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A New Law of Similarity for Profiles, Valid in the Transonic Region

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Summary.—A new law of similarity is given, valid for slender profiles in mixed transonic flow with negligible viscosity, according to which the cube of the Prandtl factor of any critical Mach number is proportional to the thickness ratio. It is shown that this rule, and that of von Kármán for flow at sonic speed, are valid for shock-waves within the range over which the shock loss is proportional to the cube of the pressure rise. Experimental pressure distributions plotted according to this rule show good agreement, except for the position of the shock-wave on the surface.

1. *Introduction.*—In this note a new law of similarity‡ is deduced for slender profiles in transonic (mixed sub and supersonic) flow with negligible viscosity. It relates flows for which the cube of the Prandtl-Glauert factor $\sqrt{(1 - M_0^2)}$ is proportional to the profile thickness ratio. A derivation is also given of a recently obtained rule of von Kármán, that at near-sonic velocity the forces on an aerofoil are proportional to the $\frac{2}{3}$ power of the thickness ratio.

The differential equation of compressible flow in two dimensions is first replaced by a simplified, approximate form valid for small transverse velocity perturbations, which is more general than the equations from which the new law is derived. This derivation uses an even more simplified form valid only for near-sonic velocities. The form assumed by the 'shock polar' and the characteristics in near-sonic flow is discussed in section 3.

In section 4 we are able, by considering a general multiplicative transformation of the potential function and the ordinates, to use the simplified equation to deduce von Kármán's result.

In section 5 a special case of the transformation leads to the new similarity law.

We discuss in section 6 the changes in shock-waves, Mach lines, etc., associated with the similarity law. It is deduced that this law, and von Kármán's result, are valid for flow with shock-waves within the range over which the increase of entropy across the shock is proportional to the cube of the pressure increase.

Any critical Mach number M_c is, in the new law, changed so that $(1 - M_c^2)^{3/2}$, the cube of the Prandtl-Glauert factor, varies as the thickness ratio.

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*The author having left the country before arrangements for publication were put in hand, this report has been revised for publication at the Royal Aircraft Establishment.

† R.A.E. Tech. Note Aero 1902, received 23rd August, 1947.

‡ Since this paper was written the writer learns that this law has also been found by G. Guderley (Die Ursache fur das Auftreten von Verdichtingsstössen in gemischten Unterschall-Uberschallströmungen, M.o.S. (A) Volkenrode Reports and Translations No. 110) and appears also to have been stated by von Kármán.

A comparison with experiments shows good agreement as to pressure distribution, but less good agreement as to shock-wave position, on aerofoils related through the similarity law.

2. *Equation for Transonic Flow.*—For two-dimensional flow with small inclination the equation for compressible flow is of the form[1]

$$(1 - M^2)\phi_{xx}' + \phi_{yy}' = 0, \qquad (1)$$

where M is the Mach number and ϕ' the potential function

$$u = \phi_x' \,; v = \phi_y'. \qquad (2)$$

In the Prandtl-Glauert rule, for the local Mach number M, the Mach number M_0 of the undisturbed flow would be taken. It is then possible to apply methods of incompressible flow to subsonic flow.

In transonic flow for $1 - M^2$ we can never take $1 - M_0^2$, because $1 - M^2$ changes sign at sonic speed and the Prandtl rule is not valid in general for transonic flow.

The Mach number depends on the absolute value of velocity w

$$we\frac{dM^2}{dw} = 2M^2\left(1 + \frac{\gamma - 1}{2}M^2\right). \qquad (3)$$

Following our approximation we put for w its x — component u and get

$$1 - M^2 = (1 - M_1^2) - 2M_1^2\left(1 + \frac{\gamma - 1}{2}M_1^2\right)\left(\frac{u}{u_1} - \right) + \ldots . \qquad (4)$$

In equation (4) u_1 is the x-component of speed at the Mach number M_1. It is not necessary that u_1 be the velocity in the undisturbed flow. Put

$$u - u_1 = \phi_x \,;\; v = \phi_y, \qquad (5)$$

equation (1) can be written

$$(1 - M_1^2)\,\phi_{xx} + \phi_{yy} = 2M_1^2\left(1 + \frac{\gamma - 1}{2}M_1^2\right)\left(\frac{u}{u_1} - 1\right)\frac{\partial}{\partial x}(u - u_1)$$

$$= 2M_1^2\left(1 + \frac{\gamma - 1}{2}M_1^2\right)\phi_x \cdot \phi_{xx}\frac{1}{u_1}. \qquad (6)$$

Putting the right-hand side of equation (6) equal to zero gives the Prandtl equation with M_1 chosen as the value in the free stream. Equation (6) differs from the Prandtl equation in that it depends very slightly on the Mach number chosen. On changing M_1 in equation (6), the first member on the left-hand and the member on the right-hand side change in the same direction. But equation (6) has the great disadvantage of not being linear.

Putting $M_1 = 1$, $u_1 = c^*$ ($c^* = u^* =$ critical velocity) we get in effect the simplified equation for compressible flow in Ref. 1.

We can get equation (6) also by approximating the mass flow in terms of the difference of speed in an equation of the second order

$$u\rho = u_1\rho_1 + \frac{d(u\rho)}{du}(u - u_1) + \tfrac{1}{2}\frac{d^2(u\rho)}{du^2}(u - u_1)^2 + \ldots .$$

We get

$$\frac{u}{u\rho} \cdot \frac{d(u\rho)}{du} = 1 - M^2$$

and

$$\frac{u^2}{u\rho}\frac{d^2(u\rho)}{du^2} = -M^2\,[3 + (\gamma - 2)M^2] = -M^2\left[1 - M^2 + 2\left(1 + \frac{\gamma - 1}{2}M^2\right)\right];$$

thus

$$\frac{u\rho}{u_1\rho_1} - 1 = (1 - M_1^2)\left(\frac{u}{u_1} - 1\right) - \tfrac{1}{2}M_1^2\,[3 + (\gamma - 2)M_1^2]\left(\frac{u}{u_1} - 1\right)^2 + \cdots \qquad (7)$$

Fig. 1 shows that for Mach number 0·833 equation (7) gives a very good approximation over a large range of velocity. The Prandtl rule corresponds to an approximation of the mass flow by a straight line. The accuracy of this rule is given by a ratio formed by the two terms of the right-hand side of equation (7). For subsonic and transonic speed we get for this ratio

$$\varepsilon = \frac{(\gamma + 1)M_1^2}{1 - M_1^2}\,\frac{\Delta u}{u_1} \quad \ldots \qquad (8)$$

Approximating the mass flow by a parabola by putting $M_1 = 1$, equation (7) becomes

$$1 - \frac{u\rho}{u^*\rho^*} = \frac{\gamma + 1}{2}\left(\frac{u}{c^*} - 1\right)^2 = \frac{1}{2\,(\gamma + 1)}\,(M^2 - 1)^2 \quad \ldots \qquad (9)$$

and equation (6) becomes

$$-(\gamma + 1)\,\frac{1}{c^*}\,\phi_x\,.\phi_{xx} + \phi_{yy} = 0\,; \quad \ldots \qquad (10)$$

where

$$\phi_x = u - c^*\,; \quad \phi_y = v.$$

3. *Shock Polar*[2] *and Characteristics for Transonic Flow.*—The terms of the first order in the shock polar expansion near Mach number $M = 1$ are

$$\frac{v^2}{c^{*2}} = \frac{\gamma + 1}{2}\left[\left(\frac{u_1 - c^*}{c^*}\right)^3 - \left(\frac{u_1 - c^*}{c^*}\right)^2\frac{u - c^*}{c^*} - \frac{u_1 - c^*}{c^*}\left(\frac{u - c^*}{c^*}\right)^2 + \left(\frac{u - c^*}{c^*}\right)^3\right] \quad \ldots \qquad (11)$$

u_1 is the velocity before the shock ($v_1 = 0$) and u, v are the velocity components after the shock.

There is no deflection of flow if

$$v = 0 \begin{cases} u_1 - c^* = u - c^*\,; \\ \text{or } c^* - u_1 = u - c^*. \end{cases} \quad \ldots \qquad (12)$$

corresponding to the Mach line and to the normal shock-wave.

The speed after the shock is sonic if the y component v^* is given by

$$\left(\frac{v^*}{c^*}\right)^2 = \left(\frac{u_1 - c^*}{c^*}\right)^3\frac{\gamma + 1}{2}. \quad \ldots \qquad (13)$$

Using equations (13) and (11) we can write

$$\left(\frac{v}{v^*}\right)^2 = 1 - \left(\frac{u - c^*}{u_1 - c^*}\right) - \left(\frac{u - c^*}{u_1 - c^*}\right)^2 + \left(\frac{u - c^*}{u_1 - c^*}\right)^3 + \cdots \quad \ldots \qquad (14)$$

According to equation (13) shock polars can be transformed into one another by increasing the v-components as the $\frac{3}{2}$ power of the differences between u-components and critical velocity c^*. The centre of the transformation is $u = c^*$, $v = 0$.

In our approximation the point with the maximum deflection is the point with the largest v-component after the shock. Differentiating equation (14) we get the maximum for subsonic flow at

$$c^* - u = \tfrac{1}{3}(u_1 - c^*); \quad \frac{v_{\max}}{v^*} = \sqrt{\left(\frac{32}{27}\right)}. \quad \ldots \qquad (15)$$

The equation of the characteristics for two-dimensional isentropic supersonic flow near the point $w = w_1$, $\vartheta = 0$ (ϑ = angle of flow) of the hodograph gives

$$\vartheta = (M_1^2 - 1)^{1/2}\left(\frac{w}{w_1} - 1\right) + \frac{1}{2}\,\frac{1 + \frac{\gamma - 1}{2} M_1^2}{(M_1^2 - 1)^{1/2}}\left(\frac{w}{w_1} - 1\right)^2 + \frac{1}{6}\,\frac{f(M_1)}{(M_1^2 - 1)^{3/2}}\left(\frac{w}{w_1} - 1\right)^3 + \ldots \quad (16)$$

where $f(M_1)$ is a rational function of M_1 with the value at critical speed

$$f(1) = -\left(\frac{\gamma + 1}{2}\right)^2.$$

Observe that the first and the second terms in the series for characteristics are equal to the corresponding terms of the series for a shock polar at the same point of the hodograph. But all these expansions are insufficient at transonic flow. The power of the Prandtl factor $\beta = \sqrt{(M_1^2 - 1)}$ in the denominator of the terms of the expansion equation (16) increases, so the convergence becomes very poor as M_1 approaches 1.

If characteristics were a good approximation for shock polars in transonic flow, we would obtain the maximum deflection at the critical speed but not at the point given by equation (15). At sonic speed, of course, characteristics in the hodograph have the direction of the velocity, so the tangent to the characteristic goes through the point $u = 0$, $v = 0$.

For transonic flow the characteristics are to be expanded at the critical speed. We find

$$\vartheta = \tfrac{2}{3}(\gamma + 1)^{1/2}\left(\frac{w}{c^*} - 1\right)^{3/2}\left[1 - \tfrac{3}{20}(5 - 2\gamma)\left(\frac{w}{c^*} - 1\right) + \cdots\right] \quad \ldots \quad (17)$$

and inverting equation (17)

$$\frac{w}{c^*} - 1 = \left(\frac{3}{2}\,\frac{1}{\sqrt{(\gamma+1)}}\,\vartheta\right)^{2/3}\left[1 + \frac{5 - 2\gamma}{10}\left(\frac{3}{2}\,\frac{1}{\sqrt{(\gamma + 1)}}\,\vartheta\right)^{2/3} + \cdots\right] \quad \ldots \quad (18)$$

Table 1 shows the first and second approximation for speed given by equation (18) and the exact value for $\gamma = 1{\cdot}405$.

TABLE 1

Expansion of Characteristics at $M = 1$

		1	2	4	6	8	10	12	14	16
ϑ (deg)		1	2	4	6	8	10	12	14	16
ϑ (radians)		0·017	0·035	0·070	0·105	0·140	0·174	0·209	0·243	0·279
$\frac{w}{c^*} - 1$	1st approx. ..	0·066	0·105	0·165	0·218	0·263	0·302	0·342	0·380	0·418
	2nd approx. ..	0·068	0·107	0·171	0·228	0·278	0·322	0·367	0·411	0·456
	exact	0·068	0·107	0·173	0·227	0·276	0·322	0·366	0·407	0·448

For small angle of inclination the first approximation is quite good. Thus we can write the equation of the characteristics for transonic flow

$$\left(\frac{v}{c^*}\right)^2 = \tfrac{4}{9}(\gamma + 1)\left(\frac{u}{c^*} - 1\right)^3 + \cdots \quad \ldots \quad (19)$$

putting v equal to zero at the critical speed. Putting v equal to zero at velocity $u = u_1$ the characteristics have the form

$$\frac{v}{c^*} = \tfrac{2}{3}(\gamma + 1)^{1/2}\left[\left(\frac{u_1}{c^*} - 1\right)^{3/2} - \left(\frac{u}{c^*} - 1\right)^{3/2}\right] + \cdots \quad \ldots \quad (20)$$

There is an essential difference between equation (20) and the shock polar equation (11).

Corresponding to equation (14) we can give equation (20) the form

$$\frac{v}{v^*} = 1 - \left(\frac{u - c^*}{u_1 - c^*}\right)^{3/2} \quad \ldots \qquad (21)$$

As for the shock polar, the equations of characteristics can be transformed into one another by increasing v^* proportionally to $(u_1 - c^*)^{3/2}$.

4. *Von Kármán's Rule Concerning the Influence of Thickness at Sonic Speed.*—We will consider here a rule formulated for the first time by von Kármán in Paris, 1946.

For a thin profile with small inclination, the equation for the potential function is given by equation (10) with the boundary condition at infinity

$$\sqrt{(x^2 + y^2)} \longrightarrow \infty : \phi_x = u - c^* \longrightarrow 0 ; \quad \phi_y = v \longrightarrow 0. \quad \ldots \qquad (22)$$

The boundary conditions at the profile should be replaced as usual by boundary conditions on the x-axis. This approximation is the better, the closer the Mach number approaches 1, because of the small change of flow across the streamlines at the speed of sound. At $y = 0$ the v-component must be given by

$$\frac{\phi_y}{c^*} = f(x). \quad \ldots \qquad (23)$$

The flow is determined by equation (10) with the boundary conditions (22) at infinity and (23) on the x-axis and by the shock-wave conditions if existing. It is possible that there are solutions depending on the Reynolds number or on the initial conditions. It is also possible that there is no steady solution.

Assuming a new potential function

$$\phi(x,y) = a.\phi'(x,y') \quad \ldots \qquad (24)$$

and a new ordinate

$$\beta.y = y' \quad \ldots \qquad (25)$$

where a and β are constant, we find the following relation between the derivatives

$$\phi_x = a.\phi_x' ; \quad \phi_y = a.\,\phi_y' = a.\beta\phi'_{y'} ;$$
$$\phi_{xx} = a.\phi'_{xx} ; \quad \phi_{yy} = a.\beta^2\phi'_{y'y'} . \quad \ldots \qquad (26)$$

The gas dynamics equation has the same form (10) for ϕ' and y' if

$$a = \beta^2. \quad \ldots \qquad (27)$$

The boundary condition at infinity are for ϕ' and y' automatically the same as for ϕ and y. The boundary conditions on the x-axis becomes

$$\frac{\phi_{y'}'}{c^*} = \frac{1}{\beta^3} f(x) = \frac{1}{a^{3/2}} f(x) = f'(x), \quad \ldots \qquad (28)$$

where $f'(x)$ is the inclination at a 'reduced' profile. For a certain shape of reduced profile we get a number of different flows depending on the β or a chosen. From equations (26), (27) and (28) the difference between velocity and critical velocity, $(u - c^*)$, is proportional to the $\frac{2}{3}$ power of the profile inclination and therefore, also of the thickness ratio t. Thus all forces and pressures are proportional to $t^{2/3}$. The v-component is proportional to t itself as expected.

Following equation (25) the distance of corresponding points from the x-axis is proportional to $t^{-1/3}$; thus corresponding points approach the x-axis with increasing thickness ratio.

According to the boundary conditions on the x-axis the profile shape changes to a first approximation by affine transformation. This approximation is not very good, because there are great changes in speed due to small changes in inclination in transonic flow. Thus the slope of the profile contour depends not only on v but also a little on $(u - c^*)$. A second approximation to the contour can be obtained if the pressure distribution is known. The same question arises in the next section.

Shock-waves will be discussed in section 6.

5. *A New Law of Similarity for Transonic Flow.*—If the speed in the undisturbed flow differs from the critical velocity within the range of validity of equation (10) (roughly $0{\cdot}6 < M < 1{\cdot}35$) the boundary conditions at infinity become

$$(x^2 + y^2)^{1/2} \longrightarrow \infty : \phi_x = u - c^* \longrightarrow u_0 - c^* ; \phi_y \longrightarrow 0. \qquad (29)$$

This becomes for the potential function ϕ' and the ordinate y'

$$(x^2 + y^2)^{1/2} \longrightarrow \infty : \phi_x' = \frac{u' - c^*}{a} \longrightarrow \frac{c^*}{\gamma + 1} ; \phi_y' \longrightarrow 0 ; \qquad (30)$$

if we put

$$a = 1 - M_0{}^2 = (\gamma + 1)\frac{u_0 - c^*}{c^*}. \qquad (31)$$

β now becomes the Prandtl factor.

Equation (27) also should be right here so equation (10) should be the same for ϕ' and y'. Following equation (31) the boundary condition on the x-axis becomes

$$\frac{\phi_{y'}'}{c^*} = \frac{1}{(1 - M_0{}^2)^{3/2}} f(x) = f'(x). \qquad (32)$$

As in section 4, we get, for the same reduced shape, the same differential equation and the same boundary conditions. As in von Kármán's rule profiles must be in an affine relationship to a first approximation. While we compare in von Kármán's rule profiles with different thicknesses in sonic flow, in this rule a smaller thickness ratio corresponds (in subsonic flow) to a higher speed in the undisturbed flow. The profiles compared must have the same reduced thickness ratio

$$t' = \frac{t}{(1 - M_0{}^2)^{3/2}} = \text{const.} \qquad (33)$$

From equation (26) and (27) the difference between speed and critical velocity $(u - c^*)$ is proportional to β^2. This is also true for all differences of velocities and pressures in the flow. The v-components change as the thickness ratio. Corresponding points approach as β^{-1} to the x-axis with increasing thickness ratio. There is no transition possible to von Kármán's rule. According to equation (33) thickness ratio becomes zero at the velocity of sound.

Note that the new rule is valid for mixed flows. We can apply it also to supersonic speed in the undisturbed flow.

The Prandtl-Glauert rule within its validity gives the same results as the new law. Following Prandtl's analogy the change in speed at corresponding points of the flow round profiles in affine relationship is proportional to the thickness ratio and to the reciprocal Prandtl factor :

$$(u - u_0) \propto \frac{t}{\beta}.$$

If the thickness ratio increases according to the new rule with the cube of β, the velocity difference increases with the square of β as required.

The law of similarity treated, and von Kármán's rule, form a bridge over the forms of Prandtl-Glauert analogy valid for pure subsonic and supersonic flow only. Note the assumption $v << c^*$ when comparing profiles.

6. *Shock Polar and Mach Line in the Law of Similarity.*—We have shown that the difference between speed and critical velocity, $(u - c^*)$, is proportional to the $\frac{2}{3}$ power of the v-component. The angle of a shock wave to the x-axis is given (Fig. 2) by

$$\tan \gamma' = \frac{u_2 - u_1}{v_2} = \frac{(u_2 - c^*) - (u_1 - c^*)}{v_2}, \qquad (34)$$

where suffixes 1 and 2 are related to the flow before and after the shock-wave and v_1 is assumed to be zero.

Transforming the profile, the inclination of a shock-wave, according to equation (34), and the changes of velocity components must change proportionally to $1/\beta$. The shock-wave has just this inclination as the corresponding points approach the x-axis according to the transformation.

Thus shock-waves are transformed into one another in flows parallel to the x-axis, as for example the shock-wave at the leading edge at supersonic speed.

The transformation is also right for a flow parallel to the x-axis after a shock-wave, as for example at the trailing edge in supersonic flow. It seems not to have been noted that the parts of the curve given by the shock equation[2] but always omitted from the shock polar diagram are the shock polars for a given state behind the shock and an unknown state before the shock (Fig. 3). Thus the shock equation has physical significance in all the parts of the hodograph within the maximum velocity $w/c^* \leqslant \left[(\gamma + 1)/(\gamma - 1)\right]^{1/2}$. In a similar way to equation (34), the angle between the shock-wave and the x-axis is now

$$\tan \bar{\gamma}' = \frac{u_1 - u_2}{v_1} = \frac{(u_1 - c^*) - (u_2 - c^*)}{v_1}. \qquad (35)$$

The curves (11) and (14) contain also the polar for a given state after the shock (Fig. 3) for transonic flow. Thus the transformation of shock-waves should be correct for flow parallel to the x-axis after the shock also.

Generally the inclination of the shock-wave to the x-axis is

$$\tan (\gamma' + \vartheta_1) = \frac{u_2 - u_1}{v_2 - v_1} = \frac{(u_2 - c^*) - (u_1 - c^*)}{v_2 - v_1}. \qquad (36)$$

Transforming $(u - c^*)$ and v according to equation (26), all transformations are correct except that of the shock polar (Fig. 4). The shock polar should be transformed by a change in direction proportional to v and by an alteration in the direction of the axis of symmetry and normal to it as in the first and second case. Within our approximation, however, this is the same as a transformation in the direction of and normal to the u-component.

Note that a normal shock-wave is expected to remain normal after transforming the flow. Thus its inclination must change as v. On the other hand the inclination of the shock-wave must change with the alteration of the ordinate proportional to $1/\beta$. Thus it is not surprising that there are small deviations from the direction perpendicular to the flow. This is no fault within our approximation. Note also that there is in general no difference between the direction of the ordinate and the direction normal to the flow in our treatment.

The inclination of the Mach line too shows small deviations when the flow in inclined. But there is only a small change in speed when the Mach angle α is changed

$$dM = \left(1 - \frac{1}{M^2}\right)^{1/2} d\alpha. \qquad (37)$$

Remember further that Mach lines in the linear supersonic theory have constant inclination. Moreover, this theory gives good results for small thickness ratio. Hence the inclination of the characteristics is calculated from the coefficients of the hyperbolic differential equation and in general a small error in the coefficient of the equation is not so important as an error of the same size in the boundary conditions.

Thus von Kármán's rule and the law of similarity treated are valid for mixed transonic flow with shock-waves and small inclinations of the streamlines.

Irrotational flow has been assumed and, therefore, small change of entropy across the flow. It is not necessary that the shock-waves themselves are assumed to be isentropic. The rules discussed are valid also for shocks having an entropy increase proportional to the cube of the pressure increase.

It has been shown how the drag of bodies can be expressed by the entropy increase in the flow[4]. Essential for the losses is the product of entropy increase in the shock and line element normal to the flow: this will be shown in a following paper. The pressure increase, like the increase in speed, is proportional to β^2. Hence the entropy increase is proportional to β^6. The ordinates alter as β^{-1}. Thus the losses change proportional to β^5. The drag changes with the the same power, being proportional to the pressure differences and the thickness ratio, equation (33).

Strong shock-waves appearing at higher Mach numbers have losses smaller than those given by the power law used. Hence the drag of thicker aerofoils with higher speed on the surface is less than expected. Thus strong shock-waves move upstream with increasing thickness ratio on corresponding profiles. (Concerning stream losses and shock-wave position see also section 5.)

7. *Critical Mach number*.—Using the law of similarity it is possible to say how certain critical flows depend on the thickness ratio of slender aerofoils. Of course the critical state must not be caused by viscosity effects but only by effects of compressibility as treated in this paper.

The example best known is the critical Mach number, that is the Mach number of the undisturbed flow causing a maximum speed on the surface of the aerofoil equal to a Mach number exactly 1. A second critical point is to be expected, if at the point with greatest thickness of the profile the mass flow of the undisturbed flow is reached. In any of these cases, assuming slender bodies, the thickness ratio of aerofoils in affine relationship is according to equation (33) proportional to the power 3 of the Prandtl factor:

$$(1 - M_0'^2)^{3/2} \propto t. \qquad \ldots \qquad (33')$$

8. *Comparison with Tests*.—In this section, we apply the law of similarity to tests made by Göthert on NACA aerofoils with maximum thickness at 30 per cent and thickness ratios between 6 per cent and 18 per cent. Choosing a reduced thickness ratio (equation (33)) of

$$t' = 0{\cdot}55$$

we get the following Mach numbers depending on the thickness ratio t

t	0·06	0·09	0·12	0·15	0·18
M_0	0·88	0·84	0·80	0·76	0·72.

The slenderest profile has very great losses measured in the wake by Göthert. Compared with this the losses of the other aerofoils are quite small. At 18 per cent thickness there are no tests at the corresponding Mach number. Thus in Fig. 5 only the aerofoils with thickness ratios of 9, 12 and 15 per cent are compared.

All the tests were made at small angles of incidence and the test points of the upper and lower surface are plotted. Taking a mean of these, we get approximately the pressure distribution at zero lift. The test points before and after the shock-wave form quite a good curve. But there are distinct deviations in the position of the shock-wave. The deviations of the two aerofoils not plotted here are of the same kind. On the profile with 6 per cent thickness ratio the shock-wave is near the trailing edge.

The shock-wave moves upstream with increasing thickness ratio. Note that the position of the shock-wave on the upper surface of the 12 per cent aerofoil, which has the smallest angle of incidence, is the same as the shock-wave position of the 15 per cent aerofoil, and that on the lower surface as the 9 per cent aerofoil.

The pressure coefficient c_p is proportional to β^2 like the pressure itself. Dividing c_p by the value of c_p at sonic speed we get the comparable pressure distribution. According to the theory the speed of sound must be reached at the same points of course.

The tests are near the limits of the applicability of our theory. The v-components should be small enough but there are large differences in speed, so it is doubtful whether equations (4), (9) and (31) are sufficiently good approximations. Comparing our expansions with the corresponding exact functions we must expect deviations at first caused by the simplification of the shock polars. The change of shock-waves in Fig. 5 is of this kind. But there are also other sources for the lack of agreement of tests and theory. Considering tests concerning critical Mach numbers we find deviations from the rule in equation (33').

Certainly the boundary layer may have an influence on pressure distribution in transonic flow. According to tests made by Ackeret, Feldman and Rott[7] we should not expect boundary-layer effects ahead of the shock-wave at the high Reynolds numbers used by Göthert. But the change in displacement thickness near the shock-wave is so important that some influence of the boundary layer on shock-wave position is quite possible.

Further it is possible that the tunnel correction applied near Mach number 1 is not sufficient. All tunnel corrections used until now have been calculated using the Prandtl-Glauert analogy. Hence the displacement caused by the local supersonic fields is not taken into consideration. Thus the Mach number of the undisturbed flow M_0 may be a little higher than calculated, especially at high Mach numbers.

Tests on very slender aerofoils with corresponding tunnel corrections should be very interesting in relation to the rule discussed. Thus tunnel height would be decreased proportionally to β^{-1} with decreasing Mach number. In this way we should get the influence of the boundary layer at high Mach numbers.

9. *Some remarks concerning Transonic Flow.*—It is evident that we can compare thicker profiles at smaller Mach number M_0 with thinner profiles at higher M_0 having mixed flow only when the shock-wave and the sonic line meet the profile at the same points.

Not less important than the critical Mach number will be another Mach number which has the same mass flow at maximum thickness and in the undisturbed flow. Between a state near this Mach number and the speed of sound the velocity at the maximum thickness is expected to fall with increasing speed in the undisturbed flow.

We must assume the flow to be of a kind such that the aerofoil produces a minimum flow displacement. Thus the mass flow at maximum thickness may never be much less than the mass flow in the undisturbed flow. In supersonic flow about slender bodies the Mach number and the mass flow at maximum ordinate and in the undisturbed flow are the same ; in pure subsonic flow the mass flow at maximum thickness is higher than in the undisturbed flow, hence near Mach number 1 the Mach number at maximum thickness is expected to approach 1 from the supersonic side if M_0 approaches 1 from the subsonic side.

Coming from higher supersonic speed we can consider the flow at sonic speed. Remember that we have at supersonic velocity on slender profiles the same state in the undisturbed flow and at maximum thickness. Upstream the velocity decreases, downstream the velocity increases and in the trailing wave resumes the velocity of the undisturbed flow. There is no connection between the flow before and after the maximum thickness. With falling M_0 the wave at the trailing edge becomes more erect but always begins on the trailing edge. Obviously it begins there also at sonic speed in the undisturbed flow. Hence it is possible to compare by von Kármán's rule different profiles at the speed of sound. On the aerofoils compared, the point with Mach number 1 must always have the same position near the maximum independently of the thickness ratio.

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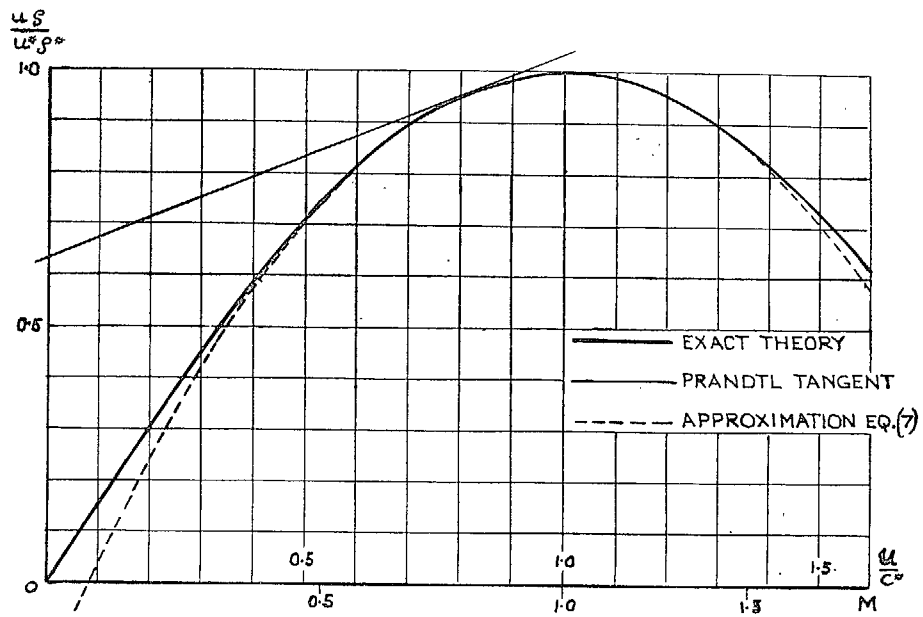

FIG. 1. Approximation for mass flow at $M_1 = 0{\cdot}833$.

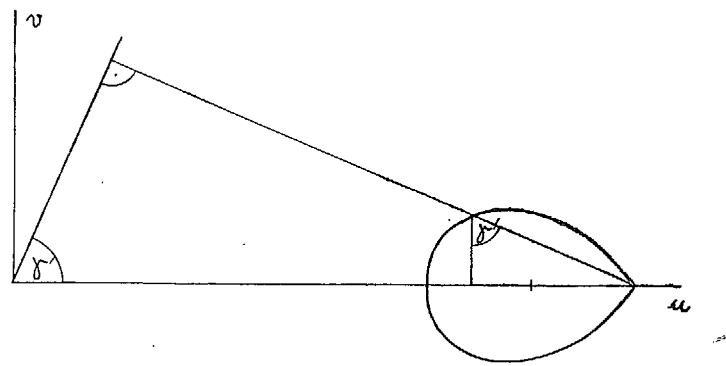

FIG. 2. Shock-wave angle in the hodograph.

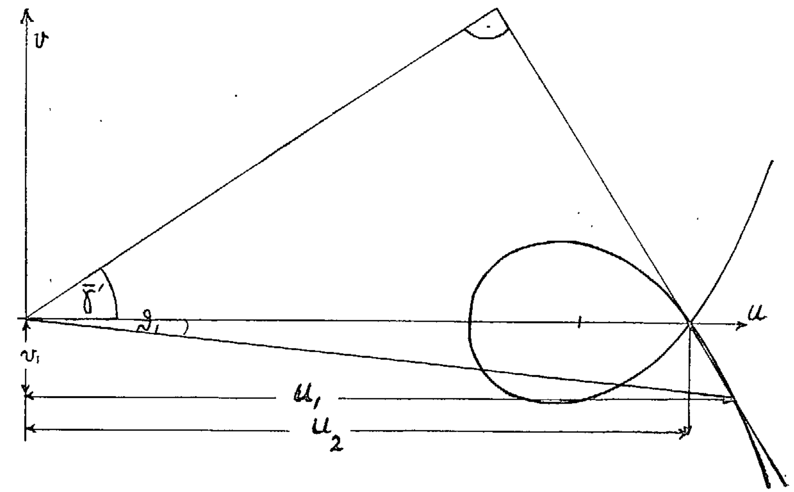

FIG. 3. Polar for a given state behind the shock.

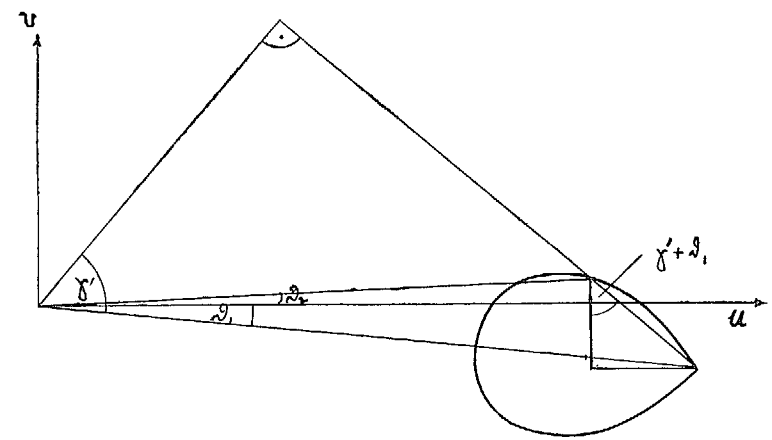

FIG. 4. Shock polar for flow at small angle of incidence.

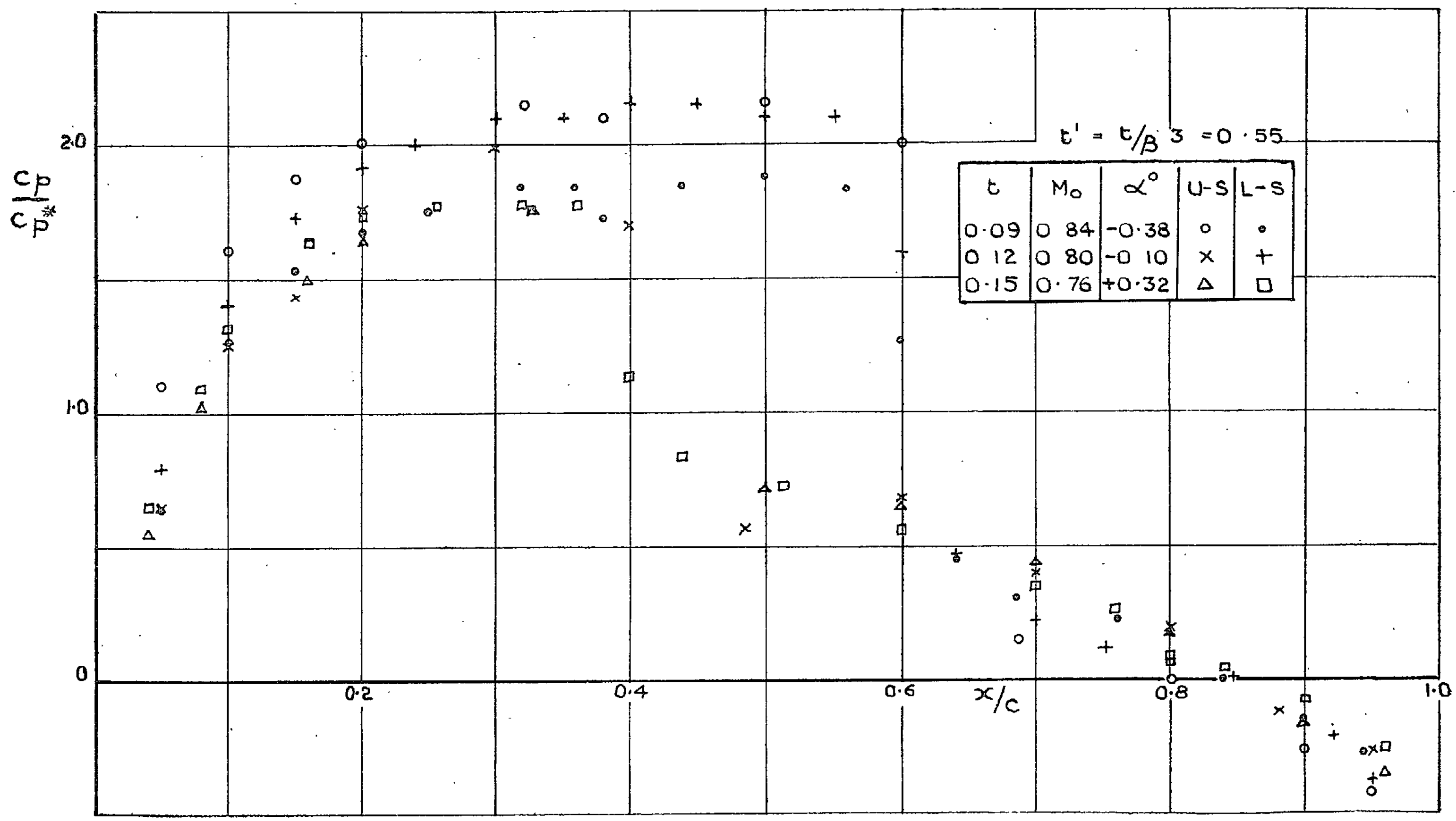

FIG. 5. Law of similarity applied to Göthert's tests.

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