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Note on the Influence of Aspect Ratio on the Variation with Mach Number of the Lift and Hinge-Moment Characteristics of a Wing and Full-Span Control

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Summary.—It is shown on the basis of the linearised theory that the effects of compressibility on the lift and hinge-moment characteristics of a wing and full-span control are functions of aspect ratio. With reduction in aspect ratio the increase of the lift characteristics with Mach number is reduced appreciably (*see* equation 12 and Table 1). The same effect is noted for the hinge-moment characteristic b_1 (equation 13). The effects on the hinge-moment characteristics b_2 and b_3 are rather more complicated (equations 14 and 15), but in many practical cases the influence of aspect ratio will be very small.

1. *Notation.*

α_1	Wing or tail plane incidence
α_2	Control setting
α_3	Tab setting
C_L	Lift coefficient
C_H	Hinge-moment coefficient
a_1, a_2, a_3	$\dfrac{\partial C_L}{\partial \alpha_1}, \dfrac{\partial C_L}{\partial \alpha_2}, \dfrac{\partial C_L}{\partial \alpha_3}$, respectively, for incompressible flow
A_1, A_2, A_3	$\dfrac{\partial C_L}{\partial \alpha_1}, \dfrac{\partial C_L}{\partial \alpha_2}, \dfrac{\partial C_L}{\partial \alpha_3}$, respectively, for compressible flow
b_1, b_2, b_3	$\dfrac{\partial C_H}{\partial \alpha_1}, \dfrac{\partial C_H}{\partial \alpha_2}, \dfrac{\partial C_H}{\partial \alpha_3}$, respectively, for incompressible flow
B_1, B_2, B_3	$\dfrac{\partial C_H}{\partial \alpha_1}, \dfrac{\partial C_H}{\partial \alpha_2}, \dfrac{\partial C_H}{\partial \alpha_3}$, respectively, for compressible flow

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* R.A.E. Tech. Note Aero. 1263, received 22nd October, 1943.

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A	Aspect ratio
U_0	Aircraft speed
w	Downwash velocity
c_0	Speed of sound in undisturbed stream
M_0	Mach number $= U_0/c_0$
β	$(1 - M_0^{\,2})^{1/2}$
γ	$(\pi A + a_{10})/(\beta\pi A + a_{10})$
μ_2	*see* equation (14)
μ_3	*see* equation (15)

Suffix $_0$ refers to infinite aspect ratio.

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2. *Introduction.*—The well-known Glauert law based on linearised theory for the effect of compressibility on the lift-curve slope of a wing of infinite span is

$$\frac{A_1}{a_1} = \frac{1}{\beta} \qquad \ldots \qquad (1)$$

Similar relations hold for the effect of compressibility on the lift and hinge-moment characteristics of controls in two-dimensional flow. For wings of finite span the relations differ from the Glauert law, and it is the purpose of this note to illustrate how they differ in practical instances. The relations will be derived on the basis of lifting-line theory using the assumption that the loading is elliptic but the argument can readily be generalised to the case of non-elliptic loading.

3. *Analysis.*—For incompressible flow we can write

$$C_L = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 \qquad \ldots \qquad (2)$$

and

$$C_H = b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 \qquad \ldots \qquad (3)$$

The downwash is given by w, where

$$\frac{w}{U_0} = \alpha_1 - \alpha_{10} = \frac{C_L}{\pi A} \qquad \ldots \qquad (4)$$

where α_{10} is the local aerodynamic incidence.

Making use of the fact that the aerodynamic characteristics of the wing and control are related to the local aerodynamic incidence by the two dimensional relations, it follows that

$$C_L = a_{10}\left(\alpha_1 - \frac{C_L}{\pi A}\right) + a_{20}\alpha_2 + a_{30}\alpha_3 \qquad \ldots \qquad (5)$$

and

$$C_H = b_{10}\left(\alpha_1 - \frac{C_L}{\pi A}\right) + b_{20}\alpha_2 + b_{30}\alpha_3. \qquad \ldots \qquad (6)$$

Hence from (2) and (5) we have

$$C_L = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = (a_{10}\alpha_1 + a_{20}\alpha_2 + a_{30}\alpha_3) \Big/ \left(1 + \frac{a_{10}}{\pi A}\right).$$

It follows that

$$a_r = a_{r0} \Big/ \left(1 + \frac{a_{10}}{\pi A}\right),\ r = 1, 2, 3. \qquad \ldots \qquad (7)$$

Likewise from (3) and (6) we have

$$C_H = b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 = \alpha_1\left(b_{10} - \frac{a_{10}\,.\,b_{10}}{\pi A + a_{10}}\right)$$
$$+ \alpha_2\left(b_{20} - \frac{a_{20}\,.\,b_{10}}{\pi A + a_{10}}\right)$$
$$+ \alpha_3\left(b_{30} - \frac{a_{30}\,.\,b_{10}}{\pi A + a_{10}}\right)$$

and hence

$$b_r = b_{r0}\left[1 - \frac{a_{r0}\,.\,b_{10}}{(\pi A + a_{10})\,b_{r0}}\right], \qquad r = 1, 2, 3. \quad \ldots \quad (8)$$

But it is shown in R. & M. 1909[1] that equation (4) for the downwash applies whether the flow is compressible or incompressible, and therefore we can similarly derive the relations

$$A_r = A_{r0}\Big/\left(1 + \frac{A_{10}}{\pi A}\right), \qquad r = 1, 2, 3, \quad \ldots \quad (9)$$

and

$$B_r = B_{r0}\left[1 - \frac{A_{r0}\,.\,B_{10}}{(\pi A + A_{10})\,B_{r0}}\right], \qquad r = 1, 2, 3. \quad \ldots \quad (10)$$

Further, the two-dimensional Glauert relation gives us

$$\frac{A_{r0}}{a_{r0}} = \frac{B_{r0}}{b_{r0}} = \frac{1}{\beta}, \qquad r = 1, 2, 3. \quad \ldots \quad (11)$$

From (7), (9) and (11) we derive immediately the first group of relations that we are seeking, *viz.*,

$$\frac{A_r}{a_r} = \frac{a_{10} + \pi A}{a_{10} + \beta\pi A}, \qquad r = 1, 2, 3. \quad \ldots \quad (12)$$
$$= \gamma, \text{ say.}$$

Likewise from (7), (8), (9), (10) and (11) we find that

$$\frac{B_1}{b_1} = \frac{a_{10} + \pi A}{a_{10} + \beta\pi A} = \gamma, \quad \ldots \quad (13)$$

$$\frac{B_2}{b_2} = \frac{\gamma}{\beta}\left\{\frac{\beta + \dfrac{a_{10}}{\pi A}\left[1 - \dfrac{b_{10}\,.\,a_{20}}{b_{20}\,.\,a_{10}}\right]}{1 + \dfrac{a_{10}}{\pi A}\left[1 - \dfrac{b_{10}\,.\,a_{20}}{b_{20}\,.\,a_{10}}\right]}\right\}$$
$$= \frac{\gamma}{\beta}\,.\,\mu_2, \text{ say,} \quad \ldots \quad (14)$$

$$\frac{B_3}{b_3} = \frac{\gamma}{\beta}\left\{\frac{\beta + \dfrac{a_{10}}{\pi A}\left[1 - \dfrac{b_{10}\,.\,a_{30}}{b_{30}\,.\,a_{10}}\right]}{1 + \dfrac{a_{10}}{\pi A}\left[1 - \dfrac{b_{10}\,.\,a_{30}}{b_{30}\,.\,a_{10}}\right]}\right\}$$
$$= \frac{\gamma}{\beta}\,.\,\mu_3, \text{ say.} \quad \ldots \quad (15)$$

4. *Discussion.*—Consider first the relations for the lifting characteristics of the wing or control given by equation (12). We see that the effect of finite aspect ratio is to reduce the ratio A_r/a_r below the two-dimensional value of $1/\beta$. To illustrate this point the following table gives the values of A_r/a_r for various values of the Mach number up to $0 \cdot 8$, and for aspect ratios of 3, 4, 6 and 8, assuming $a_{10} = 6 \cdot 0$.

TABLE 1

M	A_r/a_r				$1/\beta$
	$A = 3$	$A = 4$	$A = 6$	$A = 8$	$A = \infty$
0·2	1·012	1·014	1·016	1·017	1·022
0·4	1·054	1·060	1·068	1·072	1·092
0·6	1·139	1·157	1·179	1·193	1·25
1·8	1·327	1·371	1·436	1·477	1·667

Thus, it will be seen that for a wing of aspect ratio 6, for example, the increase of A_1/a_1 with Mach number is about two thirds of the increase for a wing of infinite aspect ratio, whilst for a tail plane of aspect ratio 3, say, the increase is only about half. It can be expected that these results will be reflected in the stability of an aeroplane with change of Mach number.

Coming now to the hinge-moment characteristics given by equations (13), (14) and (15), we see from equation (13) that B_1/b_1 is the same function of Mach number and aspect ratio as are the ratios A_r/a_r, and the above table therefore illustrates its variation with these two parameters. The expressions for the ratios B_2/b_2 and B_3/b_3 are more complicated. We may note, however, that the value of the factor μ_2 in equation (14) is determined principally by the magnitude of b_{10}/b_{20}, since a_{20}/a_{10} is normally of the order of $0 \cdot 5$ and its variation is confined between fairly narrow limits. Thus, if b_{10}/b_{20} were small then μ_2 would tend to $1/\gamma$, and then B_2/b_2 would tend to $1/\beta$. Conversely, if the value of b_{10}/b_{20} were such that $(b_{10}/a_{20})/(b_{20}/a_{10})$ were positive and comparable to unity then B_2/b_2 would be approximately given by γ. This is illustrated in Fig. 1 where the variation of B_2/b_2 with $(b_{10}/a_{20})/(b_{20}/a_{10})$ for a tail plane of aspect ratio 4 is given for various Mach numbers. The tendency with modern high speed aircraft is for b_{10}/b_{20} to be made as small as possible for the tail surface controls, in which case it is sufficiently accurate to take B_2/b_2 equal to $1/\beta$.

These remarks apply similarly to the factor μ_3, but examination of the possible variation of the ratio b_{10}/b_{30} shows that its value is never much in excess of $0 \cdot 2$, so that we may take $1/\beta$ as an acceptable approximation to B_3/b_3 for all controls.

5. *Conclusions.*—It is concluded that

$$\frac{A_1}{a_1} = \frac{A_2}{a_2} = \frac{A_3}{a_3} = \frac{B_1}{b_1} = \frac{\pi A + a_{10}}{\beta \pi A + a_{10}} = \gamma \,,$$

$$\frac{B_2}{b_2} = \frac{\gamma}{\beta}\mu_2 = \frac{\gamma}{\beta}\left\{\frac{\beta + \dfrac{a_{10}}{\pi A}\left[1 - \dfrac{b_{10} \,.\, a_{20}}{b_{20} \,.\, a_{10}}\right]}{1 + \dfrac{a_{10}}{\pi A}\left[1 - \dfrac{b_{10} \,.\, a_{20}}{b_{20} \,.\, a_{10}}\right]}\right\}$$

$$\simeq \frac{1}{\beta} \,, \text{ for tail unit controls where } b_{10}/b_{20} \text{ is small.}$$

$$\frac{B_3}{b_3} = \gamma \, . \, \mu_3 = \frac{\gamma}{\beta} \left\{ \frac{\beta + \dfrac{a_{10}}{\pi A} \left[1 - \dfrac{b_{10} \, . \, a_{30}}{b_{30} \, . \, a_{10}} \right]}{1 - \dfrac{a_{10}}{\pi A} \left[1 - \dfrac{b_{10} \, . \, a_{30}}{b_{30} \, . \, a_{10}} \right]} \right\} \simeq \frac{1}{\beta}, \text{ for all controls.}$$

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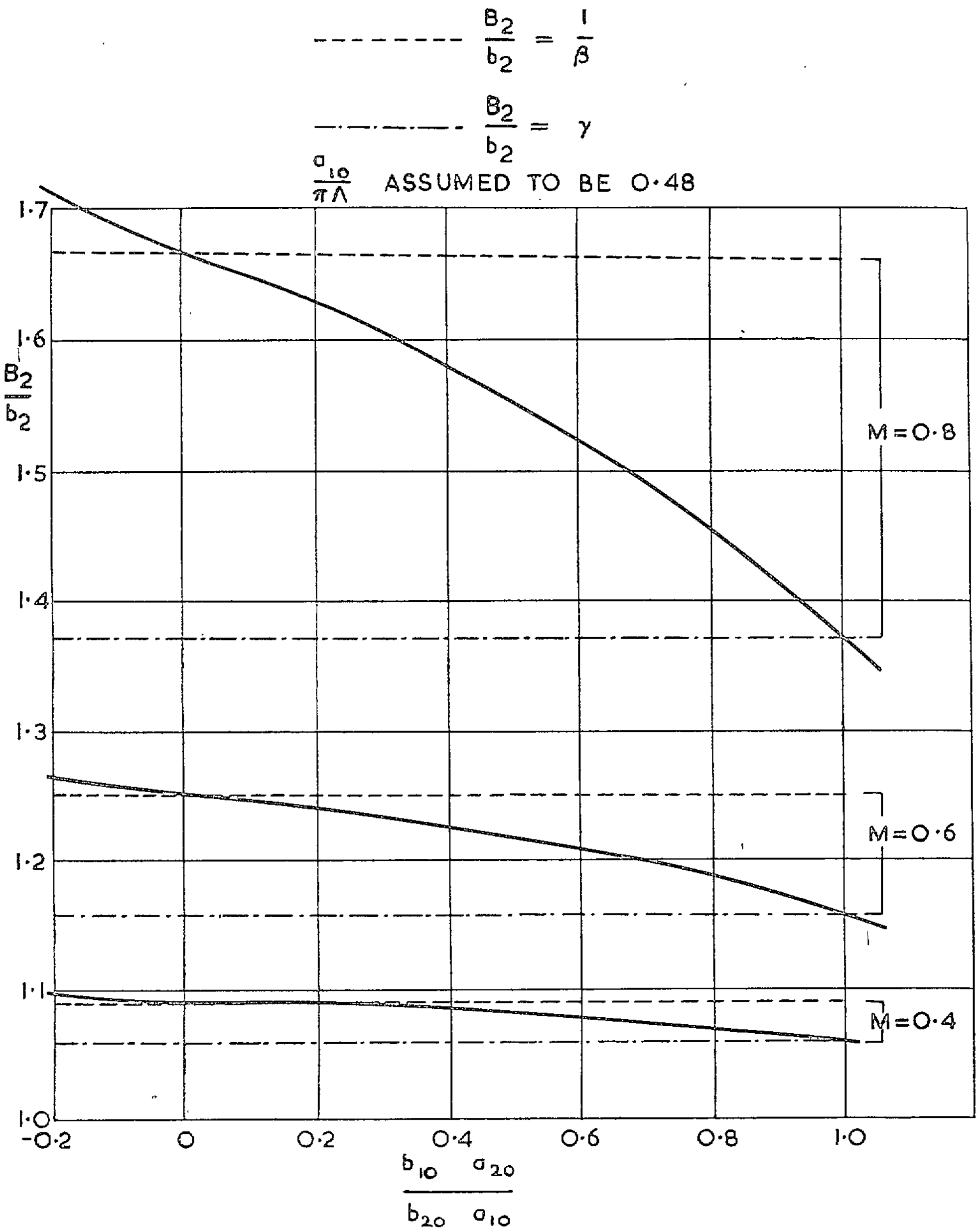

FIG. 1. Variation of compressibility factor on B_2 with $\frac{b_{10}\, a_{20}}{b_{20}\, a_{10}}$.

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