R. & M. No. 2681
(12,395)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

A Theoretical Examination of the Effect of Deadrise on Wetted Area and Associated Mass in Seaplane-Water Impacts

By

R. J. MONAGHAN, M.A.

LONDON: HER MAJESTY'S STATIONERY OFFICE
1952
PRICE 4s. 6*d*. NET

A Theoretical Examination of the Effect of Deadrise on Wetted Area and Associated Mass in Seaplane-Water Impacts

By

R. J. Monaghan, M.A.

COMMUNICATED BY THE PRINCIPAL DIRECTOR OF SCIENTIFIC RESEARCH (AIR),
MINISTRY OF SUPPLY

*Reports and Memoranda No. 2681**
March, 1949

Summary.—A theoretical examination is made of the deadrise effect on associated mass and wetted area in the two-dimensional impact case (vertical drop of an infinitely long wedge at zero attitude). Available estimates are summarised and a new theoretical formula is developed by means of an expanding prism flow which gives results for associated mass in very close agreement with those given by Wagner's[2] semi-empirical formula (on which most of the estimates of three-dimensional associated mass have so far been based). In addition the new treatment gives a formula for wetted area which is not available from Wagner's treatment except for very small values of deadrise angle.

Comparison is made between these and other formulae in the light both of theory and experiment and a brief survey is made (in Appendix I) of the assumptions involved in applying associated mass methods to motions through a free surface.

1. *Introduction.*—The usual method of approach to the seaplane-water impact problem has been to assume that during the course of an impact, momentum is transferred from the body to a fictitious 'associated' or 'virtual' mass of water† and by making assumptions about the nature of this 'mass' the motion of the body can be determined.

Various estimates have been given in the past for the value of the associated mass, depending on the assumptions made about the effects of deadrise angle and aspect ration of the wetted area on the body. (For a summary of these estimates *see* Appendix II of Ref. 3). The present note is restricted to a theoretical examination of the deadrise effect and deals with the two-dimensional case only (vertical drop of an infinitely long wedge at zero attitude). A new theoretical formula is developed which gives results in very close agreement with Wagner's[2] semi-empirical formula (on which most of the estimates of three-dimensional associated mass have been based). In addition the new treatment gives a formula for splash-up factor (splash-up is the rise of displaced water along the sides of the body) which is not available from Wagner's treatment except for very small values of deadrise angle.

Also, in Appendix I, a brief survey is made of the assumptions involved in applying associated mass methods to motions through a free surface.

This report is part of a series giving the results of an investigation of water impact forces and pressures.

* R.A.E. Tech. Note Aero. 1989, received 8th June, 1949.

† The term 'associated mass' has commonly been used by writers on this subject and for that reason is used in the present note.

2. *Available Estimates for the Two-dimensional Associated Mass for Wedges*.—As shown in Appendix I, associated mass methods can only give an approximation to the true motion of a body through a free surface and their worth is largely dependent on the correct choice of values for the associated mass to give agreement with experimental results.

The present report is restricted to a theoretical examination of the deadrise effect and this can only be made for the two-dimensional case, *i.e.*, the vertical drop of an infinitely long wedge at zero attitude. Various estimates have been made for the associated mass under these conditions and they can be summarised as follows.

2.1. *Von Kármán*.[1]—The earliest estimate appears to have been made by Von Kármán,[1] who proposed that the associated mass be taken as the mass of a semi-cylinder of water on the wetted width of the wedge as diameter. He took this wetted width to be the intersection of the wedge with the undisturbed water surface (Fig. 2b).

This value is half the value obtained from the motion of a flat plate of the same width in unbounded fluid and Von Kármán took it to apply without any correction for finite deadrise angle (θ). Thus (denoting associated mass by μM where M is the mass of the wedge) he took

$$\mu M = \varrho \frac{\pi}{2} c_0^2 = \varrho \frac{\pi}{2} h^2 \cot^2 \theta \quad \ldots \quad (1)$$

per unit length of the wedge, where $2c_0$ is the wetted width given by the intersection with the undisturbed water surface, h is the draft, (*see* Fig. 2b), and ϱ is the density of water.

2.2. *Wagner*[2].—During an impact motion there will be a rise of displaced water along the sides of the body (known as ' splash-up ') so that the actual wetted width will be greater than that given by the intersection of the body with the undisturbed water surface.

Provided that the deadrise angle θ is small, Wagner[2] considered that the flow relative to the wedge in an impact motion would be closely approximated to by the flow normal to a flat plate in unbounded fluid if at each instant

(a) the plate width was taken equal to the actual wetted width of the wedge,

and (b) the plate was taken to lie in the plane of the undisturbed free surface.

Thus, his assumed conditions are as shown in Fig 3 and the implications of these conditions are discussed in Appendix I.

From these assumptions, Wagner calculated the rise of the free surface during the course of the motion and found that for a plane-faced wedge the wetted width would be $\pi/2$ times that given by the intersection with the undisturbed water surface, *i.e.*,

$$c = \frac{\pi}{2} c_0. \quad \ldots \quad (2)$$

Also, only one side of the plate is wetted in an impact motion as compared with both sides in motion through an unbounded fluid so that the associated mass in the former case will be half that in the latter, *i.e.*,

$$\mu M = \varrho \frac{\pi}{2} c^2 = \varrho \frac{\pi^3}{8} h^2 \cot^2 \theta \qquad \text{when } \theta \text{ is small} \quad \ldots \quad (3)$$

The value of this approximation decreases as θ increases and to obtain an expression for the force on the wedge valid for all deadrise angles, Wagner chose the following method.

(1) If the motion is steady, *i.e.*, V = constant, then an exact, if laborious, solution of the impact problem for a plane-faced wedge can be made by a centre of similitude method[2]. Wagner made these calculations for a deadrise angle of 18 deg and quotes the result in Ref. 2.

(2) For limitingly small values of θ ($\theta \rightarrow 0$) a value for the vertical force in steady motion can be obtained from

$$F = V \cdot \frac{d(\mu M)}{dt} \qquad (4)$$

where μM is given by equation (3).

This gives an asymptotic curve for $\theta \rightarrow 0$.

$$F = \frac{\pi^3}{4} \frac{1}{\theta^2} \varrho V^2 h. \qquad (5)$$

(3) For very great values of θ ($\theta \rightarrow 90$ deg) the problem can be simplified to that of the immersion of a knife edge without splash-up and this can be solved exactly by conformal transformation, thus giving an asymptotic curve for $\theta \rightarrow 90$ deg.

From these three solutions Wagner then derived an empirical variation of impact force with deadrise angle by generalising equation (5) in the form

$$F = K \frac{\pi^3}{4} \frac{1}{\theta^2} \varrho V^2 h \qquad (6)$$

and by taking

$$K = \left(1 - \frac{2\theta}{\pi}\right)^2. \qquad (7)$$

From equations (4), (6) and (7) and by integration we can then obtain

$$\mu M = \varrho \frac{\pi^3}{8} \frac{1}{\theta^2} \left(1 - \frac{2\theta}{\pi}\right)^2 h^2 \qquad (8)$$

as an expression for μM valid over the whole range of θ. Equation (8) applies strictly only to steady motions (V = constant) but it has also been taken to apply to unsteady motions (V = function of time).

It should be noted that only in the region $\theta \rightarrow 0$ does equation (8) assume that the impact associated mass is half the value of some unbounded fluid associated mass. Also it is only in the same region that the splash-up factor of $\pi/2$ has been derived, so that for usual values of θ nothing is known of the wetted area.

However, in application, most later writers[3] have expressed equation (8) in the form

$$\mu M = \varrho \frac{\pi^3}{8} \cot^2 \theta \, . \, \xi_1 h^2 \qquad (8a)$$

with

$$\xi_1 = \left(\frac{\tan \theta}{\theta}\right)^2 \left(1 - \frac{2\theta}{\pi}\right)^2 \qquad (9)$$

and have regarded ξ_1 as a deadrise correction factor to the associated mass as given by equation (3). The splash-up factor has then been taken as $\pi/2$ for all deadrise angles.

Other more recent writers[4] have used equation (8) in a form equivalent to

$$\mu M = \varrho \frac{\pi}{2} (h^2 \cot^2 \theta) \, (f(\theta))^2 \qquad (8b)$$

with

$$f(\theta) = \frac{\pi}{2} \left(\frac{\tan \theta}{\theta}\right) \left(1 - \frac{2\theta}{\pi}\right) \qquad (10)$$

and have assumed that the splash-up factor for finite values of θ is given by $f(\theta)$. Theoretically, this has the advantage that $f(\theta) \to \pi/2$ as $\theta \to 0$ and $f(\theta) \to 1$ as $\theta \to \pi/2$, but the method of derivation of equation (8) cannot be said to support the variation for intermediate values of θ. For instance, there is no evidence in support of the implicit assumption that the associated mass for a wedge of finite deadrise angle is a semi-cylinder of water on the full wetted width as diameter.

2.3. *Kreps*[5].—Kreps assumed a splash-up factor of $\pi/2$ for all deadrise angles and gave a formula for associated mass in the form of equation (8a), *i.e.*,

$$\mu M = \varrho \frac{\pi^3}{8} \cot^2 \theta \,.\, \xi_1 h^2 \qquad \text{(8a)}$$

but with

$$\xi_1 = 1 - \frac{\theta}{\pi} \qquad \text{(11)}$$

instead of equation (9) for ξ_1. Equation (11) was derived from consideration of the relation between the flows without splash-up past a prism and past a flat plate.

3. *A New Treatment for the Two-dimensional Impact of a Wedge of Finite Deadrise Angle.*—Wagner's expanding plate flow of section 2.2 applies in the case of limitingly small deadrise angles. When the deadrise angle is of finite magnitude (as in the case of seaplane hull bottoms) a better approximation to the relative flow might be obtained by considering the flow past an expanding prism, derived as shown in Fig. 4. The deadrise of the prism is the same as that of the wedge and at any instant its width is equal to the wetted width of the wedge.

The mathematical solution of the flow problem is given in Appendix II. The main differences appearing when compared with Wagner's solution are

(a) The splash-up factor is now given by

$$\frac{c}{c_0} = \frac{\sqrt{\pi}}{2} \frac{\sin \theta}{\theta} \left\{ \Gamma\left(\frac{1}{2} + \frac{\theta}{\pi}\right) \Gamma\left(1 - \frac{\theta}{\pi}\right) \right\} \qquad \text{(12)}$$

where $\Gamma(n)$ denotes the complete gamma function. This reduces to Wagner's factor of $\pi/2$ as $\theta \to 0$, and $c/c_0 \to 1$ as $\theta \to \pi/2$.

An approximation, valid to within 2 per cent in the range $0 \leqslant \theta \leqslant \pi/4$ is given by

$$\frac{c}{c_0} \simeq \frac{\pi}{2}\left(1 - \frac{\theta}{\pi}\right) \qquad \text{(13)}$$

(b) The associated mass of liquid is given by

$$\mu M = \varrho c^2 \tan \theta \left\{ \frac{\pi - 2\theta}{\sin 2\theta} \cdot \frac{\pi}{\left[\Gamma\left(\frac{1}{2} + \frac{\theta}{\pi}\right) \Gamma\left(1 - \frac{\theta}{\pi}\right)\right]^2} - 1 \right\} \qquad \text{(14)}$$

$$\simeq \frac{\pi}{2} \varrho c^2 \left(1 - \frac{\theta}{\pi}\right) \qquad \text{(15)}$$

over the practical range of θ as compared with Wagner's value

$$\mu M = \frac{\pi}{2} \varrho c^2 \qquad \text{(3)}$$

Thus there is a deadrise effect both on splash-up and on associated mass.

Comparison of the two expressions (equations (3) and (15)) for the associated mass of a flat plate and of a prism respectively at the same wetted width ($2c$) shows that the effect of deadrise is to introduce the Kreps factor[5].

$$\xi_1 = 1 - \frac{\theta}{\pi} \qquad \ldots \qquad (11)$$

as given in section 2.3.

On the other hand, the splash-up factor as given by equation (12) or (13) is smaller than the Wagner value of $\pi/2$ as used by Kreps, so that the associated mass in terms of draft is given by (from equations (15) and (13))

$$\mu M = \varrho \frac{\pi^3}{8} \cot^2 \theta \left(1 - \frac{\theta}{\pi}\right)^3 h^2 \qquad \ldots \qquad (16)$$

The objection can be raised that, in the treatment of this section, the flow is diverted by the prism too early (at the point P_1 in Fig. 4a) as compared with the flow past the wedge (which is diverted at point P_2 of Fig. 4a). This is the case, but

(a) the treatment is in any case only approximate,

and (b) in the Wagner flat plate treatment the flow is diverted much too late (at the point P_3), so that it is considered that the suggested method might give a closer approximation to the true conditions than consideration of the flat plate flow has provided.

4. *Comparison of Results.*—Fig. 5 shows a comparison of the various estimates for the two-dimensional associated mass.

The expanding prism analogy of the present note gives a formula for associated mass

$$\mu M = \varrho \frac{\pi^3}{8} \cot^2 \theta \, . \left(1 - \frac{\theta}{\pi}\right)^3 h^2 \qquad \ldots \qquad (16)$$

whose results (represented by the full line) are in very close agreement with those obtained from Wagner's semi-empirical formula

$$\mu M = \varrho \frac{\pi^3}{8} \frac{1}{\theta^2} \left(1 - \frac{2\theta}{\pi}\right)^2 h^2 \qquad \ldots \qquad (8)$$

over the range of $\theta \leqslant \pi/4$. The latter formula includes the exact centre-of-similitude solution at $\theta = 18$ deg, and has been the basis of the majority of three-dimensional associated mass estimates.

Therefore, the expanding prism analogy might be expected to give a good approximate representation of events during the impact. In particular, it gives an expression

$$\frac{c}{c_0} = \frac{\pi}{2} \left(1 - \frac{\theta}{\pi}\right) \qquad \ldots \qquad (13)$$

for the splash-up factor in the range $0 \leqslant \theta \leqslant \pi/4$, whereas Wagner's theory[2] only gives this factor for limitingly small values of θ. This factor is of importance in determining wetted areas, instant of chine immersion, etc., and its variation is shown in Fig. 6.

The effect on associated mass of neglecting the variation of the splash-up factor by taking it as $\pi/2$ for all deadrise angles, while using the deadrise correction factor from prism to flat-plate flow as derived in the present report, is shown by the broken line in Fig. 5 which corresponds to Kreps's[5] formula

$$\mu M = \rho \frac{\pi^3}{8} \cot^2 \theta \left(1 - \frac{\theta}{\pi}\right) h^2 \qquad \ldots \qquad \text{(8a) and (11)}$$

The results are everywhere greater than those given by Wagner[2] (equation (5)) or the formula of the present note (equation (16)).

However, experimental (three-dimensional) measurements of wetted areas for $\theta = 20$ deg have so far shown no variation of splash-up factor with deadrise angle and support the value $\pi/2$ throughout, both in impact and in planing motions. Also, analysis[6] of three-dimensional impact results from the N.A.C.A. for values of θ up to 40 deg would seem to support a deadrise variation of associated mass

$$\cot^2\theta\left(1 - \frac{\theta}{\pi}\right)$$

as in Kreps's formula, rather than the variation

$$\cot^2\theta\left(1 - \frac{\theta}{\pi}\right)^3$$

of the present report (which agrees with the variation from Wagner's formula, *cf.* Fig. 5).

Thus two-dimensional theory supports the variation

$$\frac{c}{c_0} = \frac{\pi}{2}\left(1 - \frac{\theta}{\pi}\right) \qquad \ldots \qquad (13)$$

while three-dimensional experimental results support the constant value

$$\frac{c}{c_0} = \frac{\pi}{2} \,.$$

The discrepancy would best be investigated by a series of two-dimensional impact tests.

Also shown in Fig. 6 is the variation

$$\frac{c}{c_0} = \frac{\pi}{2}\,\frac{\tan\theta}{\theta}\left(1 - \frac{2\theta}{\pi}\right) \qquad \ldots \qquad (10)$$

for splash-up factor advanced by Milwitzky[4], which would give even smaller wetter areas and later chine immersion than the variation suggested in this report.

5. *Conclusions*.—(1). The expanding prism analogy of the present report gives a theoretical formula for the two-dimensional associated mass of a wedge in very close agreement with Wagner's[2] semi-empirical formula. It is

$$\mu M = \varrho\,\frac{\pi^3}{8}\cot^2\theta\left(1 - \frac{\theta}{\pi}\right)^3 h^2$$

in the range $0 \leqslant \theta \leqslant \pi/4$.

(2). It also gives an expression for the variation of splash-up factor with deadrise angle which is not available from Wagner's[2] work. This is

$$\frac{c}{c_0} = \frac{\pi}{2}\left(1 - \frac{\theta}{\pi}\right)$$

in the range $0 \leqslant \theta \leqslant \pi/4$, which reduces to the Wagner value of $\pi/2$ as $\theta \to 0$.

(3). So far, experimental (three-dimensional) results have not shown any variation of splash-up factor with deadrise. In effect, they support Kreps's[5] result for associated mass

$$\mu M = \varrho\,\frac{\pi^3}{8}\cot^2\theta\left(1 - \frac{\theta}{\pi}\right) h^2$$

in preference to the formula of conclusion 1 or Wagner's formula (which give almost identical results).

(4). There is no support for the recently advanced variation

$$\frac{c}{c_0} = \frac{\pi}{2} \frac{\tan \theta}{\theta} \left(1 - \frac{2\theta}{\pi}\right)$$

of splash-up factor. (This expression was derived from Wagner's formula for associated mass by assuming that the associated mass for a wedge of finite deadrise angle is a semi-cylinder of water on the wetted width as diameter.)

LIST OF SYMBOLS

M	Mass of body
μM	Associated mass of water
ϱ	Density of water
θ	Deadrise angle
h	Draft with respect to undisturbed free surface
$2c_0$	Wetted width at intersection of wedge with undisturbed free surface
$2c$	Actual wetted width of wedge
V	Vertical velocity
F	Vertical force
ξ_1	Deadrise correction factor to associated mass

The symbols in the appendices are defined as they occur.

REFERENCES

No.	*Author*	*Title, etc.*
1	T. Von Kármán	The Impact on Seaplane Floats during Landing. N.A.C.A. Technical Note No. 321. 1929.
2	H. Wagner	Phenomena associated with Impact and Gliding on a Liquid Surface. A.R.C. 2575. *Z.A.M.M.* Vol. 12. 1932.
3	R. J. Monaghan	A Review of the Essentials of Impact Force Theories for Seaplanes and Suggestions for Approximate Design Formulae. A.R.C. 11,245. November, 1947. (To be published.)
4	B. Milwitzky..	A Generalised Theoretical and Experimental Investigation of the Motions and Hydrodynamic Loads experienced by V-bottom Seaplanes during Step-landing Impacts. N.A.C.A. Technical Note No. 1516. February, 1948.
5	R. Kreps	Experimental Investigation of Impact in Landing on Water. N.A.C.A. Technical Memo. No. 1046. CAHI Translation. 1943.
6	R. J. Monaghan and P. R. Crewe ..	Formulae for estimating the Forces in Seaplane-Water Impacts without Rotation or Chine Immersion. A.R.C. 12,399. January, 1949. (To be published.)

APPENDIX I

Application of Associated Mass Methods to the Two-dimensional Impact Problem

A two-dimensional impact is the vertical impact of an infinitely long wedge at zero attitude. The flow in any cross-section can then be taken as two-dimensional, as in Fig. 1.

At touch-down suppose the (vertical) velocity of the body (Mass M) is V_0. The liquid is at rest, so that the total momentum of the system is MV_0. As the body penetrates the surface it sets up motion in the liquid so that the liquid gains momentum and if no external forces are acting and viscosity is neglected, then the body must lose a corresponding amount in order to satisfy the law of conservation of momentum. Thus, if V is the velocity of the body at some later time, 't', we can write

$$MV_0 = MV + \mu M \,.\, V \qquad (1)$$

where $\mu M \,.\, V$ represents the liquid momentum and as yet no assumptions have been made about the form of μM.

Now, in general, the total momentum of the liquid is given by

$$\underline{B} = \int \varrho \underline{v} \, d\tau \qquad (2)$$

where $d\tau$ is an element of volume (*see* Fig. 1) with density ϱ and vectorial velocity $\underline{v}$, and the integral is taken over the whole volume of the liquid. (In our case, $\underline{B}$ will be vertical.)

The flow is 'potential' since it has been generated from rest by the normal pressures applied to the liquid by the surface of the body and no other forces are acting. Hence

$$\underline{v} = -\operatorname{grad} \phi \qquad (3)$$

and substituting in equation (2) we get

$$\underline{B} = -\int \varrho \operatorname{grad} \phi \, d\tau \qquad (4)$$

which by Gauss's theorem becomes the surface integral

$$\underline{B} = \int_S \varrho \phi \underline{n} \, ds = \int_{S_B} \varrho \phi \underline{n} \, ds + \int_{S_W} \varrho \phi \underline{n} \, ds + \int_{S_\infty} \varrho \phi \underline{n} \, ds \qquad (5)$$

where $\underline{n}$ is the inwards drawn normal,

S_B is the wetted surface of the body,

S_W is the free surface of the liquid,

and S_∞ is the surface at ∞ (*see* Fig. 1).

On S_∞, ϕ is zero so that equation (5) becomes

$$\underline{B} = \int_{S_B} \varrho \phi \underline{n} \, ds + \int_{S_W} \varrho \phi \underline{n} \, ds \,. \qquad (6)$$

In the motion of a body in unbounded fluid, the second integral in equation (6) disappears so that

$$\underline{B} = \int_{S_B} \varrho \phi \underline{n} \, ds \,. \qquad (6a)$$

i.e., the momentum of the liquid can be obtained by an integration over the body surface alone.

Furthermore, the potential function ϕ can be expressed in the form

$$\phi = \underline{V} \cdot \underline{\Phi} \qquad \ldots \quad (7)$$

where $\underline{V}$ is the velocity of the body,

and $\underline{\Phi}$ is a geometrical function independent of velocity.

Equation (7) is the basis for the whole of the associated mass treatment of the motion of a body in unbounded fluid[2, 3], and leads to an expression of the form $\mu M.V$ for the component of liquid momentum in the direction of a 'principal axis' of the energy ellipsoid, where μM has the dimensions of mass and is determined by the geometry of the body alone, and V is the velocity component in that direction.

This is no longer the case in the presence of a free surface. Even if it were possible to express ϕ in the form of equation (7), with Φ dependent on body shape alone, the second integral of equation (6) would still involve knowledge of the free surface shape which depends on the history of the motion.

If associated mass methods are to be applied to the motion of a body through a free surface it is therefore necessary to assume as an approximation that

$$\phi = 0$$

on the free surface in which case equation (6a) applies.

Thus the momentary flow conditions (relative to the body) are assumed to be as shown in Fig. 2, where the free surface $\phi = 0$ in Fig. 2b is equivalent to the plane of symmetry $\phi = 0$ in Fig. 2a. Thus the flows in the lower half-planes of the two motions are subject to the same boundary conditions and hence the two flows are identical in that region. In the free surface problem only half of the body surface is wetted as compared with the whole surface wetted in the unbounded fluid problem and hence the associated mass in the former case is only half that obtained in the latter.

It can be shown that the motion in Fig. 2 is along a principal axis of the energy ellipsoid so that equation (1) can be applied for the momentum balance, with μM a function of the geometry of the body alone.

A further approximation must still be made. The condition $\phi = 0$ on the free surface implies that the free surface is flat, whereas it will actually be 'splashed-up'* in the region of the body, somewhat as in Fig. 1. For that reason Wagner[4] takes the condition $\phi = 0$ to apply along the line of the undisturbed free surface (as in Fig. 3) and calculates his results accordingly. The same approximation is taken in the present report.

APPENDIX II

Mathematical Details of the Expanding Prism Analogy to the Two-dimensional Impact of a Wedge of Finite Deadrise Angle

In this case, when the deadrise angle θ is finite, a better approximation to the relative flow than Wagner's expanding flat plate solution might be obtained by considering an expanding prism flow, derived as shown in Fig. 4a.

* 'Splash-up' is the rise of displaced water along the sides of the body.

Thus the unbounded relative flow problem is that of the flow of a stream of velocity V past a prism of deadrise angle θ and momentary width $2c$. This is taken as the z-plane, with origin and axes as shown in Fig. 4b, and the flow in the z-plane can be transformed into a flow past a flat plate in a ζ-plane (Fig. 4c) as follows.

Transformation Between the z- and ζ-Planes.—It is required to transform the prism in the z-plane into a flat plate in the ζ-plane, so that the points $[\pm\, c,\, 0]$ go to the points $[\pm\, c,\, 0]$ and the points $[0,\, \pm\, ic \tan \beta]$ go to the origin.

The Schwarz-Christoffel transformation gives

$$\frac{dz}{d\zeta} = K\zeta^{2\theta/\pi}\,(\zeta^2 - c^2)^{-\theta/\pi} = K\zeta^{2n}\,(\zeta^2 - c^2)^{-n} \quad \ldots \quad (1)$$

where $n = \theta/\pi$, $z = x + iy$, $\zeta = \xi + i\eta$.

If $|\zeta| < c$, then

$$\frac{dz}{d\zeta} = \frac{K}{e^{ni\pi}} \frac{\left(\frac{\zeta^2}{c^2}\right)^n}{\left(1 - \frac{\zeta^2}{c^2}\right)^n} \quad \ldots \quad (2)$$

Put $\zeta^2/c^2 = \tau$, then $d\zeta = c\,d\tau/2\tau^{1/2}$ and

$$dz = \frac{Kc}{2\,e^{ni\pi}} \frac{\tau^{n-1/2}}{(1-\tau)^n}\,d\tau \quad \ldots \quad (3)$$

When $\zeta = 0$, $z = \pm\, ic \tan \theta$

$$\text{Therefore, } z \pm ic \tan \theta = \frac{Kc}{2e^{ni\pi}} \int_0^\tau \tau^{-n1/2}\,(1 - \tau)^{-n}\,d\tau \quad \ldots \quad (4)$$

gives the transformation between the z- and ζ-planes *via* the τ-plane

$$\tau = \frac{\zeta^2}{c^2}.$$

Also, y positive corresponds to η positive and infinity in the z-plane to infinity in the ζ-plane. Now η=0, $|\xi| < c$ corresponds to the faces of the prism in the z-plane. Also η=0 implies τ real, and for this case the integral on the right hand side of equation (4) is solvable in terms of the incomplete Beta function giving (for positive y)

$$z - ic \tan \theta = \frac{Kc}{2e^{ni\pi}}\, B_\tau\,(p,\, q) \quad \ldots \quad (5)$$

where $p = n + \frac{1}{2}$

$q = 1 - n$.

However, evaluation of $B_\tau\,(p,\, q)$ is handicapped by the fact that $n = \theta/\pi \leqslant \frac{1}{2}$, therefore p and q are both in the range $[\frac{1}{2},\, 1]$, and tables of the incomplete Beta function are only available for values of p and q equal to 0·5, 1·0, 2·0, etc. It was not considered appropriate to investigate further numerical solutions in this report.

Evaluation of the Constant K in the Transformation.—Since $\zeta = c$ ($\tau = 1$) implies that $z = c$, then from equation (5)

$$c\,(1 - i \tan\theta) = \frac{Kc}{2e^{i\theta}}\, B\,(n + \tfrac{1}{2},\, 1 - n),$$

i.e.,

$$K = \frac{2}{B\,(n + \frac{1}{2},\, 1 - n)\,.\,\cos\theta} \qquad (6)$$

$$= \frac{\sqrt{\pi}}{\Gamma\,(n + \frac{1}{2})\,\Gamma\,(1 - n)\,.\,\cos\theta} \qquad (7)$$

where $n = \theta/\pi$

and Γ denotes the Gamma function.

Correspondence of Flow Fields in the z- and ζ-Planes.—Take

$$-\frac{dw}{dz} = ve^{-i\beta}$$

and

$$-\frac{dw}{d\zeta} = v_1 e^{-i\beta_1}$$

Then

$$ve^{-i\beta} = v_1 e^{-i\beta_1} \Big/ \frac{dz}{d\zeta} \qquad (8)$$

At ∞,

$$\operatorname*{Lim}_{\zeta \to \infty} \left(\frac{dz}{d\zeta}\right) = \operatorname*{Lim}_{\zeta \to \infty} \left[K\zeta^{2\theta/\pi}\,(\zeta^2 - c^2)^{-\theta/\pi}\right] \text{ (from (1))}$$

$$= \operatorname*{Lim}_{\zeta \to \infty} \left[K\left(1 - \frac{c^2}{\zeta^2}\right)^{-\theta/\pi}\right]$$

$$= K\,.$$

Hence, at infinity in the two planes

$$v_1 = Kv \qquad (9)$$

$$\beta_1 = \beta \qquad (10)$$

so that if there is a uniform stream V in the z-plane in the direction of the negative y-axis, there is a uniform stream KV in the ζ-plane in the direction of the negative η-axis.

Thus the complex relative flow potential is given by

$$w = -\,iKV\,\sqrt{(\zeta^2 - c^2)}\,. \qquad (11)$$

Superimposing a velocity V in the direction of the positive y-axis on the whole system in the z-plane, we obtain

$$w = iVz - iKV\,\sqrt{(\zeta^2 - c^2)} \qquad (12)$$

as the complex potential for the flow caused by a prism moving in liquid at rest at infinity, the relation between z and ζ being given by the integral of 1.

Momentum of the Liquid.—On the surface of the prism, $\zeta = \xi$ and $|\xi| < c$, hence

$$w = iV\,(x + iy) + KVc\sqrt{\left(1 - \frac{\xi^2}{c^2}\right)} \qquad \ldots \quad (12)$$

hence

$$\phi = KVc\sqrt{\left(1 - \frac{\xi^2}{c^2}\right)} - Vy \quad \text{on } S_B \, . \qquad \ldots \quad (14)$$

Now the momentum of the liquid is given by

$$\underline{B} = \varrho \int_{S_B} \phi \underline{n}\, ds$$

hence the vertical component, downwards, is

$$B_y = \varrho \int_{S_B} \phi\, dx$$

which from equation (14) becomes

$$B_y = \varrho \int_{-c}^{+c} KVc\sqrt{\left(1 - \frac{\xi^2}{c^2}\right)} \cdot \frac{dx}{d\xi}\, d\xi - \varrho V \int_{-c}^{+c} y\, dx$$

$$= 2\varrho \int_{-0}^{+1} KVc\sqrt{(1 - \tau)}\, \frac{dx}{d\tau}\, d\tau - \varrho V \int_{-c}^{+c} y\, dx \qquad \ldots \quad (15)$$

if τ is now taken equal to $\dfrac{\xi^2}{c^2}$ ·

From equation (3)

$$\frac{dx}{d\tau} = \frac{Kc\cos\theta}{2}\; \frac{\tau^{n-1/2}}{(1-\tau)^n}$$

hence

$$2\varrho KVc \int_0^1 \sqrt{(1-\tau)}\, . \frac{dx}{d\tau}\, d\tau = \varrho K^2Vc^2 \cos\theta \int_0^1 \tau^{n-1/2}\,(1-\tau)^{1/2-n}\, d\tau$$

$$= \varrho K^2Vc^2 \cos\theta \, .\, B\,(n + \tfrac{1}{2}, \tfrac{3}{2} - n)$$

$$= \varrho Vc^2 \left\{\frac{\pi - 2\theta}{\sin 2\theta} \cdot \frac{\pi}{[\Gamma\,(n + \frac{1}{2})\,\Gamma\,(1 - n)]^2}\right\} \tan\theta$$

also

$$-\varrho V \int_{-c}^{+c} y\, dx \doteq -\varrho Vc^2 \tan\theta$$

therefore

$$B_y = \varrho Vc^2 \tan\theta \left\{\frac{\pi - 2\theta}{\sin 2\theta} \cdot \frac{\pi}{[\Gamma\,(n + \frac{1}{2})\,\Gamma\,(1 - n)]^2} - 1\right\} \qquad \ldots \quad (16)$$

$$\simeq \frac{\pi}{2}\, \varrho Vc^2 \left(1 - \frac{\theta}{\pi}\right)$$

and

$$F_y = \frac{DB_y}{Dt} = B_y\left\{\frac{1}{V}\,\frac{dV}{dt} + \frac{2}{c}\,\frac{dc}{dt}\right\}. \qquad \ldots \quad (17)$$

Splash-up.—Considering the flow relative to the prism, then

$$w = -iKV\sqrt{(\zeta^2 - c^2)} \qquad (11)$$

and

$$\frac{dz}{d\zeta} = K\zeta^{2n}(\zeta^2 - c^2)^{-n} \qquad (1)$$

hence

$$\frac{dw}{dz} = -iV\zeta^{1-2n}(\zeta^2 - c^2)^{n-1/2}. \qquad (18)$$

On the undisturbed free surface

$$\zeta = \xi \text{ and } |\xi| > c$$

hence the resultant velocity is vertical and of magnitude

$$v_n = V\left(1 - \frac{c^2}{\xi^2}\right)^{n-1/2}. \qquad (19)$$

Then the elevation of the water above the keel at position ξ at time t is given by

$$\eta = \int_0^t v_n\,dt = \int_{c=0}^{c\leqslant x} \frac{u(c)\,dc}{\left(1 - \frac{c^2}{\xi^2}\right)^{1/2-n}} \qquad (20)$$

where $u(c) = V\Big/\frac{dc}{dt}$. (21)

At the surface of the body $\xi = x = c$, and therefore

$$\eta_b = \int_0^{\xi=x} \frac{u(c)\,dc}{\left(1 - \frac{c^2}{\xi^2}\right)^{1/2-n}} = x\tan\theta \qquad (22)$$

from the geometry of the body. Solving this integral equation by the same method as used by Wagner for his flat plate motion, *i.e.*, putting

$$u(c) = a_0 + a_1 c + a_2 c^2 + \ldots + a_r c^r + \ldots \qquad (23)$$

we find that for a straight-sided wedge,

$$\eta_b = x\tan\theta = \frac{a_0}{2}xB\left(\tfrac{1}{2}, n + \tfrac{1}{2}\right) + b_1 x^2 + b_2 x^3 + \ldots \quad \text{(from (22))}$$

from which

$$a_0 = \frac{2}{\sqrt{\pi}}\,\frac{\theta}{\sin\theta}\,\frac{\tan\theta}{\Gamma\left(\frac{1}{2} + \frac{\beta}{\pi}\right)\Gamma\left(1 - \frac{\beta}{\pi}\right)} = \frac{2K\theta}{\pi} \qquad (24)$$

and

$$b_1 = b_2 = \ldots = 0$$

$\left\{\text{When } \theta = 0 \text{ this reduces to the Wagner value of } \frac{2\theta}{\pi}\right\}$.

Now

$$u(c) = a_0 = \frac{V}{dc/dt} = \frac{dh}{dc}$$

hence

$$\frac{h}{c} = a_0 \quad \ldots \quad (25)$$

Also $h/c_0 = \tan \theta$ where c_0 is the wetted width without splash-up.

Therefore, the splash-up factor

$$\frac{c}{c_0} = \frac{\sqrt{\pi}}{2} \frac{\sin \theta}{\theta} \left\{ \Gamma\left(\tfrac{1}{2} + \frac{\theta}{\pi}\right) \Gamma\left(1 - \frac{\theta}{\pi}\right) \right\} \quad \ldots \quad (26)$$

$$\doteq \frac{\pi}{2}\left(1 - \frac{\theta}{\pi}\right) \qquad \text{for } 0 \leqslant \theta \leqslant \frac{\pi}{4} \quad \ldots \quad (27)$$

The maximum error involved in using the approximate factor given by equation (27) is less than 2 per cent in the range specified.

Also, as $\theta \to \dfrac{\pi}{2}$, $\dfrac{c}{c_0} \to 1$ (from equation (26)).

Further Work.—Pressure distributions over the wedge could be obtained by using the flow potential given by equation (12) instead of Wagner's flat plate potential, but the work is complicated by lack of solutions of equation (4) for the transformation between the z- and ξ-planes.

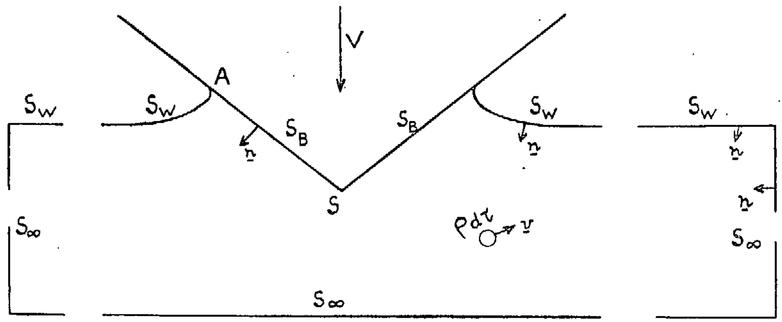

FIG. 1. Cross-section of the vertical impact of a wedge at zero attitude. (Ref. Appendix I.)

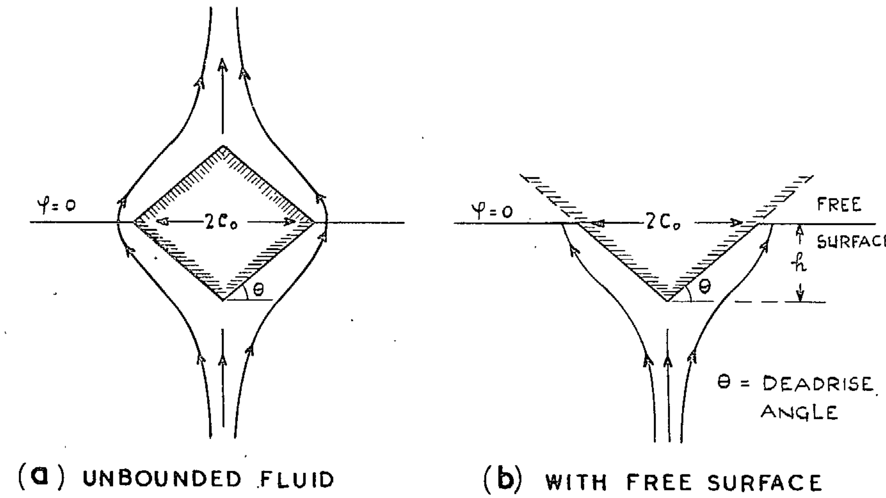

FIG. 2. Application of associated mass methods. Assumed flows relative to body. (Ref. Appendix I.)

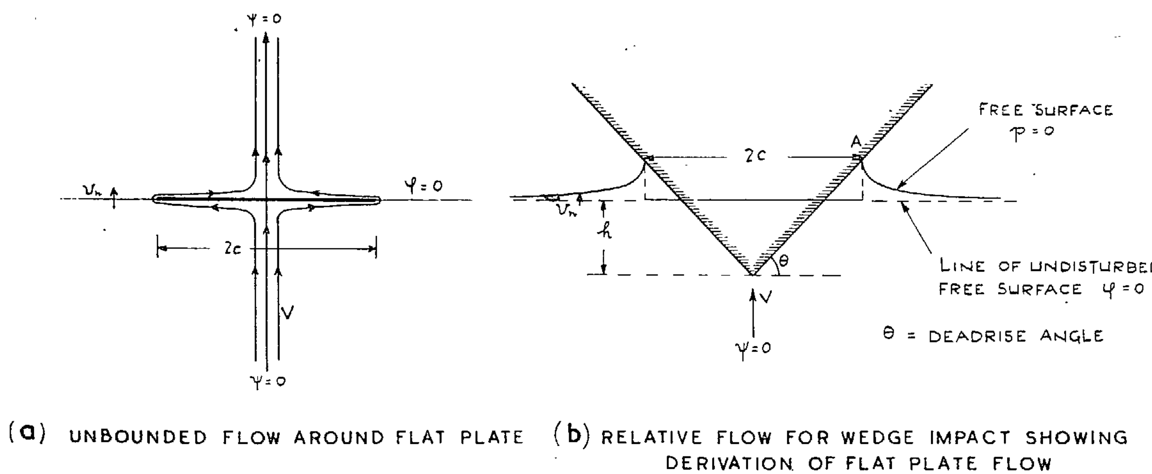

FIG. 3. Associated mass treatment of two-dimensional wedge impact. Wagner's approximation for the relative flow when θ is small.

(22752) Wt. 15-680 K9 3/52 F.M. & S.

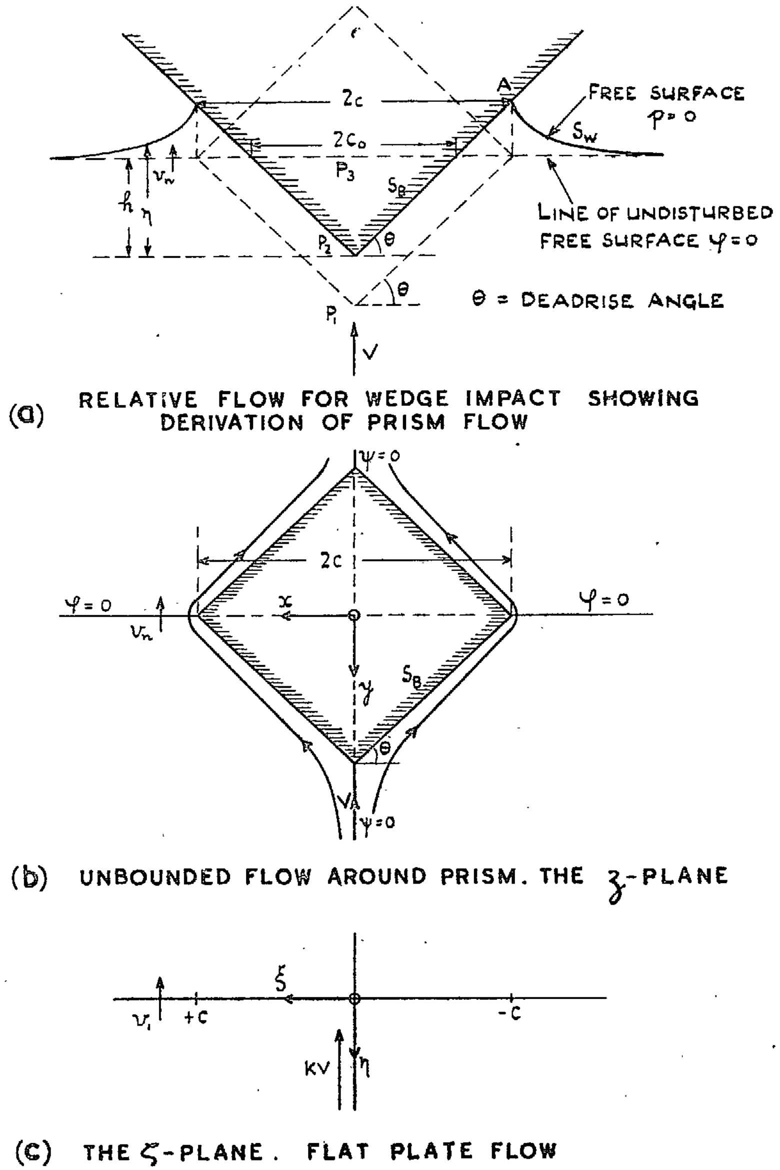

FIG. 4. Suggested associated mass treatment of wedge impact when θ is finite. (Compare with Fig. 3.)

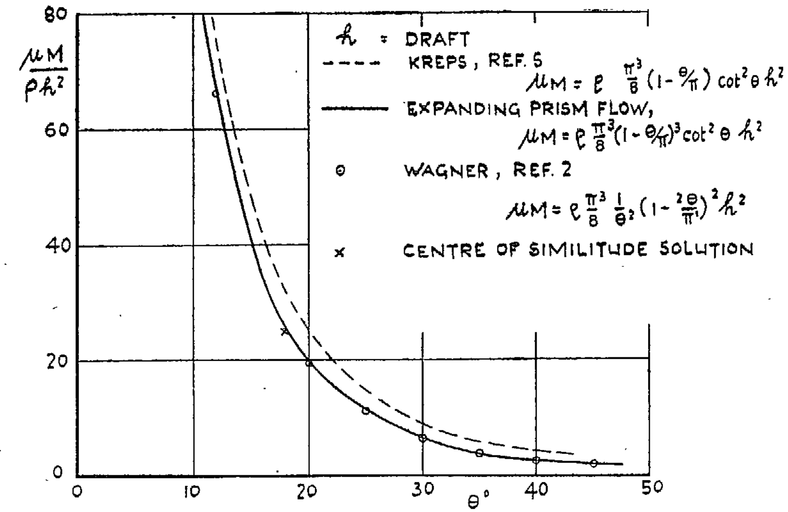

FIG. 5. Variation of the two-dimensional associated mass with deadrise angle.

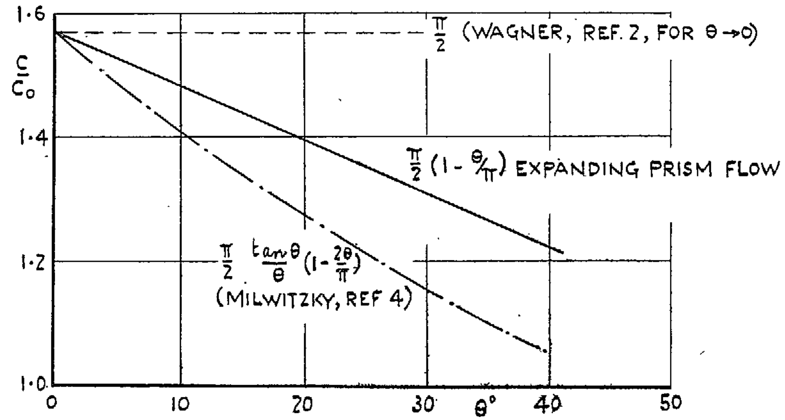

FIG. 6. Comparison of various estimates for the splash-up factor.

PRINTED IN GREAT BRITAIN

R. & M. No. 2681
(12,395)
A.R.C. Technical Report

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)—

1934–35 Vol. I. Aerodynamics. *Out of print.*
Vol. II. Seaplanes, Structures, Engines, Materials, etc. 40*s.* (40*s.* 8*d.*)

1935–36 Vol. I. Aerodynamics. 30*s.* (30*s.* 7*d.*)
Vol. II. Structures, Flutter, Engines, Seaplanes, etc. 30*s.* (30*s.* 7*d.*)

1936 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40*s.* (40*s.* 9*d.*)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 50*s.* (50*s.* 10*d.*)

1937 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40*s.* (40*s.* 10*d.*)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 60*s.* (61*s.*)

1938 Vol. I. Aerodynamics General, Performance, Airscrews. 50*s.* (51*s.*)
Vol. II. Stability and Control, Flutter, Structures, Seaplanes, Wind Tunnels, Materials. 30*s.* (30*s.* 9*d.*)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50*s.* (50*s.* 11*d.*)
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63*s.* (64*s.* 2*d.*)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50*s.* (51*s.*)

Certain other reports proper to the 1940 *volume will subsequently be included in a separate volume.*

ANNUAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL—

1933–34	1*s.* 6*d.* (1*s.* 8*d.*)
1934–35	1*s.* 6*d.* (1*s.* 8*d.*)
April 1, 1935 to December 31, 1936.	4*s.* (4*s.* 4*d.*)
1937	2*s.* (2*s.* 2*d.*)
1938	1*s.* 6*d.* (1*s.* 8*d.*)
1939–48	3*s.* (3*s.* 2*d.*)

INDEX TO ALL REPORTS AND MEMORANDA PUBLISHED IN THE ANNUAL TECHNICAL REPORTS, AND SEPARATELY—

April, 1950 R. & M. No. 2600. 2*s.* 6*d.* (2*s.* 7½*d.*)

INDEXES TO THE TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL—

December 1, 1936 — June 30, 1939.	R. & M. No. 1850.	1*s.* 3*d.* (1*s.* 4½d.)
July 1, 1939 — June 30, 1945.	R. & M. No. 1950.	1*s.* (1*s.* 1½*d.*)
July 1, 1945 — June 30, 1946.	R. & M. No. 2050.	1*s.* (1*s.* 1½*d.*)
July 1, 1946 — December 31, 1946.	R. & M. No. 2150.	1*s.* 3*d.* (1*s.* 4½*d.*)
January 1, 1947 — June 30, 1947.	R. & M. No. 2250.	1*s.* 3*d.* (1*s.* 4½*d.*)

Prices in brackets include postage.

Obtainable from
HER MAJESTY'S STATIONERY OFFICE
York House, Kingsway, LONDON, W.C.2 423 Oxford Street, LONDON, W.1
P.O. Box 569, LONDON, S.E.1
13a Castle Street, EDINBURGH, 2 1 St. Andrew's Crescent, CARDIFF
39 King Street, MANCHESTER, 2 Tower Lane, BRISTOL, 1
2 Edmund Street, BIRMINGHAM, 3 80 Chichester Street, BELFAST
or through any bookseller.

S.O. Code No. 23-2681