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By

A. R. CURTIS, B.A.,
of the Aerodynamics Division, N.P.L.

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Summary.—Some minor developments in the technique of Thwaites' Numerical Method of Aerofoil Design[1] are described. In particular, the process of obtaining the camber-line ordinates from the Goldstein[2] Approximation I velocity distribution is discussed in detail; the relevant tables of constants are given.

An opportunity is taken to include a complete set of 20-point tables of Conjugation Factors needed in any actual application of the Numerical Methods. The theory underlying these tables is given by Watson in R. & M. 2716[3].

1. *Reasons for Enquiry.*—One of the most efficient methods of ensuring a large low-drag C_L range for given thickness, is to assume flat velocity distributions over the front portion of the upper surface at the top of this range, and over the front portions of the lower surface at the bottom of this range.

When Thwaites' Numerical Method[1] is used to design an aerofoil on this basis, it is necessary to have a numerical procedure for obtaining the camber-line ordinates from the function $g_i(\theta)$, and it is a convenient method of performing this operation that constitutes the main substance of this note.

2. *Notation.*—The notation used is that of Ref. 1 and 2. All symbols used have the same significance as in these reports. Only *additional* symbols are defined in the text.

Where $\pm$ signs occur, the upper sign refers to the upper surface of the aerofoil and the lower sign to the lower surface.

3. *The Method of Obtaining the Camber-line Ordinates.*—The relation between $y_c(\theta)$ and $g_i(\theta)$ is usually stated as

$$2y_c(\theta) \equiv - A_0(\cos\theta - 1) + \int_0^\theta \sin\theta \operatorname{conj}[g_i(\theta)]\, d\theta \quad \ldots \quad (1)$$

where $$A_0 = -\tfrac{1}{2}\int_0^\pi \sin\theta \operatorname{conj}[g_i(\theta)]\, d\theta. \quad \ldots \quad (2)$$

This corresponds to the relations for the symmetric case, *viz.*:—

$$2y_s(\theta) \equiv \int_0^\theta \operatorname{conj}[g_s(\theta)\sin\theta]\, d\theta \quad \ldots \quad (3)$$

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which Watson[3] has shown, on the basis of work by Germain[4], to be expressible in the form

$$(y_s)_p = \sum_{r=0}^{N} K_{pr} \,[g_s \sin \theta]_r \qquad \ldots \quad (4)$$

where $(y_s)_p$ denotes the value of $y_s(\theta)$ at $\theta = \frac{p\pi}{N}$ with a similar notation for $g_s \sin \theta$, and the K_{pr} are readily calculable coefficients consisting merely of the sum or difference of $N + 1$ fundamental coefficients ζ

where
$$\begin{cases} \zeta_r = -\dfrac{1}{N}\sum\limits_{s=1}^{N-1} \cos \dfrac{rs\pi}{N} + \dfrac{(-1)^{r+1}}{2N^2} \quad r = 0 \text{ or } N \\ \zeta_0 = \zeta_N = 0. \end{cases} \qquad \ldots \quad (5)$$

It has been the practice in the past to draw a distinction between the forms of the integrands occurring in equations (1) and (3) above; *i.e.*, the processes of conjugation and multiplication by $\sin \theta$ have not been regarded as commutative. But if the two processes are carried out in either order on a finite Fourier Series, the results will be found the same.

Hence values of $y_c(\theta)$, except for a constant term and a term in $\cos \theta$, may be obtained by a similar process except that the odd function $g_s \sin \theta$ is replaced by the even function $g_i \sin \theta$, and we have the result

$$\left(y_c + \frac{A_0}{2}\cos \theta + K\right)_p = \sum_{r=0}^{N} K_{pr}' \,[g_i \sin \theta]_r \qquad \ldots \quad (6)$$

The coefficients K_{pr}' are different from the K_{pr} occurring in the symmetrical case, but are also expressible in terms of the same $N + 1$ fundamental constants ζ_r. $A_0/2$ and the arbitrary constant K in equations (6) are obtained from the conditions that $y_c(0)$ and $y_c(\pi)$ are both zero.

In order to obtain $C_{L\,\text{opt}}$, α_{opt}, C_{M0} and β it is necessary to determine the coefficients A_1 and A_2 in the Fourier Series for g_i, namely

$$A_1 = \frac{2}{\pi}\int_0^{\pi} g_i \sin \theta \, d\theta$$

$$A_2 = \frac{2}{\pi}\int_0^{\pi} g_i \sin 2\theta \, d\theta.$$

With the assumed finite Fourier Series for g_i they are most easily obtained as

$$A_1 = \frac{2}{N}\sum_{r=1}^{N-1} (g_i \sin \theta)_r$$

$$A_2 = \frac{4}{N}\sum_{r=1}^{N-1} (g_i \sin \theta)_r \cos \theta_r$$

on the assumption that $g_i \sin \theta$ vanishes at $\theta = 0$ and $\theta = \pi$. If this is not the case, one half of the appropriate terms is added.

4. *The Numerical Procedure (General)*.—The equation used for obtaining the functions $g_s(\theta) \sin \theta$ and $g_i(\theta) \sin \theta$ from which the aerofoil ordinates are computed is

$$\sin \theta \,\{1 + g_s(\theta) + g_i(\theta)\}$$

$$= \pm\, q/U \,\frac{\sqrt{(\psi^2 + \sin^2 \theta)}}{1 + \frac{1}{2}C_0^{\,2}} - C_L\left(\frac{1}{2\pi} + \frac{\cos \theta}{a_0}\right) + \frac{C_{L\,\text{opt}}}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)(1 + \cos \theta) \qquad (7)$$

If q/U is postulated on the upper surface at $C_L = (C_L)_u$ and on the lower surface at $C_L = (C_L)_l$, then

$$2g_i(\theta)\sin\theta = \left[\frac{q/U\,\sqrt{(\psi^2+\sin^2\theta)}}{1+\frac{1}{2}C_0^2}\right]_u - \left[\frac{q/U\,\sqrt{(\psi^2+\sin^2\theta)}}{1+\frac{1}{2}C_0^2}\right]_l$$

$$-\left[(C_L)_u + (C_L)_l\right]\left(\frac{1}{2\pi}+\frac{\cos\theta}{a_0}\right) + C_{L\,\mathrm{opt}}\,(1+\cos\theta)\left(\frac{1}{a_0}+\frac{1}{2\pi}\right) \qquad (8)$$

and

$$2g_s(\theta)\sin\theta + 2\sin\theta = \left[\frac{q/U\,\sqrt{(\psi^2+\sin^2\theta)}}{1+\frac{1}{2}C_0^2}\right]_u + \left[\frac{q/U\,\sqrt{(\psi^2+\sin^2\theta)}}{1+\frac{1}{2}C_0^2}\right]_l$$

$$-\left[(C_L)_u - (C_L)_l\right]\left[\frac{1}{2\pi}+\frac{\cos\theta}{a_0}\right]. \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

In equations (8) and (9) the value of $C_{L\,\mathrm{opt}}$ should be consistent with the values of $g_i(\theta)\sin\theta$, *i.e.*, the relation $C_{L\,\mathrm{opt}} = 2\int_0^\pi g_i(\theta)\sin\theta\,d\theta$ must at any rate be approximately satisfied.

The best way of ensuring this is to use the relation

$$\left(\frac{\pi}{a_0}+\frac{1}{2}\right)C_{L\,\mathrm{opt}} = \frac{2\pi}{N}\sum_{r=1}^{N-1}(g_i\sin\theta)_r. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Since, in practice, it is usually only necessary to postulate q/U (and hence $g_i(\theta)$) over the front portion of the wing, suitable values of $g_i \sin\theta$ may be inserted over the rear (paying due regard to ' smoothness ' at the join) to ensure that relation (10) is in fact satisfied.

Over the rear portion of the section $g_s(\theta)$ and $g_i(\theta)$ are best made linear in x $(= \frac{1}{2}(1-\cos\theta))$, the former assuming a suitable negative value at the trailing edge to produce a very small trailing edge radius of curvature, and the latter assuming the value zero at the trailing edge, or at any rate a non-negative value.

To ascertain what value to give $g_s(\theta)$ at $\theta = \pi$, use is made of the relation

$$(2\rho_T)^{1/2} = \frac{1}{\pi}\int_0^\pi g_s(\theta)\,(1-\cos\theta)\,d\theta$$

or to the accuracy of the numerical method $= \frac{1}{N}\sum_{r=1}^{N-1}(g_s\sin\theta)_r\left(\tan\frac{\theta r}{2}\right)$.

Values of ψ, a_0 and C_0 have to be estimated; this is best effected by comparison with previously designed sections of a similar nature.

Suppose an n per cent thick aerofoil is required with $C_{L\,\mathrm{opt}}$ in the middle of the C_L range, and that the velocity distribution q/U is to be constant over the front part of the upper surface at $C_L = (C_L)_u$ and over the front part of the lower surface at $C_L = (C_L)_l$; the values of q/U necessary to satisfy these requirements are given approximately by the equation

$$q/U \simeq 1 + \frac{n}{100} + \frac{(C_L)_{u\,\mathrm{or}\,l}}{6} \pm \frac{C_{L\,\mathrm{opt}}}{3}.$$

This is a *very rough rule* to which the following limitations and extensions apply.

(i) If $C_{L\,\mathrm{opt}}$ be less than the middle of the C_L range q/U should be reduced somewhat on the upper surface, and increased on the lower.

(ii) It is usually not advisable to aim at absolutely the highest attainable C_L range for given thickness as this will result in a small leading edge radius of curvature with resultant loss of $C_{L\,\text{opt}}$.

(iii) A much more reliable procedure is to use data from a previously designed section.

5. *An Example.*—Previous experience indicated that it should be possible to design a $12\frac{1}{2}$ per cent thick cambered section with a C_L range of about 0 to 0·3. Any further insistence on C_L range performance would undoubtedly be at the expense of leading-edge radius of curvature, with consequent loss in $C_{L\,\text{opt}}$. The section and relevant final Approximation III velocity distributions are given in Fig. 1.

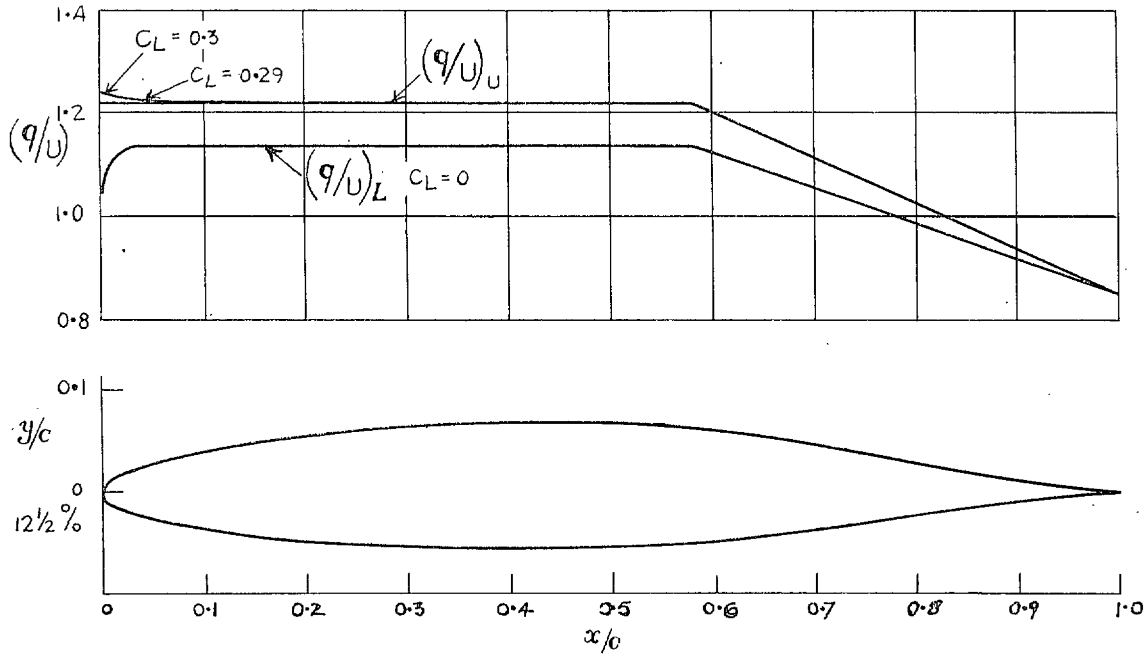

FIG. 1.

Ordinates of the section and a summary of the calculations are given in Table A. It is to be noted that this aerofoil was only designed to test the efficacy of the Method of the previous paragraphs.

Acknowledgements.—Acknowledgements are due to Miss C. Tracey who computed Table 2, and worked out the example.

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2	S. Goldstein (Part III with E. J. Richards)	Approximate Two-dimensional Aerofoil Theory. Parts I to IV. A.R.C. Reports 5804 (May, 1942); 6156 (September, 1942); 6225 (October, 1942); 8548 (March, 1945). (To be published.)
3	E. J. Watson	Formulae for the Computation of the Functions Employed for Calculating the Velocity Distributions about a Given Aerofoil. R. & M. 2176. May, 1945.

TABLE A

$\frac{2 0\theta}{\pi}$	Guessed Values of ψ		Assumed Values per q/U		From Eqn. (9)	From Eqn. (8)	Via Table 1	Via Table 2 $A_0 = 0\cdot00126$ $K = -0.00391$	Final Shape. y_s has been scaled up to $12\frac{1}{2}\%$ y_c is left unaltered	
	ψ_u	ψ_l	$C_L = 0\cdot3$ $(q/U)_u$	$C_L = 0$ $(q/U)_l$	$g_s \sin\theta$	$g_i \sin\theta$	y_s	y_c	y_u	y_l
0	0·13	0·13	1·20	0·97	0	0	0	0	0	0
1	0·1315	0·1285	1·20	1·05	0·02631	0·00100	0·00889	0·00021	0·01052	0·01010
2	0·1330	0·1270	1·20	1·10	0·03022	0·00275	0·01674	0·00076	0·02017	0·01865
3	0·1345	0·1255	1·20	1·12	0·04700	0·00585	0·02443	0·00167	0·03000	0·02666
4	0·1360	0·1240	1·20	1·12	0·06429	0·01157	0·03193	0·00289	0·03992	0·03414
5	0·1375	0·1225	1·20	1·12	0·08176	0·01690	0·03881	0·00420	0·04921	0·04081
6	0·1390	0·1210	1·20	1·12	0·09798	0·02169	0·04474	0·00547	0·05736	0·04642
7	0·1405	0·1195	1·20	1·12	0·11211	0·02583	0·04942	0·00659	0·06391	0·05073
8	0·1420	0·1180	1·20	1·12	0·12358	0·02920	0·05257	0·00749	0·06846	0·05348
9	0·1435	0·1165	1·20	1·12	0·13198	0·03172	0·05389	0·00806	0·07056	0·05444
10	0·1450	0·1150	1·20	1·12	0·13705	0·03336	0·05310	0·00828	0·06986	0·05330
11	0·1450	0·1035	1·20	1·12	0·13790	0·03456	0·04960	0·00802	0·06554	0·04950
12					0·09852	0·02633	0·04214	0·00700	0·05587	0·04187
13					0·04957	0·02077	0·03260	0·00579	0·04360	0·03202
14	Both g_s and g_i are made linear in x over this range. $g_s(\pi) = -0\cdot125$ $g_i(\pi) = 0$				+ 0·00921	0·01537	0·02306	0·00450	0·03124	0·02224
15					− 0·01980	0·01054	0·01470	0·00325	0·02030	0·01380
16					− 0·03634	0·00654	0·00818	0·00213	0·01162	0·00736
17					− 0·04038	0·00355	0·00374	0·00121	0·00555	0·00313
18					− 0·03362	0·00156	0·00124	0·00054	0·00198	0·00090
19					− 0·01892	0·00048	0·00018	0·00014	0·00040	0·00012
20					0	0	0	0	0	0

TABLE A—*continued*

$\frac{20\theta}{\pi}$	Via Table 3	Via Table 4	Via Table 5	Via Table 6			Final Approximation III Velocities		
	ε_s	ε_c	ε_s'	ε_c'	F_u	F_l	$C_L = 0$ $(q/U)_l$	$C_L = 0 \cdot 3$ $(q/U)_u$	$C_L = 0 \cdot 29$ $(q/U)_u$
0	0	− 0·01638	+ 0·04661	0	8·59458	8·59458	0·25867	0·49075	0·46577
1	0·00787	− 0·01595	+ 0·04918	0·00424	5·59240	5·64200	1·09113	1·23883	1·22270
2	0·01153	− 0·01535	− 0·00540	0·00341	3·25832	3·27217	1·13666	1·22934	1·22013
3	0·01051	− 0·01467	+ 0·00645	0·00633	2·34591	2·33585	1·14061	1·22069	1·21428
4	0·01304	− 0·01328	0·01764	0·01079	1·86714	1·84050	1·14057	1·22314	1·21828
5	0·01618	− 0·01140	0·02715	0·01277	1·58041	1·55065	1·14082	1·22447	1·22061
6	0·02128	− 0·00929	0·03382	0·01416	1·39744	1·36673	1·14020	1·22487	
7	0·02696	− 0·00698	0·04132	0·01500	1·28194	1·25140	1·14081	1·22569	
8	0·03415	− 0·00455	0·04856	0·01625	1·21243	1·18060	1·14040	1·22618	
9	0·04238	− 0·00191	0·05735	0·01691	1·17897	1·14662	1·14100	1·22594	
10	0·05215	+ 0·00079	0·06785	0·01839	1·17825	1·14293	1·14051	1·22558	
11	0·06408	0·00396	0·08119	0·02072	1·21102	1·16994	1·13888	1·22465	
12	0·07520	0·00676	0·05335	0·01439	1·22034	1·19139	1·09904	1·16447	
13	0·08043	0·00874	+ 0·01480	0·01168	1·25423	1·22907	1·04352	1·09667	
14	0·08001	0·01039	− 0·01976	0·00886	1·33297	1·31203	0·99357	1·03470	
15	0·07458	0·01157	− 0·04873	0·00664	1·47885	1·46093	0·95080	0·98208	
16	0·06484	0·01248	− 0·07360	0·00461	1·73088	1·71606	0·91478	0·93718	
17	0·05188	0·01308	− 0·09227	0·00286	2·19362	2·18198	0·88715	0·90195	
18	0·03592	0·01338	− 0·10682	0·00141	3·16797	3·16011	0·86691	0·87566	
19	0·01916	0·01357	− 0·11021	0·00136	6·23593	6·29191	0·85417	0·86004	
20	0	0·013727	− 0·13033	0	—	—	—	—	

APPENDIX I

The purpose of all the following six 20-point tables is to obtain the values of one function of θ at the points corresponding to $\theta = r\pi/20$, $r = 0, 1, \ldots\ldots 20$ from the values of a given function of θ at these same stations. Denote by $F(\theta_r)$ the value of the given function when $\theta = \theta_r$, with a corresponding notation for $G(\theta)$, the function to be obtained. Then if $r \leqslant 10$, $G(\theta_r)$ is obtained by placing a column of all the values of $F(\theta)$ against the column headed r, forming the products of adjacent numbers in the two columns, and adding. If $r \geqslant 10$, the procedure is the same, except that the column of values of $F(\theta)$ is first written out in reverse order, and is placed against the column in the table with the appropriate r at the bottom. In some cases $-r$ is written at the bottom; this means that when the cross multiplication described above has been performed, the final result must have its sign changed.

Table 1 gives values of y_s	from values of	$g_s \sin\theta$
Table 2 gives values of $y_c + \frac{1}{2}A_0 \cos\theta + K$		$g_i \sin\theta$
Table 3 gives values of ε_s		ψ_s
Table 4 gives values of ε_c		ψ_c
Table 5 gives values of ε_s'		ψ_s
Table 6 gives values of ε_c'		ψ_c

Corresponding tables for 40-point calculations have also been constructed, but are not included as their use really entails specialised equipment other than calculating machines.

TABLE 1

Factors for obtaining y_s from $g_s \sin \theta$

$20\frac{\theta}{\pi}$	$y(1)$	$y(2)$	$y(3)$	$y(4)$	$y(5)$	$y(6)$	$y(7)$	$y(8)$	$y(9)$	$y(10)$
0										
1	0·060838	0·028836	0·016608	0·012523	0·009533	0·007833	0·006428	0·005457	0·004606	0·003953
2	0·028836	0·077446	0·041359	0·026141	0·020356	0·015961	0·013290	0·011034	0·009410	0·007965
3	0·016608	0·041359	0·086979	0·049192	0·032569	0·025813	0·020567	0·017243	0·014393	0·012278
4	0·012523	0·026141	0·049192	0·093407	0·054649	0·037175	0·029766	0·023926	0·020111	0·016803
5	0·009533	0·020356	0·032569	0·054649	0·098013	0·058602	0·040534	0·032634	0·026335	0·022120
6	0·007833	0·015961	0·025813	0·037175	0·058602	0·101372	0·061470	0·042943	0·034643	0·027962
7	0·006428	0·013290	0·020567	0·029766	0·040534	0·016470	0·103781	0·063479	0·044571	0·035924
8	0·005457	0·011034	0·017243	0·023926	0·032634	0·042943	0 063479	0·105409	0·064760	0·045515
9	0·004606	0·009410	0·014393	0·020111	0·026335	0·034643	0·044571	0·064760	0·106353	0·065384
10	0·003953	0·007965	0·012278	0·016802	0·022120	0·027963	0·035924	0·045515	0·065384	0·106662
11	0·003359	0·006821	0·010374	0·014287	0·018430	0·023401	0·028907	0·036548	0·045824	0·065384
12	0·002868	0·005768	0·008830	0·012002	0·015568	0·019374	0·024025	0·029216	0·036548	0·045515
13	0·002409	0·004877	0·007396	0·010111	0·012946	0·016192	0·019683	0·024025	0·028907	0·035924
14	0·002009	0·004037	0·006158	0·008340	0·010735	0·013255	0·016192	0·019374	0·023401	0·027963
15	0·001628	0·003290	0·004981	0·006782	0·008649	0·010735	0·012946	0·015568	0·018430	0·022120
16	0·001281	0·002572	0·003914	0·005290	0·006782	0·008340	0·010111	0·012002	0·014287	0·016802
17	0·000944	0·001905	0·002881	0·003914	0 004981	0·006158	0·007396	0·008830	0·010374	0·012278
18	0·000624	0·001253	0·001905	0·002572	0·003290	0·004037	0·004877	0·005768	0·006821	0·007965
19	0·000309	0·000624	0·000944	0·001281	0·001628	0·002009	0·002409	0·002868	0·003359	0·003953
20	0	0								
	$y(19)$	$y(18)$	$y(17)$	$y(16)$	$y(15)$	$y(14)$	$y(13)$	$y(12)$	$y(11)$	$y(10)$

TABLE 2

Factors for obtaining $y_c + \frac{1}{2}A_0 \cos\theta + K$ from $g_i \sin\theta$

	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	0·089318	0·048148	0·028480	0·019312	0·011872	0·006789	0·002339	−0·001044	−0·004089	−0·006501	−0·008695
1	0·096295	0·117798	0·067460	0·040352	0·026102	0·014211	0·005746	−0·001750	−0·007545	−0·012784	−0·016955
2	0·056960	0·067460	0·101191	0·054937	0·030819	0·018268	0·007783	+0·000288	−0·006357	−0·011498	−0·016143
3	0·038624	0·040352	0·054937	0·091657	0·047104	0·024391	0·012811	0·003177	−0·003665	−0·009715	−0·014366
4	0·023744	0·026102	0·030819	0·047104	0·085230	0·041647	0·019785	0·008858	−0·000182	−0·006532	−0·012125
5	0·013579	0·014211	0·018268	0·024391	0·041647	0·080623	0·037693	0·016426	+0·005990	−0·002591	−0·008542
6	+0·004677	+0·005746	0·007783	0·012811	0·019785	0·037693	0·077264	0·034826	0·014017	+0·003981	−0·004219
7	−0·002087	−0·001750	+0·000288	+0·003177	+0·008858	0·016426	0·034826	0·074855	0·032817	0·012389	+0·002701
8	−0·008177	−0·007545	−0·006357	−0·003665	−0·000182	+0·005990	0·014017	0·032817	0·073227	0·031536	0·011445
9	−0·013002	−0·012784	−0·011498	−0·009715	−0·006532	−0·002591	+0·003981	0·012389	0·031536	0·072284	0·030912
10	−0·017391	−0·016955	−0·016143	−0·014366	−0·012125	−0·008452	−0·004219	+0·002701	0·011445	0·030912	0·071974
11	−0·020908	−0·020750	−0·019823	−0·018552	−0·016375	−0·013753	−0·009822	−0·005163	+0·002076	0·011136	0·030912
12	−0·024108	−0·023776	−0·023159	−0·021832	−0·020180	−0·017655	−0·014696	−0·010446	−0·005472	+0·002076	0·011445
13	−0·026644	−0·026517	−0·025785	−0·024787	−0·023113	−0·021124	−0·018280	−0·015006	−0·010446	−0·005163	+0·002701
14	−0·028926	−0·028653	−0·028145	−0·027066	−0·025730	−0·023737	−0·021433	−0·018280	−0·014696	−0·009822	−0·004219
15	−0·030662	−0·030555	−0·029933	−0·029089	−0·027690	−0·026040	−0·023737	−0·021124	−0·017655	−0·013753	−0·008542
16	−0·032183	−0·031943	−0·031498	−0·030558	−0·029398	−0·027690	−0·025730	−0·023113	−0·020180	−0·016375	−0·012125
17	−0·033223	−0·033126	−0·032567	−0·031808	−0·030558	−0·029089	−0·027066	−0·024787	−0·021832	−0·018552	−0·014366
18	−0·034070	−0·033847	−0·033436	−0·032567	−0·031498	−0·029933	−0·028145	−0·025785	−0·023159	−0·019823	−0·016140
19	−0·034472	−0·034379	−0·033847	−0·033126	−0·031943	−0·030555	−0·028653	−0·026517	−0·023776	−0·020750	−0·016955
20	−0·017344	−0·017236	−0·017035	−0·016612	−0·016091	−0·015331	−0·014463	−0·013322	−0·012054	−0·010452	−0·008695
	(20)	(19)	(18)	(17)	(16)	(15)	(14)	(13)	(12)	(11)	(10)

TABLE 3

Factors for obtaining $\varepsilon(\theta)$ from $\psi(\theta)$ when $\psi(\theta)$ is even

$20\frac{\theta}{\pi}$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	+0·635310	0	0·208265	0	0·120711	0	0·081593	0	0·058543	0
1	0	+0·843575	0	0·328976	0	0·202304	0	0·140135	0	0·101246
2	—0·427045	0	+0·756021	0	0·289858	0	0·179253	0	0·124297	0
3	0	—0·514599	0	+0·716903	0	0·266807	0	0·163415	0	0·112233
4	—0·087554	0	—0·553717	0	+0·693852	0	0·250969	0	0·151351	0
5	0	—0·126672	0	—0·576768	0	+0·678014	0	0·238905	0	0·141422
6	—0·039118	0	—0·149723	0	—0·592606	0	+0·665950	0	0·228976	0
7	0	—0·062169	0	—0·165561	0	—0·604670	0	+0·656021	0	0·220269
8	—0·023051	0	—0·078007	0	—0·177625	0	—0·614599	0	+0·647314	0
9	0	—0·038889	0	—0·090071	0	—0·187554	0	—0·623306	0	+0·639245
10	—0·015838	0	—0·050953	0	—0·100000	0	—0·196261	0	—0·631375	0
11	0	—0·027902	0	—0·060882	0	—0·108707	0	—0·204330	0	—0·639245
12	—0·012064	0	—0·037831	0	—0·069589	0	—0·116776	0	—0·212200	0
13	0	—0·021993	0	—0·046538	0	—0·077658	0	—0·124646	0	—0·220269
14	—0·009929	0	—0·030700	0	—0·054607	0	—0·085528	0	—0·132715	0
15	0	—0·018636	0	—0·038769	0	—0·062477	0	—0·093597	0	—0·141422
16	—0·008707	0	—0·026705	0	—0·046639	0	—0·070546	0	—0·102303	0
17	0	—0·016776	0	—0·034575	0	—0·054708	0	—0·079253	0	—0·112233
18	—0·008069	0	—0·024646	0	—0·042644	0	—0·063415	0	—0·089103	0
19	0	—0·015939	0	—0·032715	0	—0·051351	0	—0·073344	0	—0·101246
20	—0·003935	0	—0·012004	0	—0·020711	0	—0·030640	0	—0·042704	0
	—(19)	—(18)	—(17)	—(16)	—(15)	—(14)	—(13)	—(12)	—(11)	—(10)

TABLE 4

Factors for obtaining $\varepsilon(\theta)$ from $\psi(\theta)$ when $\psi(\theta)$ is odd

$20\,\frac{\theta}{\pi}$	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	0	+0·6353102	0	0·2082650	0	0·1207107	0	0·0815926	0	0·0585425	0
1	−1·2706204	0	+0·4270452	0	0·0875543	0	0·0391181	0	0·0230501	0	0·0158385
2	0	−0·8435752	0	+0·5145995	0	0·1266724	0	0·0621682	0	0·0388886	0
3	−0·4165300	0	−0·7560209	0	+0·5537176	0	0·1497225	0	0·0780067	0	0·0509526
4	0	−0·3289757	0	−0·7169028	0	+0·5767677	0	0·1655610	0	0·0900707	0
5	−0·2414214	0	−0·2898576	0	−0·6938527	0	+0·5926062	0	0·1776250	0	0·1000000
6	0	−0·2023033	0	−0·2668075	0	−0·6780142	0	+0·6046702	0	0·1875543	0
7	−0·1631852	0	−0·1792532	0	−0·2509690	0	−0·6659502	0	+0·6145995	0	0·1962611
8	0	−0·1401351	0	−0·1634147	0	−0·2389050	0	−0·6560209	0	+0·6233063	0
9	−0·1170850	0	−0·1242966	0	−0·1513507	0	−0·2289757	0	−0·6473141	0	+0·6313751
10	0	−0·1012465	0	−0·1122326	0	−0·1414214	0	−0·2202689	0	−0·6392453	0
11	−0·0854080	0	−0·0891825	0	−0·1023033	0	−0·1327146	0	−0·2122001	0	−0·6313751
12	0	−0·0733440	0	−0·0792532	0	−0·0935965	0	−0·1246458	0	−0·2043299	0
13	−0·0612800	0	−0·0634147	0	−0·0705464	0	−0·0855277	0	−0·1167756	0	−0·1962611
14	0	−0·0513507	0	−0·0547079	0	−0·0624776	0	−0·0776575	0	−0·1087068	0
15	−0·0414214	0	−0·0426439	0	−0·0466391	0	−0·0546074	0	−0·0695887	0	−0·1000000
16	0	−0·0327146	0	−0·0345751	0	−0·0387689	0	−0·0465386	0	−0·0608819	0
17	−0·0240078	0	−0·0246458	0	−0·0267049	0	−0·0307001	0	−0·0373818	0	−0·0509526
18	0	−0·0159390	0	−0·0246458	0	−0·0186361	0	−0·0219933	0	−0·0279025	0
19	−0·0078702	0	−0·0080688	0	−0·0087068	0	−0·0099293	0	−0·0120640	0	−0·0158385
20	0	−0·0039351	0	−0·0120039	0	−0·0207107	0	−0·0306400	0	−0·0427040	0
	−(20)	−(19)	−(18)	−(17)	−(16)	−(15)	−(14)	−(13)	−(12)	−(11)	−(10)

TABLE 5

Factors for obtaining $\varepsilon'(\theta)$ from $\psi(\theta)$ when $\psi(\theta)$ is even

$20\frac{\theta}{\pi}$	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	10	—4·061191	0	—0·458743	0	—0·170711	0	—0·091573	0	—0·059272	0
1	—8·122382	10	—4·519934	0	—0·629454	0	—0·262284	0	—0·150845	0	—0·102508
2	0	—4·519934	10	—4·231902	0	—0·550316	0	—0·229983	0	—0·134809	0
3	—0·917486	0	—4·231902	10	—4·152764	0	—0·518015	0	—0·213947	0	—0·125961
4	0	—0·629454	0	—4·152764	10	—4·120463	0	—0·501979	0	—0·205099	0
5	—0·341421	0	—0·550316	0	—4·120463	10	—4·104427	0	—0·493131	0	—0·200000
6	0	—0·262284	0	—0·518015	0	—4·104427	10	—4·095579	0	—0·488032	0
7	—0·183147	0	—0·229983	0	—0·501979	0	—4·095579	10	—4·090480	0	—0·485184
8	0	—0·150845	0	—0·213947	0	—0·493131	0	—4·090480	10	—4·087632	0
9	—0·118544	0	—0·134809	0	—0·205099	0	—0·488032	0	—4·087632	10	—4·086346
10	0	—0·102508	0	—0·125961	0	—0·200000	0	—0·485184	0	—4·086346	10
11	—0·086473	0	—0·093660	0	—0·120862	0	—0·197152	0	—0·483898	0	—4·086346
12	0	—0·077624	0	—0·088561	0	—0·118014	0	—0·195866	0	—0·483898	0
13	—0·068776	0	—0·072525	0	—0·085713	0	—0·116728	0	—0·195866	0	—0·485184
14	0	—0·063677	0	—0·069677	0	—0·084426	0	—0·116728	0	—0·197152	0
15	—0·058579	0	—0·060829	0	—0·068391	0	—0·084426	0	—0·118014	0	—0·200000
16	0	—0·055730	0	—0·059543	0	—0·068391	0	—0·085713	0	—0·120862	0
17	—0·052882	0	—0·034444	0	—0·059543	0	—0·069677	0	—0·088561	0	—0·125961
18	0	—0·051596	0	—0·054444	0	—0·060829	0	—0·072525	0	—0·093660	0
19	—0·050310	0	—0·051596	0	—0·055730	0	—0·063677	0	—0·077624	0	—0·102508
20	0	—0·025155	0	—0·026441	0	—0·029289	0	—0·034388	0	—0·043236	0
	(20)	(19)	(18)	(17)	(16)	(15)	(14)	(13)	(12)	(11)	(10)

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TABLE 6

Factors for obtaining $\varepsilon'(\theta)$ from $\psi(\theta)$ when $\psi(\theta)$ is odd

$20\frac{\theta}{\pi}$	(0)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	10	—4·0611911	0	—0·4587431	0	—0·1707107	0	—0·0915735	0	—0·0592722	0
1	0	10	—3·6024480	0	—0·2880324	0	—0·0791372	0	—0·0323013	0	—0·0160359
2	0	—3·6024480	10	—3·8904804	0	—0·3671696	0	—0·1114385	0	—0·0483372	0
3	0	0	—3·8904804	10	—3·9696176	0	—0·3994709	0	—0·1274744	0	—0·0571854
4	0	—0·2880324	0	—3·9696176	10	—4·0019189	0	—0·4155068	0	—0·1363226	0
5	0	0	—0·3671696	0	—4·0019189	10	—4·0179548	0	—0·4243550	0	—0·1414214
6	0	—0·0791372	0	—0·3994709	0	—4·0179548	10	—4·0268030	0	—0·4294538	0
7	0	0	—0·1114385	0	—0·4155068	0	—4·0268030	10	—4·0319018	0	—0·4323022
8	0	—0·0323013	0	—0·1274744	0	—0·4243550	0	—4·0319018	10	—4·0347502	0
9	0	0	—0·0483372	0	—0·1363226	0	—0·4294538	0	—4·0347502	10	—4·0360363
10	0	—0·0160359	0	—0·0571854	0	—0·1414214	0	—0·4323022	0	—4·0360363	10
11	0	0	—0·0248841	0	—0·0622842	0	—0·1442698	0	—0·4335883	0	—4·0360363
12	0	—0·0088482	0	—0·0299829	0	—0·0651326	0	—0·1455559	0	—0·4335883	0
13	0	0	—0·0139470	0	—0·0328313	0	—0·0664187	0	—0·1455559	0	—0·4323022
14	0	—0·0050988	0	—0·0167954	0	—0·0341174	0	—0·0664187	0	—0·1442698	0
15	0	0	—0·0079472	0	—0·0180815	0	—0·0341174	0	—0·0651326	0	—0·1414214
16	0	—0·0028484	0	—0·0092333	0	—0·0180815	0	—0·0328313	0	—0·0622842	0
17	0	0	—0·0041345	0	—0·0092333	0	—0·0167954	0	—0·0299829	0	—0·0571854
18	0	—0·0012861	0	—0·0041345	0	—0·0079472	0	—0·0139470	0	—0·0248841	0
19	0	0	—0·0012861	0	—0·0028484	0	—0·0050988	0	—0·0088482	0	—0·0160359
20	0	—0·0251548	0	—0·0264409	0	—0·0292893	0	—0·0343881	0	—0·0432363	0
	(20)	(19)	(18)	(17)	(16)	(15)	(14)	(13)	(12)	(11)	(10)

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