N.A.R.

R. & M. No. 2726
(12,887)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

The Numerical Solution of Two-dimensional Fluid Motion in the Neighbourhood of Stagnation Points and Sharp Corners

By

L. C. Woods, M.Sc., B.E. (N.Z.), D.Phil.,
Engineering Laboratory, Oxford University

LONDON: HER MAJESTY'S STATIONERY OFFICE
1952
PRICE 3s. 6d. NET

The Numerical Solution of Two-dimensional Fluid Motion in the Neighbourhood of Stagnation Points and Sharp Corners

By

L. C. Woods, M.Sc., B.E. (N.Z.), D. Phil.,

Engineering Laboratory, Oxford University

*Reports and Memoranda No. 2726**

October, 1949

Summary.—Methods are given in this paper of dealing with singularities of functions satisfying certain two-dimensional partial differential equations. For a numerical solution the differential equations are replaced by difference equations on a square mesh. Log $(1/q)$ where q is the velocity, becomes infinite at stagnation points, sharp corners, sinks, *etc.*, while the conjugate function θ (flow direction) becomes multi-valued. The method consists in finding a series expansion for the function (log $1/q$ or θ) in the neighbourhood of the singularity. This expansion is then used to find relationships between the function values at points of the mesh adjacent to the singularity. A method of working directly in the transformed flow plane (in which the aerofoil is a slit), and thus avoiding irregular squares on the boundary, is also given. The method is developed for incompressible flow, but an approximation suitable for compressible flow is given.

Introduction.—Differential equations for incompressible and compressible flow can be replaced by difference equations on a square mesh. Thom[1], Southwell[2] and Emmons[3] have used this principle to determine the flow in channels and about aerofoils, but the finite difference method breaks down in the neighbourhood of a stagnation point or sharp corner. At a stagnation point the second space derivatives of the velocity potential become infinite, while at a sharp corner where the velocity itself becomes infinite, the first derivatives of the potential function are infinite. Both Southwell and Emmons, who used equations involving the stream function, appear to have neglected the singularity at the stagnation point. The stagnation points and their exact location, represent boundary conditions too important to be neglected (for example, by using a grid in the physical (x, y)-plane, so arranged that these points are avoided). The author has found, for instance, that the movement of the front stagnation point 1/1000 of the chord distance for a small nose radius aerofoil at a given angle of incidence, can alter the magnitude of the velocity peak by more than ten per cent. The difficulties at these singular points become more serious when equations involving log $(1/q)$, where q is the velocity, are used instead of the stream function, since log $(1/q)$ becomes infinite both at the stagnation points and the sharp corners. Methods[4] have been given for problems in which the derivatives of the function in question become infinite at one or more points in the field, and this paper will deal with the case for which the function itself becomes infinite or multivalued at points in the field. Log $(1/q)$ becomes infinite at sharp corners, stagnation points, sinks, sources and vortices. All these can be included in the same general method. This involves finding a series expansion for log $(1/q)$ in the neighbourhood of the singularity, which allows relations to be established between the values of log $(1/q)$ at the points of the grid neighbouring the infinity.

The flow direction θ becomes multi-valued at these singularities, but the method enables the exact character of θ near the singularity to be determined, and the difficulty overcome.

* Oxford University Engineering Laboratory Report No. 27.

In addition to the results mentioned above for $\log (1/q)$ and θ, the following related results are given:—

1. Calculation of $s = \int d\phi/q$, (where ϕ is the potential function, and s the distance along a streamline) near the singularity. This integration is an essential part of the method of working in the (ϕ, ψ) plane (*see* Section 2 below).
2. Calculation of the trailing edge angle from a given velocity distribution curve. This is a useful step in the case when the aerofoil profile has to be calculated from the given velocity curve.

Symbols

(x, y)		the physical flow plane.
(ϕ, ψ)		the transformed flow plane, such that the aerofoil boundary is on $\psi = 0$ (ψ is the stream function).
(r, ε)		polar co-ordinates in the (ϕ, ψ) plane.
q		velocity vector magnitude.
L	$=$	$\log (1/q)$.
θ		angle between x-axis and the velocity vector.
f	$=$	$L + i\theta$.
t		change in direction of boundary at a point.
R		radius of curvature of the boundary.
h	$=$	t/π.
n		interval of the square mesh.
s		distance along a streamline or boundary.
X		residual of the Relaxation Process.
τ		trailing-edge angle.

1. *The Velocity Field in the Neighbourhood of a Singularity.*—Referring to Fig. 1, the Schwarz-Christoffel Transformation from the z-plane to the w-plane is $\dfrac{dz}{dw} = \dfrac{1}{K} e^{i\lambda} w^{-t/\pi}$, where K and λ are real constants, and the origin is taken at the corner A.

but
$$\frac{dz}{dw} = e^{i\theta}/q = e^{i\lambda}/K \times w^{-h}, \quad \ldots \qquad (1)$$
where $h \equiv t/\pi$.

Therefore $f \equiv \log\left(\dfrac{1}{q}\right) + i\theta \equiv L + i\theta = h \log \dfrac{K}{w} + i\lambda$.

Thus, since $w = re^{i\varepsilon}$, $L = h \log \dfrac{K}{r}$, $\theta = (\lambda - h\varepsilon)$.

When $r = K$ then $L = 0$, and since $\varepsilon = 0$ on AC, λ is the angle between the x-axis and AC. These results imply that in the neighbourhood of a sharp corner or stagnation point on an aerofoil we can write,

$$f = -h \log w + a + bw + cw^2 + \ldots \qquad (2)$$

or $$L + i\theta = -h \log r + a_1 + b_1 r + c_1 r^2 + \ldots - (h\varepsilon - \lambda + b_2 r + c_2 r^2 + \ldots) \qquad (3)$$

where $a_1, b_1 \ldots, b_2, c_2 \ldots$ are real, and functions of ε only.

'$a + bw + cw^2 + \ldots$' represents the contribution to f due to the remainder of the aerofoil, and so sufficiently close to the singularity $-h \log w$ will be the dominating term in the series.

Integrating (1) we find $z = \dfrac{1}{K}\dfrac{w^{1-h}}{1-h} + C$, and since the origin in the z-plane transforms into the origin in the w-plane, $C = 0$. Using this in (2) we find for the z-plane that f can be expanded in the series

$$f = -\frac{h}{1-h} \log z + a_3 + b_3 z + c_3 z^2 + \quad \ldots \qquad (4)$$

This equation could be made the basis of a method of dealing with singularities in the (x, y) or z-plane. It would be similar to that described below for the (ϕ, ψ)-plane. The (ϕ, ψ)-plane however has two advantages over the (x, y)-plane:—

1. Irregular stars[2], which involve the use of interpolation formulae in the Relaxation Process, are avoided on the boundary.
2. By confining the circulation to one of a series of discrete values, it is possible to arrange that each of the leading and trailing edge stagnation points coincide with a mesh point in the square grid in the (ϕ, ψ)-plane (*see* Ref. 5). In the (x, y)-plane this is usually only possible with a symmetrical aerofoil at zero incidence.

The one disadvantage is that the boundary conditions in the (ϕ, ψ)-plane depend upon the solution, but this can be easily overcome as described in section 2 below.

We conclude this section by briefly noting that sources, sinks and vortices can also be represented by equations of the type (2). Dealing with a sink as typical of these, we have $V = B/r$, where V is the velocity due to the sink, B is a known constant proportional to sink strength, and r is radius from the sink centre. Thus near the sink:—

$$L = \log r + a + br + cr^2 + \ldots .$$

This would find some application on boundary-layer suction aerofoils.

2. *Boundary Conditions.*—Now, since f is an analytic function (except at the stagnation points and sharp corners), it satisfies $\nabla^2 f = 0$, *i.e.*, $\nabla^2 L = \nabla^2 \theta = 0$, in both the z- and w-planes. The Cauchy-Riemann Equations also hold, *i.e.*, in the w-plane,

$$\frac{\partial L}{\partial \phi} = \frac{\partial \theta}{\partial \psi}, \quad \frac{\partial L}{\partial \psi} = -\frac{\partial \theta}{\partial \phi}. \qquad (5)$$

On the aerofoil boundary θ is specified, or can be calculated from the aerofoil co-ordinates.

Now $$\frac{\partial L}{\partial \psi} = -\frac{\partial \theta}{\partial s}\frac{\partial s}{\partial \phi} = +\frac{1}{Rq} \qquad (6)$$

since $\partial\phi/\partial s = q$, and $\partial s/\partial\theta = -R$ (radius of curvature of boundary). The other boundary conditions are the locations of the stagnation points and sharp corners if any. The rear stagnation point will be fixed at the trailing edge (Joukowski Condition), whereas the location of the front stagnation point will depend upon the circulation or incidence required. The non-linear boundary conditions (6) can be dealt with as follows:—

Assuming initially a distribution of q against ϕ in the w-plane, and that $\phi = 0$ at the trailing edge, we find the distance $\bar{s}$ from the trailing edge to any particular potential $\bar{\phi}$ by integrating thus:—

$$\bar{s} = \int_0^{\bar{\phi}} \frac{d\phi}{q} \qquad (7)$$

since $q = \partial\phi/\partial s$. The front stagnation point is maintained at a constant potential of such a value that it falls on a mesh point of the square grid. In an actual calculation, the integral in (7) is replaced by a summation, and the potentials $\bar{\phi}$ are the potentials of the lines of the grid cutting the boundary. Integrating right round the aerofoil enables c (chord distance) to be found from the known relationship between the perimeter and the chord of the aerofoil; thus s/c for each grid point on the boundary can be found. If the non-dimensional graph c/R against s/c has been plotted, then $1/R$ is quickly obtained for each grid point. Using the assumed values of q and the derived values of $1/R$ an approximate boundary condition $\partial L/\partial\psi = + 1/Rq$ is found for each point.

This method may seem crude, and it would seem that the resulting values of $\partial L/\partial\psi$ would be quite wide of the true values. However, as an overestimate in the value of q reduces the values of c subsequently determined, the product qc remains reasonably close to the true value. Furthermore, along the greater part of the aerofoil chord, R varies slowly.

Using these approximate boundary conditions, the usual relaxation treatment of $\nabla^2 L = 0$ is now applied. This will result in new values of q along the aerofoil, which are then used in exactly the same manner as described above, to determine a new boundary condition $\partial L/\partial\psi$. This leads to a new q distribution on the aerofoil surface, and so on. . . . The process converges rapidly, and even assuming $q = 1$ initially, only two or three integrations along the surface are required.

Fig. 2(b) shows a typical square 0546 in the field. The values of L are progressively changed so as to reduce the residuals X_2 defined by $X_2 = L_4 + L_6 + L_0 + L_5 - 4L_2$ to a minimum. On the boundary at a typical point 0 we can write:—

$$L_1 + L_3 - 2L_0 = n^2\left(\frac{\partial^2 L}{\partial\phi^2}\right)_0 + \frac{2n^4}{4!}\left(\frac{\partial^4 L}{\partial\phi^4}\right)_0 + \dots$$

$$2L_2 - 2L_0 = 2n\left(\frac{\partial L}{\partial\psi}\right)_0 + n^2\left(\frac{\partial^2 L}{\partial\psi^2}\right)_0 + \frac{n^3}{3}\left(\frac{\partial^3 L}{\partial\psi^3}\right)_0 + \dots$$

i.e.,
$$L_1 + L_3 + 2L_2 - 4L_0 - 2n\left(\frac{\partial L}{\partial\psi}\right)_0 \simeq n^2\nabla^2 L = X_0 = 0. \quad \dots \quad (8)$$

Using (6) we have

$$L_1 + L_3 + 2L_2 - 4L_0 - \frac{2ne^{L_0}}{R_0} = X_0 \quad \dots \quad (9)$$

which can be written, with less accuracy, (since $L_0 << 1$)

$$L_1 + L_3 + 2L_2 - \left(4 + \frac{2n}{R_0}\right) L_0 - \frac{2n}{R_0} = X_0. \quad \dots \quad (10)$$

3. *The θ Field near a Stagnation Point or Sharp Corner*.—From (3), $\theta = -h\varepsilon + \lambda + br + \dots$. Referring to Fig. 3, we see that λ is the value of θ for the tangent TT′ to the stagnation streamline. TT′ bisects the trailing edge angle τ. The value of θ_4 depends upon ε, *i.e.*, upon the direction in the w-plane from which point 4 is approached. For the square marked 1234, the appropriate direction is 2–0–4, for which $\varepsilon = \pi/2$. Therefore $\theta_4 = -h(\pi/2) + \lambda = -t/2 + \lambda, = -\tau/4 + \lambda$, since in this case $t = \frac{1}{2}\tau$, and so for the residual at 0

$$X_0 = \theta_1 + \theta_2 + \theta_3 + \theta_4 - 4\theta_0 = \theta_1 + \theta_2 + \theta_3 + \lambda - \frac{\tau}{4} - 4\theta_0 \quad \dots \quad (11)$$

For other more involved formulae, such as Bickley's '20' formula[8], the value to be taken for θ_4 depends only upon ε. For instance if the square in question were centred at point 3, $\varepsilon = (3\pi)/4$, and $\theta_4 = -(3\tau)/8 + \lambda$, *etc.*

Thom has used an approximation for θ near a re-entrant corner. This is $\theta_1 + \theta_2 + \theta_3 = 3\theta_0$. Consideration of Fig. 3 shows that approximately,

$$\theta_1 + \theta_3 \simeq 2\theta_0 \simeq 2\theta_2 \simeq 2\,(-\tau/4 + \lambda), \text{ therefore } \theta_1 + \theta_2 + \theta_3 \simeq 3\theta_0.$$

For a sharp corner projecting into the fluid the only change required is in the sign of τ.

4. *The L Field near a Stagnation Point or Sharp Corner.*—From equation (3), with origin at the stagnation point,

$$L = -h \log (\phi^2 + \psi^2)^{1/2} + a_1 + b_1 (\phi^2 + \psi^2)^{1/2} + c_1 (\phi^2 + \psi^2) + \ldots.$$

Thus on $\psi = 0$, $L = -h \log \phi + p_1 + s_1\phi + \frac{t_1}{2}\phi^2 + \ldots.$ .. .. .. (12)

where p_1, s_1 .. are now constants, since ε is constant ($= 0$).
Similarly on $\phi = 0$,

$$L = -h \log \psi + p_2 + s_2\psi + (t_s/2)\,\psi^2 + \ldots \qquad (13)$$

From (12) $\dfrac{\partial L}{\partial \phi} = \dfrac{h}{\phi} + s_1 + t_1\phi + \ldots.,\quad \dfrac{\partial^2 L}{\partial \phi^2} = \dfrac{h}{\phi^2} + t_1 + \ldots.,$

$$\frac{\partial^3 L}{\partial \phi^3} = -\frac{2!h}{\phi^3} + \ldots\ldots.., \qquad \ldots\ldots.., \qquad \frac{\partial^n L}{\partial \phi^n} = (-)^n \frac{(n-1)!h}{\phi^n} + \ldots\ldots.$$

Using these results, and referring to Fig. (3), we now determine:—

a. *L_5 in terms of L_6.*

$$L_5 = L_6 + n\left(\frac{\partial L}{\partial \phi}\right)_6 + \frac{n^2}{2!}\left(\frac{\partial^2 L}{\partial \phi^2}\right)_6 + \ldots\ldots. \quad (n \text{ is the mesh interval}).$$

$$\simeq L_6 + n\,(s_1 + t_1\phi) + (n^2/2!)\,t_1 - h\left[\left(\frac{h}{\phi}\right) - \frac{1}{2}\left(\frac{h}{\phi}\right)^2 + \frac{1}{3}\left(\frac{h}{\phi}\right)^3 - \ldots\ldots\right]$$

$$= L_6 + (ns_1 + n\phi t_1 + (n^2/2!)\,t_1) - h \log\left(1 + \frac{n}{\phi}\right), \text{ if } -1 < \frac{n}{\phi} < 1.$$

But $\phi = -2n$, since ϕ refers to point 6.

Therefore $L_5 \simeq L_6 + h \log_e 2 - \frac{\phi}{2}\,s_1 - \frac{3}{8}\,\phi^2 t_1.$ .. .. .. (14)

b. *L_5 in terms of L_3.*

$$L_3 = L_5 + n\left(\frac{\partial L}{\partial \psi}\right)_5 + \frac{n^2}{2!}\left(\frac{\partial^2 L}{\partial \psi^2}\right)_5 + \ldots\ldots. \quad \text{Now} \left(\frac{\partial L}{\partial \psi}\right)_5 = \frac{e^{L_5}}{R_5},$$

and from Laplace's equation we have $\dfrac{\partial^3 L}{\partial \psi^3} = -\dfrac{\partial^2}{\partial \phi^2}\left(\dfrac{\partial L}{\partial \psi}\right)$

$$= -\frac{\partial^2}{\partial \phi^2}\left(\frac{e^L}{R}\right) = -\frac{1}{R}\frac{\partial^2}{\partial \phi^2}(e^L), \text{ assuming } R \text{ constant near the stagnation point.}$$

Therefore $\quad \dfrac{\partial^3 L}{\partial \psi^3} = -\dfrac{1}{R}\left[e^L\left(\dfrac{\partial^2 L}{\partial \phi^2}\right) + e^L\left(\dfrac{\partial L}{\partial \phi}\right)^2\right] \simeq -\dfrac{e^L}{R}\left(\dfrac{h}{\phi} + \dfrac{h^2}{\phi^2}\right) = -\dfrac{e^L}{R}\dfrac{h(h+1)}{\phi^2}.$

Similarly, $\quad \dfrac{\partial^5 L}{\partial \phi^5} \simeq +\dfrac{e^L}{\phi^4}\dfrac{h(h+1)}{R}\{h(h+1) + 2 \times 3\}$, *etc.*

Also $\quad \dfrac{\partial^2 L}{\partial \psi^2} = -\dfrac{\partial^2 L}{\partial \phi} \simeq -\dfrac{h}{\phi^2}, \quad \dfrac{\partial^4 L}{\partial \psi^4} = \dfrac{\partial^4 L}{\partial \phi^4} = \dfrac{3!h}{\phi^4}$, *etc.*

$$\text{Thus } L_3 \simeq L_5 + \frac{ne^L}{R} - \frac{nh(h+1)\,e^L}{R}\left[\frac{1}{3!}\left(\frac{n}{\phi}\right)^2 - \frac{3!}{5!}\left(\frac{n}{\phi}\right)^4 + \frac{5!}{7!}\left(\frac{n}{\phi}\right)^6 - \ldots\ldots\right]$$

$$-\frac{h}{2}\left[\left(\frac{n}{\phi}\right)^2 - \frac{1}{2}\left(\frac{n}{\phi}\right)^4 + \frac{1}{3}\left(\frac{n}{\phi}\right)^6 - \ldots\ldots\right]$$

omitting powers of $h > 2$, as $h_{max} = \frac{1}{2}$.

$$\text{Therefore } L_3 = L_5 + \frac{ne^L}{R} - \frac{nh(h+1)}{R}e^L\left\{\tfrac{1}{2}\log\left(1 + \left(\frac{n}{\phi}\right)^2\right) + \left(\frac{\phi}{n}\right)\tan^{-1}\left(\frac{n}{\phi}\right) - 1\right\}$$

$$-\frac{h}{2}\log\left(1 + \frac{n}{\phi}\right).$$

When $\quad \phi = n$, as at point 5:—

$$L_3 = L_5 + \left(\frac{\phi}{Rq}\right)_5 - \frac{\phi h(h+1)}{Rq}\left(\tfrac{1}{2}\log 2 + \frac{\pi}{4} - 1\right) - \frac{h}{2}\log 2.$$

$$= L_5 + \left(\frac{\phi}{Rq}\right)_5\{1 - 0{\cdot}131h(h+1)\} - 0{\cdot}346h. \qquad (15)$$

c. *L_0 in terms of L_1 and L_3.*

$$L_1 + L_3 - 2L_0 = n^2\left(\frac{\partial^2 L}{\partial \phi^2}\right)_0 + \frac{2n^4}{4!}\left(\frac{\partial^4 L}{\partial \phi^4}\right)_0 + \ldots\ldots$$

$$= -n^2\left(\frac{\partial^2 L}{\partial \psi^2}\right) + \frac{2n}{4!}\left(\frac{\partial^4 L}{\partial \psi^4}\right)_0 - \ldots\ldots$$

$$= -n^2 t_2 - \left(\frac{n}{\psi}\right)^2 h + \tfrac{1}{2}\left(\frac{n}{\psi}\right)^4 - \ldots\ldots, \text{ from (13).}$$

Therefore $L_1 + L_3 - 2L_0 = -n^2 t_2 - h\log\left[1 + \left(\dfrac{n}{\psi}\right)^2\right]$. When $\psi = n$, as at point 0, then:—

$$2L_0 = L_1 + L_3 + 0{\cdot}693h + n^2 t_2. \qquad (16)$$

Equation (15) can be used to calculate L_5 from L_3 during the relaxation of the field, and equation (14) is a useful check. The constants s, t_1, t_2, are normally very small and can usually be neglected. If necessary they could be eliminated by using similar equations for other neighbouring points.

In Ref. 5 Thom suggests an alternative method of dealing with the infinity at the stagnation point. He uses the function $F = \log (q_a/q_c) + i\,(\theta_c - \theta_a)$, where subscript a refers to the aerofoil and c to the cylinder from which it is transformed. However (3) shows that F is finite only if $h_a = h_c$. While this is true at a rounded nose it is not true at the trailing edge. The method would be useful to transform the known flow around one aerofoil to that around another with the same trailing edge angle.

5. *Integration near a Singularity*.—In determining the location of the feet of the equipotential lines upon the aerofoil surface it is necessary to integrate the expression $ds = d\phi/q$, which becomes infinite at a stagnation point. We proceed as follows:—

On the surface near the singularity,

$$L = -h \log \phi + a + b\phi + \ldots\ldots \qquad (17)$$

Put $e^a = A$, and we find approximately $q = \dfrac{\phi^h}{A\,(1 + b\phi)}$.

Therefore $\displaystyle\int_0^{\bar\phi} \frac{d\phi}{q} = A\,\frac{\bar\phi^{1-h}}{1-h}\left(1 + b\,\frac{1-h}{2-h}\,\bar\phi\right)$. Now $A = \dfrac{\bar\phi}{\bar q\,(1 + b\bar\phi)}$, where $\bar q$ is the value of q at $\bar\phi$.

$$\text{Therefore} \int_0^{\bar\phi} \frac{d\phi}{q} = \frac{\bar\phi}{\bar q\,(1-h)}\left(1 - \frac{b\bar\phi}{2-h}\right) = \frac{\bar\phi}{\bar q\,(1-h)}\left(1 - \frac{\delta}{2-h}\right) \qquad (18)$$

where, from (17), δ is the first difference of the function $L + h \log \phi$ on the mesh at $\bar\phi$ (found by extrapolation). Generally δ can be ignored.

6. *Calculation of Trailing-Edge Angle from a Given Velocity Distribution along the Chord*.—When it is required to calculate the aerofoil profile corresponding to a given velocity distribution at a specified free stream Mach number, it is necessary to calculate the trailing-edge angle so that the methods given above can be employed. We start from equation (4) and find that $\dfrac{q''}{q'} \simeq \left(\dfrac{x''}{x'}\right)^{\frac{h}{1-h}}$ where $h = \dfrac{\tau}{2\pi}$, and (q', x'), (q'', x'') are points on the given (q, x) -curve selected as near to the trailing edge as the data permits. (x in this case is measured from the trailing edge.)

If $g = \dfrac{\log q''/q'}{\log x''/x'}$, then $\dfrac{\tau}{2\pi - \tau} = g$. Therefore the trailing edge can be calculated from

$$\tau = \frac{2\,\pi g}{1 + g}\,. \qquad (19)$$

A few results are shown in Fig. 4, which verify equation (19). The numbers attached to each point are the A.R.C. references from which pressure distributions were taken. The 'true angle' was obtained from the aerofoil co-ordinates and is actually the angle between the tangents to the profile drawn at the smaller of x'', x', say x'. Of course the smaller x' the nearer will this angle be to the actual trailing-edge angle.

Another useful formula is $\delta\phi_1 \simeq (1 - h)\,q\,\delta x_1$ .. .. (20)

where $\delta\phi_1$ and δx_1 are small increments measured from the stagnation point.

This follows since $\delta\phi_1 \simeq \int_0^{\delta x_1} q \, dx = \int_0^{\delta x_1} q_1 \left(\frac{x}{\delta x_1}\right)^{\frac{h}{1-h}} dx$

$$= (1 - h) \, q_1 \, \delta x_1.$$

7. *The Appropriate Value to take for ' n '.*—It is clear, that unless the mesh size n is small, certainly less than the radius of curvature near the nose, the above formulae will have little accuracy. Normally, near a rounded nose the value of $h = \tau/(2\pi) = \pi/(2\pi) = \frac{1}{2}$ should be taken, but on a coarse mesh, this value of h would be too large, and so would exaggerate the influence of the stagnation point. The field on the coarse mesh due to the stagnation point is like that due to a nose angle much less than π. This is not usually the case in the neighbourhood of the trailing edge since R is quite large and varies slowly, except at the edge itself when $R = 0$.

Now if τ_A and τ_H are the leading and trailing-edge angles respectively, then

$$\oint d\theta + \tau_A + \tau_H = 0,$$

is simply one of the conditions that the profile is closed. With a finite number of mesh points, say m of them, the condition becomes

$$\sum_{i=1}^{m} \theta_i + \tau_A + \tau_H = 0,$$

or since $\delta\theta = - \frac{\partial L}{\partial \psi} \delta\phi = - \frac{\delta\theta}{Rq}$, (*see* (6))

$$\tau_A \equiv 2\pi h = - \sum_{i=1}^{m} \left(\frac{\delta\phi}{Rq}\right)_i - \tau_H. \qquad (21)$$

Equation (21) must be satisfied regardless of the real values of τ_A and τ_H, otherwise the profile will not be closed.

Use of (21) is based on the assumption that τ_H can be taken equal to the trailing-edge angle. On a coarse mesh near the trailing edge this may not be quite accurate, and it is better to treat τ_H and τ_A in exactly the same way as follows. τ_H is made equal to the sum of the $\left(\frac{\delta\phi}{Rq}\right)_i (= \sum_i \delta\theta_i)$ between the points of contact of parallel tangents on the aerofoil and the trailing edge. τ_A is selected to satisfy (21), and then it will obviously be equal to the sum of the $\left(\frac{\delta\phi}{Rq}\right)_i$ between the points of contact of the tangents and the leading edge.

8. *Compressible Flow.*—It can be shown that[7]

$$\frac{\partial\theta}{\partial n} - (1 - M^2) \frac{\partial L}{\partial s} = 0,$$

$$\frac{\partial\theta}{\partial s} + \frac{\partial L}{\partial n} = 0,$$

where L is now the log of the compressible velocity, and M the local Mach number. If α is the angle between the compressible and incompressible flow vectors at a point, then transforming to the incompressible flow grid

(ϕ, ψ) we have $$\frac{\partial\theta}{\partial n} = \frac{\partial\theta}{\partial\phi} \frac{\partial\phi}{\partial n} + \frac{\partial\theta}{\partial\psi} \frac{\partial\psi}{\partial n}$$

$$= q_i \left(\cos\alpha \frac{\partial\theta}{\partial\psi} - \sin\alpha \frac{\partial\theta}{\partial\phi}\right) \textit{ etc.},$$

where q_i is the incompressible velocity. Retaining only the first powers of α, since for most cases it is quite small, we find after some calculation:—

$$\nabla^2 L = \frac{\partial}{\partial\phi}\left(M^2 \frac{\partial L}{\partial\phi}\right) + \frac{\partial}{\partial\phi}\left(M^2\alpha \frac{\partial L}{\partial\psi}\right) + \frac{\partial}{\partial\psi}\left(M^2\alpha \frac{\partial L}{\partial\phi}\right) \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Initially we solve $\nabla^2 L = \frac{\partial}{\partial\phi}\left(M^2 \frac{\partial L}{\partial\phi}\right)$ .. .. (23)

and by an integration through the field α is found, enabling the full equation to be solved. The method has been described in full in another paper[9]. Making the usual approximation of small pertubation theory, *i.e.*, that M^2 is equal to M_∞^2 (the undisturbed stream Mach number) we write (23) with less accuracy

$$\beta^2 \frac{\partial^2 L}{\partial\phi^2} + \frac{\partial^2 L}{\partial\psi^2} = 0, \text{ where } \beta = (1 - M_\infty^2)^{1/2}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

The affine transformation $\phi_j = \lambda\phi$, $\psi_j = \lambda\beta\psi$, $L_j = \nu L$, where λ and ν are constants transforms (24) into:—

$$\frac{\partial^2 L_j}{\partial\phi_j^2} + \frac{\partial^2 L_j}{\partial\psi_j^2} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

Thus if $L_j = \mathrm{f}\,(\psi_j, \phi_j)$ is a solution of (25), then $L = \frac{1}{\nu}\,\mathrm{f}\,(\lambda\phi, \lambda\beta\psi)$ is a solution of (24). Now from equation (3), for a corner $\delta\theta$ on an aerofoil in incompressible flow, we can write:—

$$L + i\theta = -\frac{\delta\theta}{2\pi}\log(\phi^2 + \psi^2) + a_1 + b_1(\phi^2 + \psi^2)^{1/2} + \ldots - i\left(\frac{\delta\theta}{\pi}\tan^{-1}\frac{\psi}{\theta} - \lambda + \ldots\right)$$

Thus a solution of (25) is

$$L_j = \frac{\delta\theta_j}{2\pi}\log(\phi_j^2 + \psi_j^2)^{1/2} + \ldots .$$

But $\delta\theta_j = \frac{\partial L_j}{\partial\psi_j}\,\delta\phi_j = \frac{\nu}{\beta}\left(\frac{\partial L}{\partial\psi}\,\delta\phi\right) = \frac{\nu}{\beta}\,\delta\theta$, and so the solution of (24) in the neighbourhood of a corner $\delta\theta$ is

$$L = -\frac{\delta\theta}{2\beta\pi}\log(\phi^2 + \beta^2\psi^2) + \ldots . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

From this equation we can conclude that most of the formulae for incompressible flow, given above, may be modified to give approximate results for compressible flow by replacing h by h/β. This does not of course apply to section 3 which remains unchanged.

For aerofoils at an angle of incidence, in addition to the modification given above, the stagnation point moves as the Mach number of the stream is increased. This movement must be calculated by finding α at the stagnation point (integrating along the stagnation stream line). The stagnation point will thus move off a grid point and interpolation formulae are required.

9. *Conclusions.*—The formulae given in this paper have all been employed and have been found (in those cases where exact solutions, or solutions by other methods, were available) to be sufficiently accurate. The case of an asymmetrical aerofoil at a given angle of incidence and at free stream Mach numbers of 0, 0·4, 0·65, and 0·75 was one of the problems selected, and the results are reasonably close to experimental curves provided by the National Physical Laboratory. (R. & M. 2727.)

10. *Acknowledgements.*—The method described in Section 2 is similar in principle to that given by Thom in R. & M. 1604. The author expresses his gratitude to Professor Thom for his valuable advice and criticism given during the preparation of this paper.

REFERENCES

No.	Author	Title, etc.
1	Thom and Klanfer	Compressible Flow Past an Aerofoil. (O.U.E.L. 26). A.R.C. 12,476. 22nd July, 1949. (Unpublished.)
2	R. V. Southwell	Relaxation Methods in Theoretical Physics. Oxford University Press. 1946.
3	H. Emmons	The Numerical Solution of Compressible Flow Problems. N.A.C.A. Tech. Note No. 932. 1944.
4	H. Motz	The Treatment of Singularities of Partial Differential Equations by Relaxation Methods. *Quarterly of Applied Mathematics.* Vol. IV, No. 4. January, 1947.
5	A. Thom	A Cambered Aerofoil Treated by Squares Method. (O.U.E.L. 10). A.R.C. 11,652. 20th July, 1948. (Unpublished.)
6	A. Thom	Arithmetical Solutions of Equations of the Type $\nabla^4 f$ = const. R. & M. 1604. March, 1944.
7	Liepmann and Puckett	Introduction to Aerodynamics of a Compressible Fluid. Wiley & Sons. 1947.
8	W. G. Bickley	Finite Difference Formulae for the Square Lattice. *The Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 1, Part 1. March, 1947.
9	L. C. Woods	Two-dimensional Aerofoil Design in Compressible Flow. R. & M. 2731.
10	L. C. Woods and A. Thom	A New Relaxation treatment of the Compressible Two-dimensional Flow about on Aerofoil with Circulation. R. & M. 2727. March, 1950.

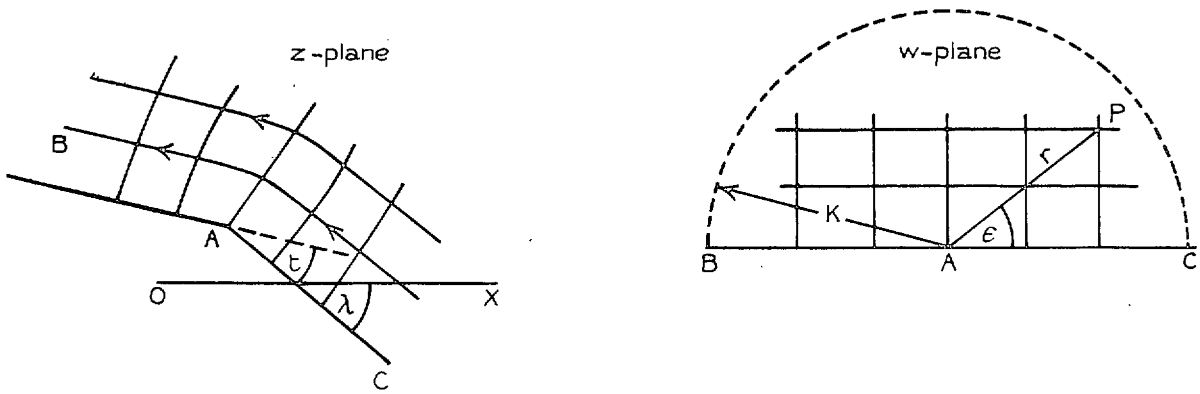

Fig. 1.

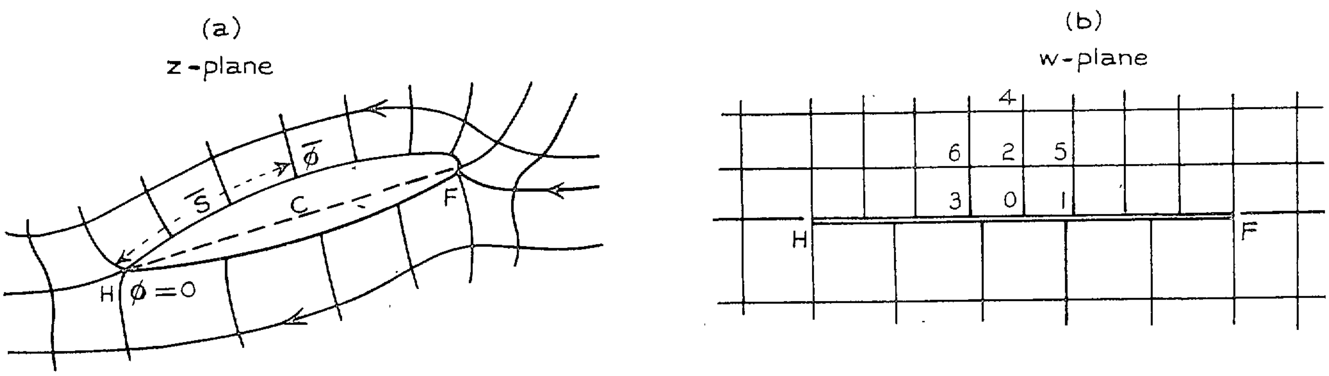

Fig. 2.

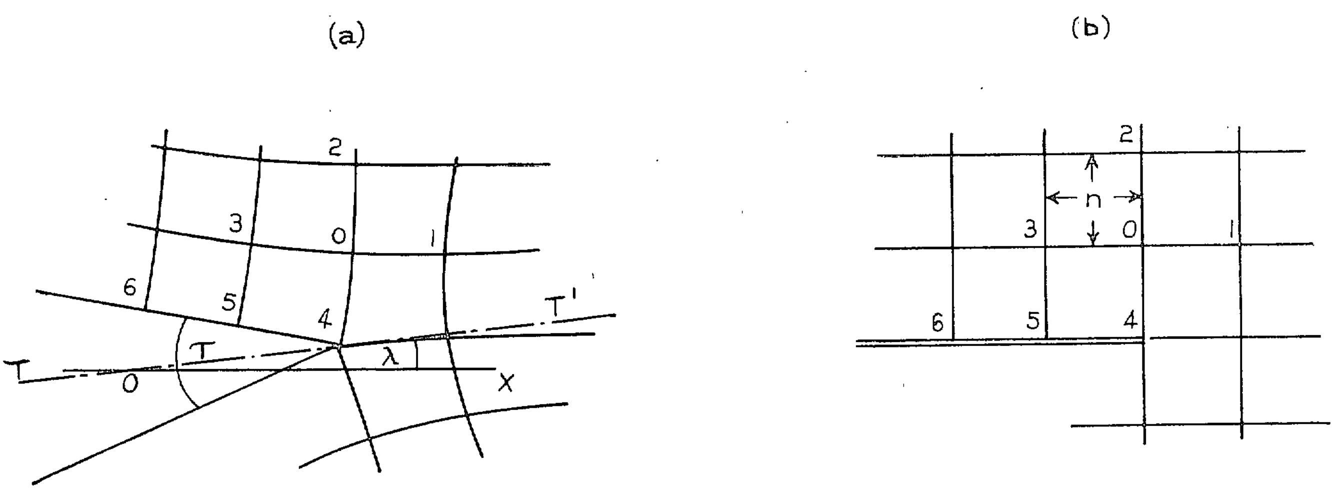

Fig. 3.

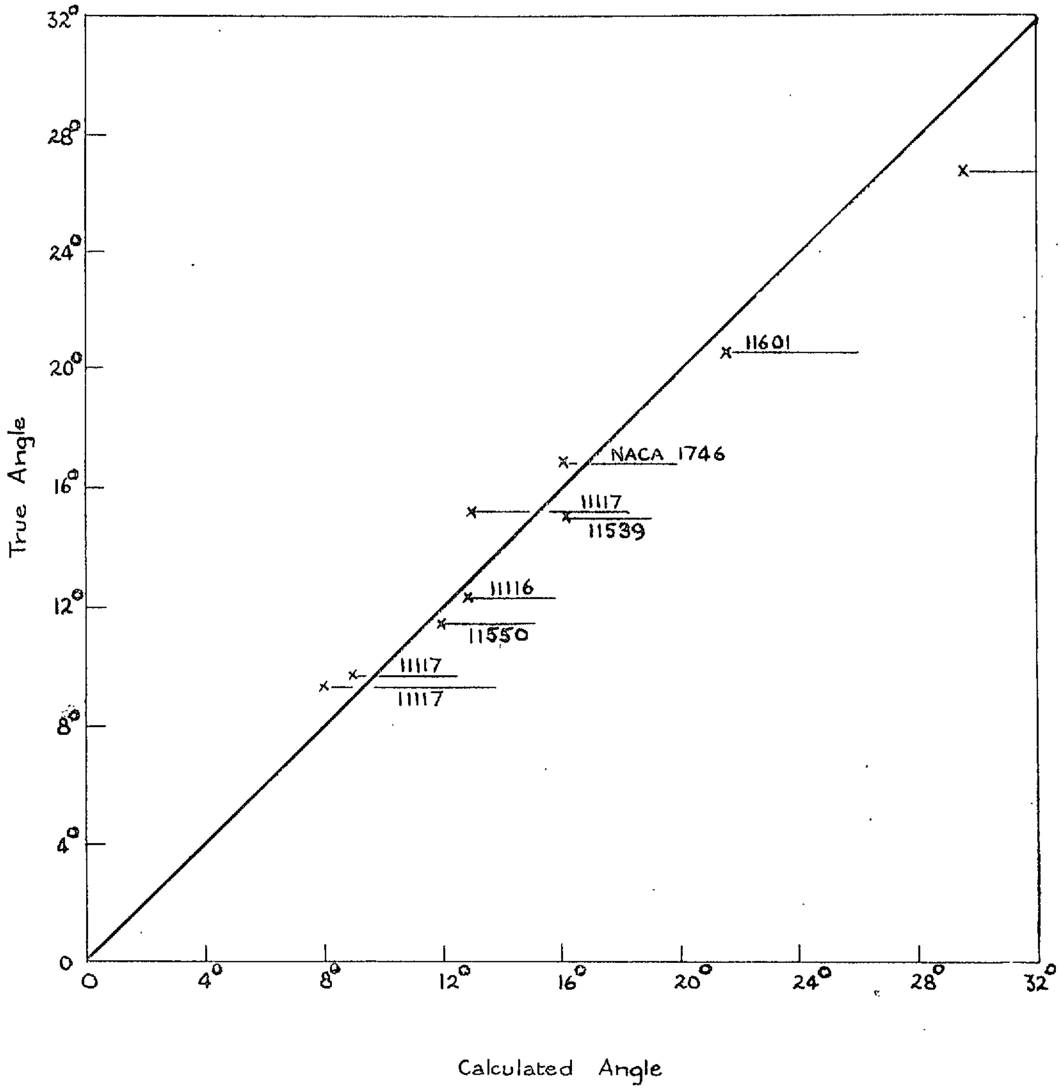

FIG. 4. Trailing edge angles deduced from pressure curves.

(23105) Wt. 15-680 K9 5/52 F. M. & S.

PRINTED IN GREAT BRITAIN

R. & M. No. 2726
(12,887)
A.R.C. Technical Report

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)—

1934–35 Vol. I. Aerodynamics. *Out of print.*
Vol. II. Seaplanes, Structures, Engines, Materials, etc. 40*s.* (40*s.* 8*d.*)

1935–36 Vol. I. Aerodynamics. 30*s.* (30*s.* 7*d.*)
Vol. II. Structures, Flutter, Engines, Seaplanes, etc. 30*s.* (30*s.* 7*d.*)

1936 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40*s.* (40*s.* 9*d.*)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 50*s.* (50*s.* 10*d.*)

1937 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40*s.* (40*s.* 10*d.*)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 60*s.* (61*s.*)

1938 Vol. I. Aerodynamics General, Performance, Airscrews. 50*s.* (51*s.*)
Vol. II. Stability and Control, Flutter, Structures, Seaplanes, Wind Tunnels, Materials. 30*s.* (30*s.* 9*d.*)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50*s.* (50*s.* 11*d.*)
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63*s.* (64*s.* 2*d.*)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50*s.* (51*s.*)

Certain other reports proper to the 1940 *volume will subsequently be included in a separate volume.*

ANNUAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL—

1933–34 1*s.* 6*d.* (1*s.* 8*d.*)
1934–35 1*s.* 6*d.* (1*s.* 8*d.*)
April 1, 1935 to December 31, 1936. 4*s.* (4*s.* 4*d.*)
1937 2*s.* (2*s.* 2*d.*)
1938 1*s.* 6*d.* (1*s.* 8*d.*)
1939–48 3*s.* (3*s.* 2*d.*)

INDEX TO ALL REPORTS AND MEMORANDA PUBLISHED IN THE ANNUAL TECHNICAL REPORTS, AND SEPARATELY—

April, 1950 R. & M. No. 2600. 2*s.* 6*d.* (2*s.* 7½*d.*)

INDEXES TO THE TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL—

December 1, 1936 — June 30, 1939. R. & M. No. 1850. 1*s.* 3*d.* (1*s.* 4½d.)
July 1, 1939 — June 30, 1945. R. & M. No. 1950. 1*s.* (1*s.* 1½*d.*)
July 1, 1945 — June 30, 1946. R. & M. No. 2050. 1*s.* (1*s.* 1½*d.*)
July 1, 1946 — December 31, 1946. R. & M. No. 2150. 1*s.* 3*d.* (1*s.* 4½*d.*)
January 1, 1947 — June 30, 1947. R. & M. No. 2250. 1*s.* 3*d.* (1*s.* 4½*d.*)

Prices in brackets include postage.

Obtainable from

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, LONDON, W.C.2 423 Oxford Street, LONDON, W.1
P.O. Box 569, LONDON, S.E.1
13a Castle Street, EDINBURGH, 2 1 St. Andrew's Crescent, CARDIFF
39 King Street, MANCHESTER, 2 Tower Lane, BRISTOL, 1
2 Edmund Street, BIRMINGHAM, 3 80 Chichester Street, BELFAST

or through any bookseller.

S.O. Code No. 23-2726