1 - \beta^2)}{\pi\alpha}. \qquad (31)$$



(a) CROSS – SECTION OF THE CYLINDER



(b) CYLINDER, SHOWING NOTATION



(c) CYLINDER WITH BOTH ENDS FREE AND PREVENTED FROM ROTATING AT THE MID – SECTION

FIGS. 1a, 1b and 1c. The cylinder, showing notation.

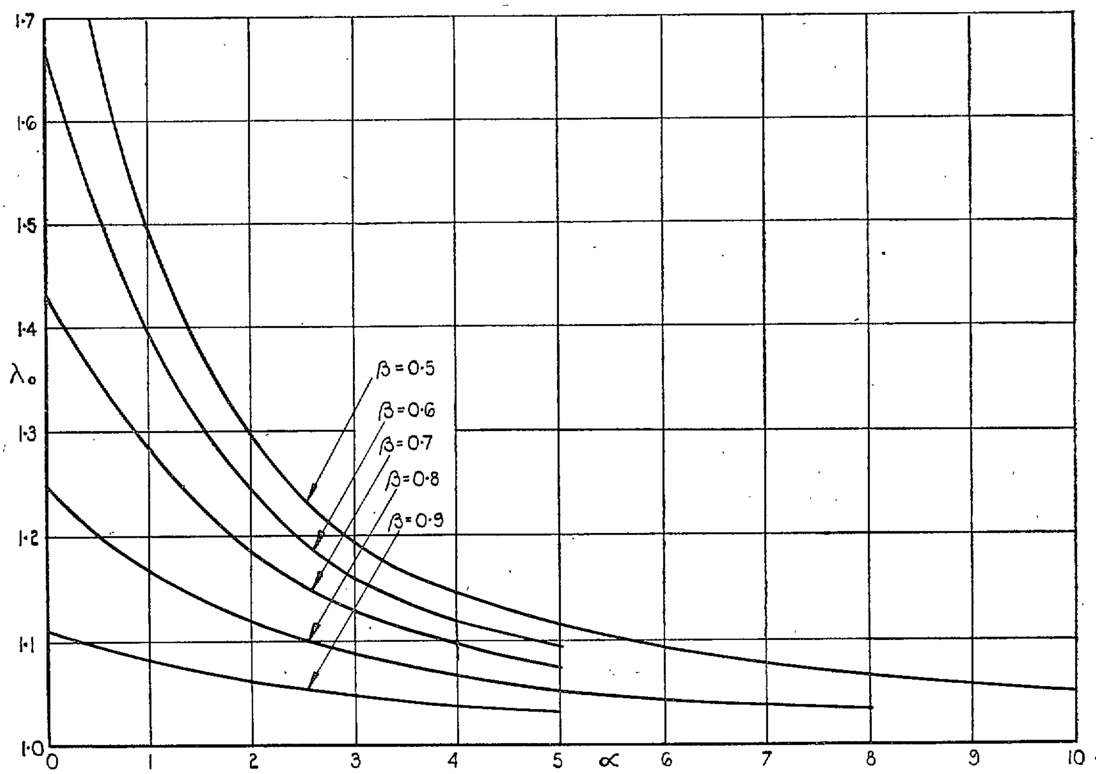


FIG. 2. Frequencies for symmetrical vibration—fundamental mode.

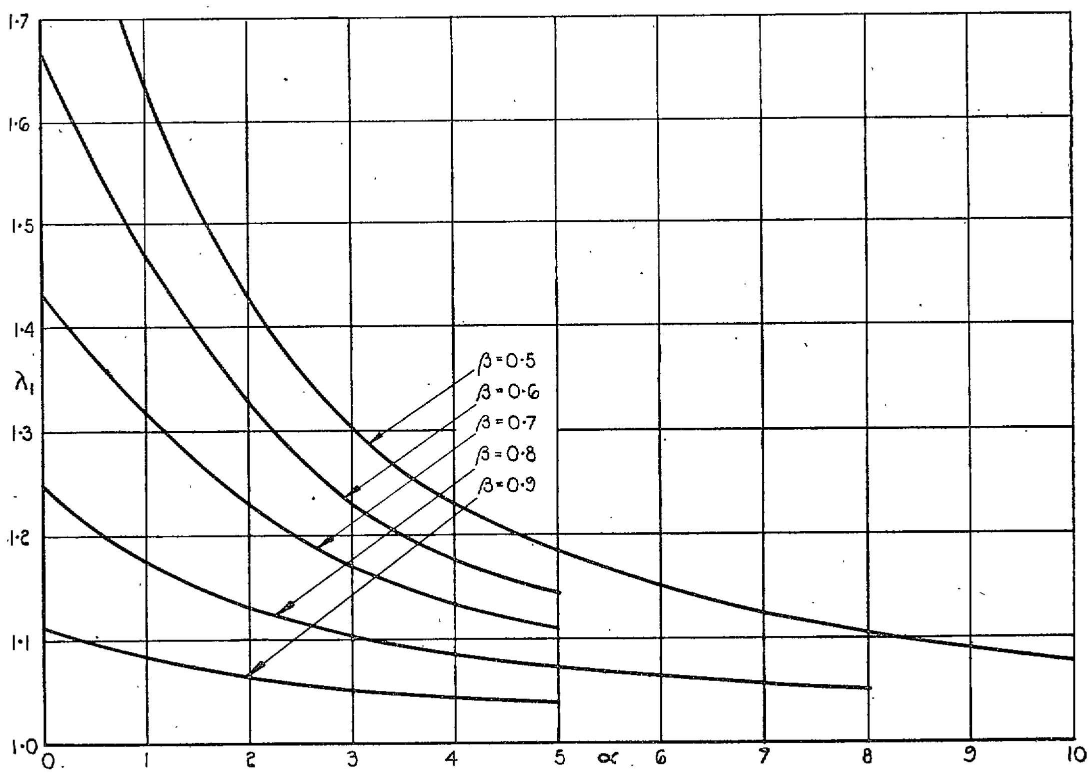


FIG. 3. Frequencies for symmetrical vibration—first harmonic.

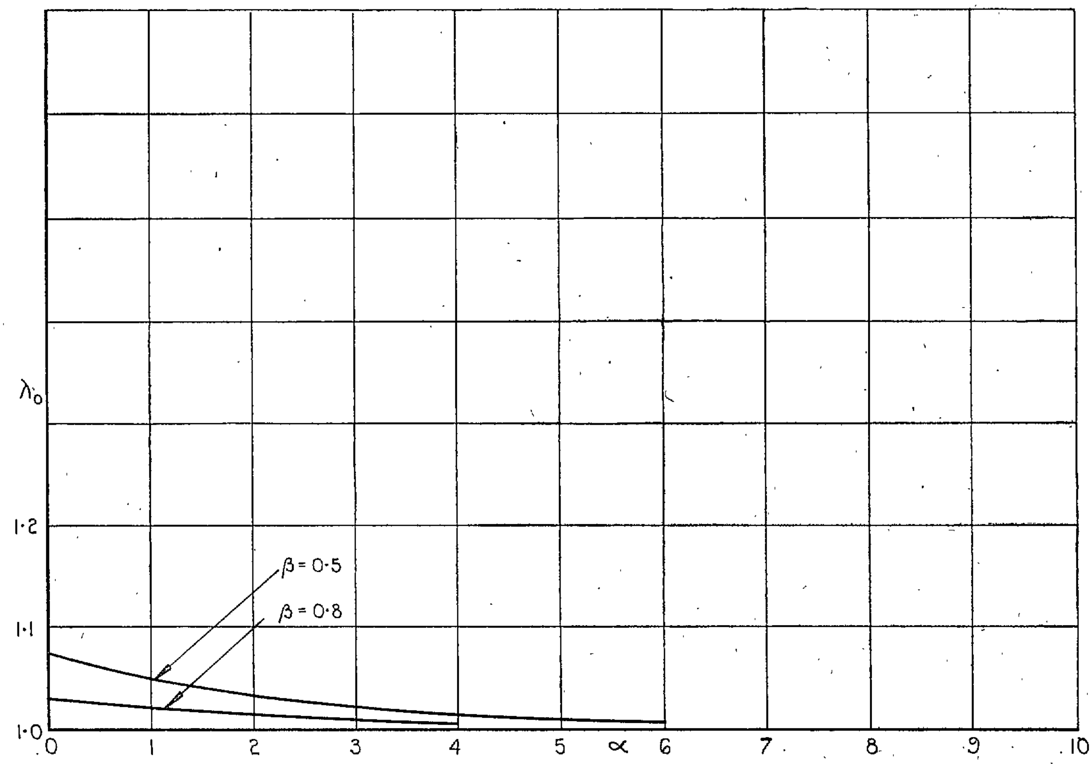


FIG. 4. Frequencies for anti-symmetrical vibration—fundamental mode.

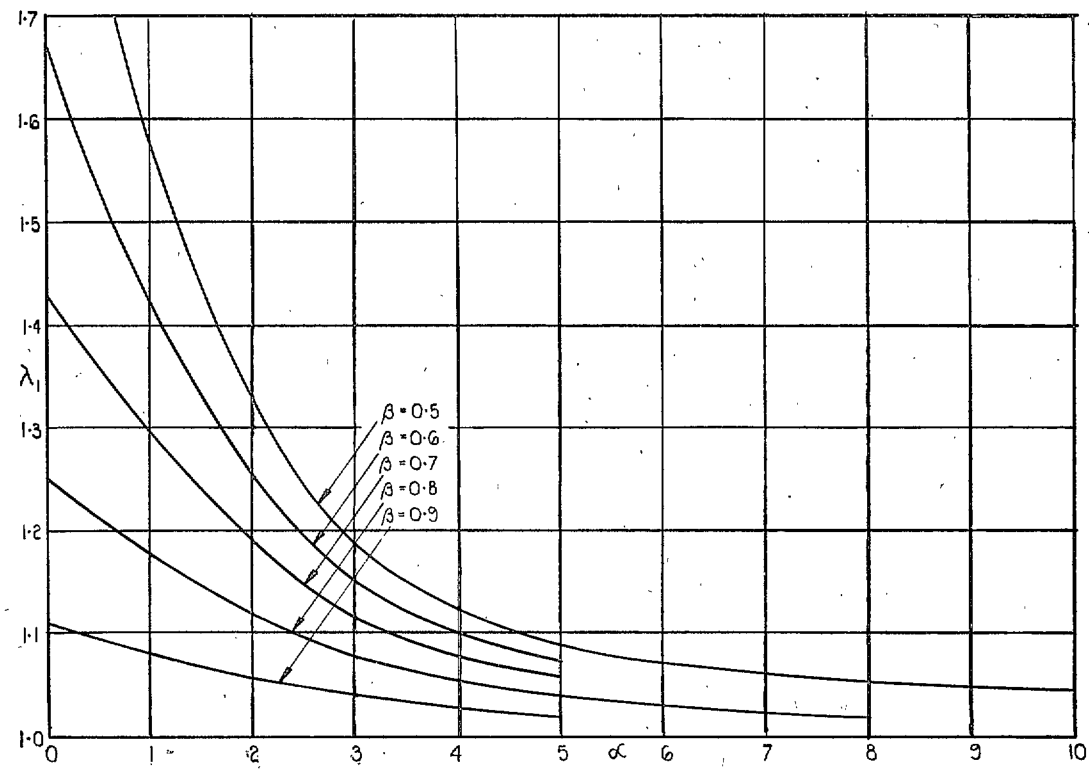


FIG. 5. Frequencies for anti-symmetrical vibration—first harmonic.

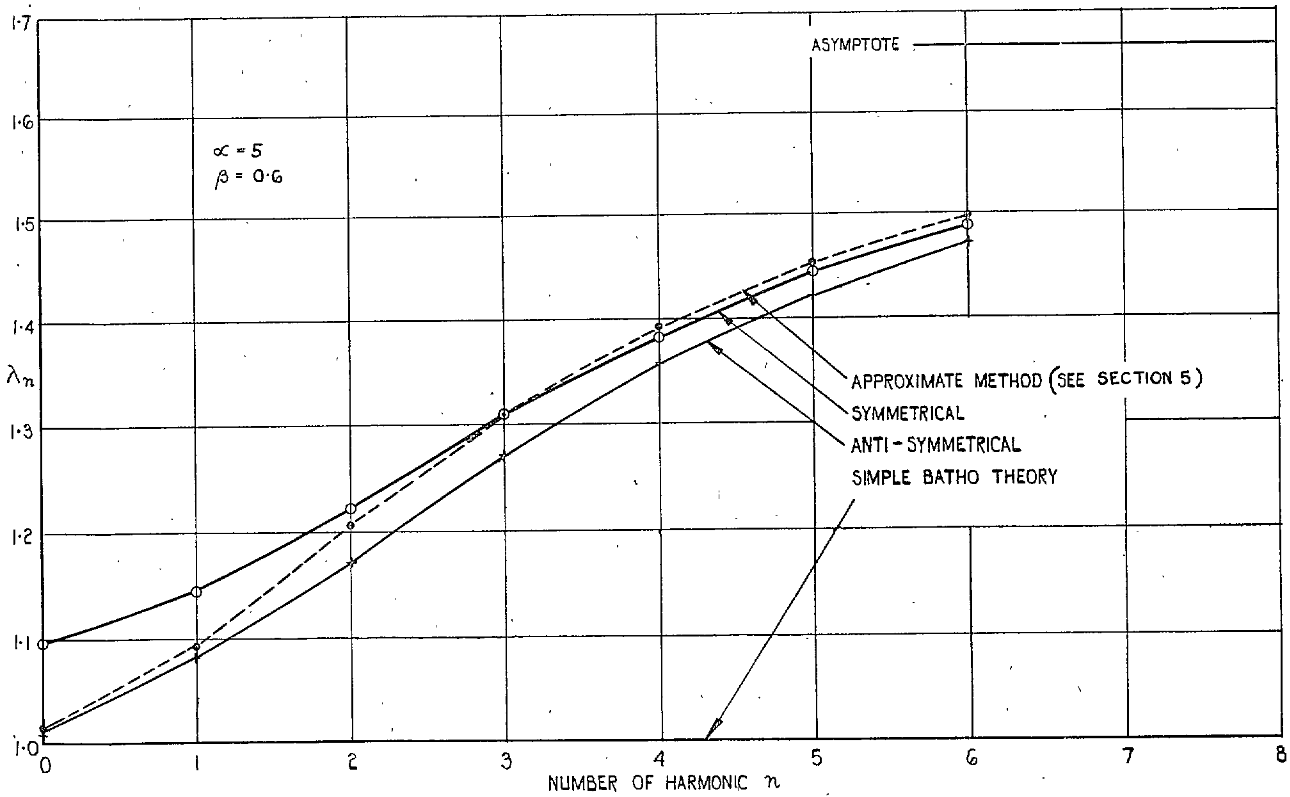


FIG. 6. Frequencies in different modes. $\alpha = 5$.

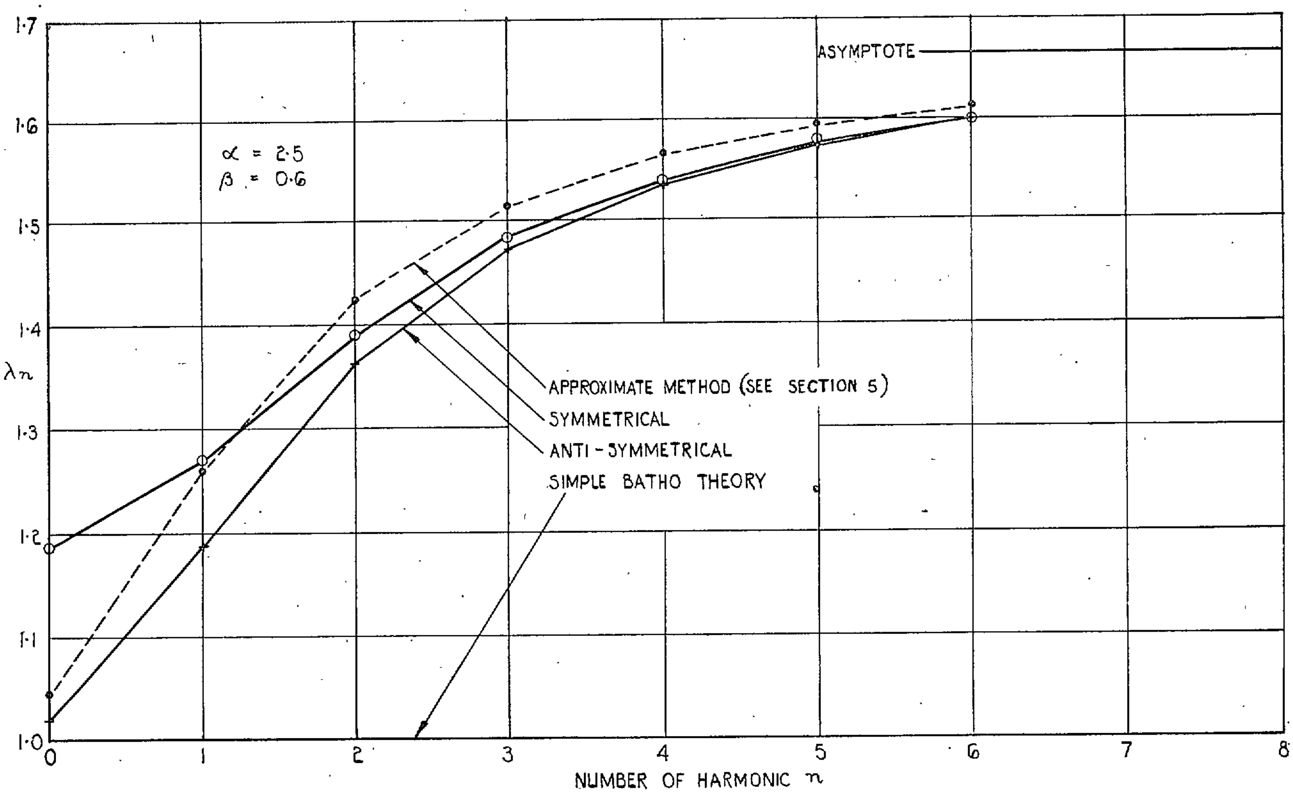
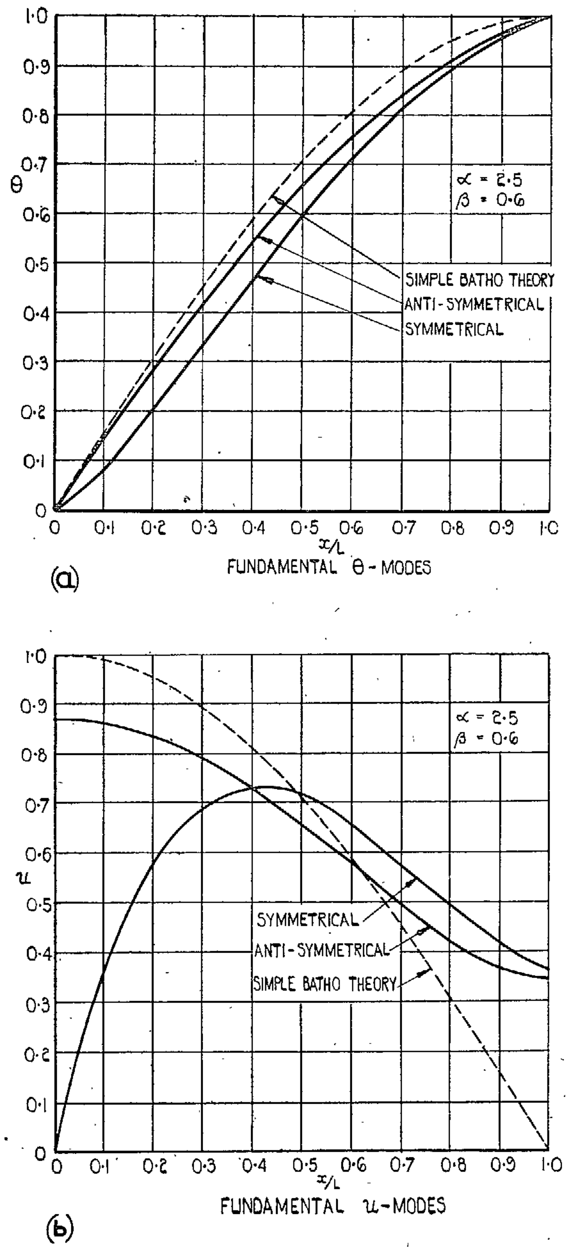
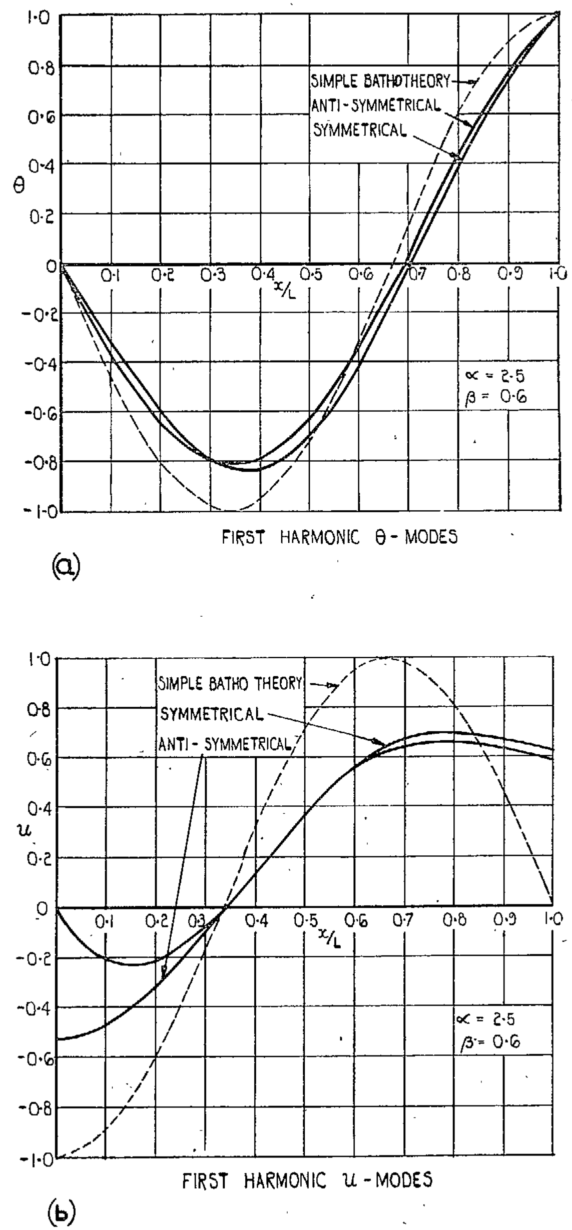


FIG. 7. Frequencies in different modes. $\alpha = 2 \cdot 5$.

Wt. 17/680 K.9 5/55 G.G. 34-262



FIGS. 8a and 8b. Fundamental modes of vibration.



FIGS. 9a and 9b. First harmonic modes of vibration.

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