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Summary.—A theory for thin wings of any plan form describing simple harmonic oscillations of small amplitude in a supersonic air stream is developed. It is based on the use of the generalised Green's Theorem in conjunction with particular solutions which vanish over the characteristic cone with vertex at any point in the field of flow.

The theory can be used to calculate the aerodynamic forces acting on fluttering wings when the modes of distortion are known.

1. *Introduction.*—The problem of determining the aerodynamic forces acting on oscillating wings of any plan form in a uniform supersonic stream is considered. Formal solutions are obtained for the case of thin wings on the basis of linearised theory. In two dimensions, experimental evidence[1] has shown that the linear theory[2] is inadequate and that the thickness of the aerofoil should be taken into account[3]. This is probably true also in three dimensions but, as a first step towards the solution of the thick wing problem, the linearised theory for thin wings is developed in this paper. For wings with leading and trailing edges inclined at all points at angles to the direction of flow greater than the Mach angle, the solution is readily derived. In the more difficult case of wings with edges inclined to the stream at angles less than the Mach angle, the problem is reduced to one of finding the solution of an integral equation of a type similar to that which arises in incompressible flow.

In the limiting case when the frequency of oscillation is zero, the formulae derived in this paper lead to those already assumed for steady motion[4, 5].

2. *Basic Equations.*—The velocity potential ϕ due to a small disturbance caused by a wing oscillating in an air stream V satisfies the equation

$$\frac{\partial^2\phi}{\partial t_1^2} + 2V\frac{\partial^2\phi}{\partial x_1\,\partial t_1} + V^2\frac{\partial^2\phi}{\partial x_1^2} = V_0^2\left[\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial y_1^2} + \frac{\partial^2\phi}{\partial z_1^2}\right],$$

where V_0 is the velocity of sound, and where x_1, y_1, z_1 define the position of any moving point of the stream at time t_1. Let c represent a unit of length and assume that

$$x_1 = cX\cot\mu,\quad y_1 = cY,\quad z_1 = cZ,\quad t_1 = \frac{cT}{V},$$

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where $M = V/V_0$ and $\sin\mu = 1/M$. In the new co-ordinates X, Y, Z, T, the equation can be rewritten as

$$M^2\frac{\partial^2\phi}{\partial T^2} + \frac{2M^2}{\cot\mu}\frac{\partial^2\phi}{\partial X\,\partial T} = \frac{\partial^2\phi}{\partial Y^2} + \frac{\partial^2\phi}{\partial Z^2} - \frac{\partial^2\phi}{\partial X^2}\,. \qquad \ldots \quad \ldots \quad \ldots \quad (1)$$

For simple harmonic motion, let $\phi = \phi' e^{i\lambda T}$, where $\lambda = pc/V$, and use the transformation

$$\phi' = \Phi e^{-i\lambda \sec^2\mu\,.\,\cot\mu\,.\,X}\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Substitution in equation (1) then yields, when the exponential factor is suppressed,

$$\frac{\partial^2\Phi}{\partial Y^2} + \frac{\partial^2\Phi}{\partial Z^2} - \frac{\partial^2\Phi}{\partial X^2} - k^2\Phi = 0\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

where $k = \lambda \sec\mu$. For steady motion, $k = 0$.

The velocity potential ϕ of the disturbance must be zero at the wave front*, and for the problems considered in this paper $\partial\phi/\partial Z$, the normal velocity, is known at the wing surface. Hence the boundary conditions associated with equation (3) are $\Phi = 0$ over the wave front, and $\partial\Phi/\partial Z$ known over the wing.

Let X_0, Y_0, Z_0 be the co-ordinates of a particular point in the field, and let $x = X_0 - X$, $y = Y_0 - Y$, $z = Z_0 - Z$, $r^2 = y^2 + z^2$. Then the characteristic cone Γ with vertex at X_0, Y_0, Z_0 is defined by

$$x^2 - r^2 = 0\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

The solution of equation (3) with the boundary conditions stated is derived by the application of the generalised Green's Theorem[6,7] to the volume enclosed by Γ and the wave front (*see* Figs. 1, 2). When Φ and hence ϕ are determined, the lift distribution $l(x_1, y_1)$ is given by the usual formula

$$l(x_1, y_1) = 2\rho_0\left(\frac{\partial\phi}{\partial t_1} + V\frac{\partial\phi}{\partial x_1}\right), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

where ρ_0 is the air density of the undisturbed stream.

3. *Generalised Green's Theorem*.—Let

$$L(\Phi) \equiv \frac{\partial^2\Phi}{\partial Y^2} + \frac{\partial^2\Phi}{\partial Z^2} - \frac{\partial^2\Phi}{\partial X^2} - k^2\Phi\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and let Ψ denote a particular solution of equation (3). The theorem then gives the following integral relation between the required solution Φ and Ψ, namely

$$\int_V [\Psi L(\Phi) - \Phi L(\Psi)]\,d\tau = \int_S \left[\Phi\frac{\partial\Psi}{\partial\nu} - \Psi\frac{\partial\Phi}{\partial\nu}\right] dS\,. \qquad \ldots \quad \ldots \quad \ldots \quad (7)$$

---

* The wave front is regarded as the envelope of the Mach cones with vertices on the leading edge.

The integral on the left is taken over the volume enclosed by the surface S excluding, however, any points of singularity which may be within the region. The integral on the right is taken over the external surface S and the surfaces enclosing the singularities. In equation (7), the symbol ν refers to the co-normal which has direction cosines $-l$, m, n, where l, m, n are the direction cosines of the inward drawn normal at any point of the surface. It then follows that the operator

$$\frac{\partial}{\partial \nu} \equiv -l\frac{\partial}{\partial X} + m\frac{\partial}{\partial Y} + n\frac{\partial}{\partial Z} . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

The normal and the co-normal have the same projection on the $X = 0$ plane.

Since Φ and Ψ are solutions of equation (3), $L(\Phi) = L(\Psi) = 0$, and equation (7) reduces to

$$\int_S \left(\Phi\frac{\partial \Psi}{\partial \nu} - \Psi\frac{\partial \Phi}{\partial \nu}\right) dS = 0 . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

For the problems considered in this paper the surface S includes parts of Γ and the wave front as shown in Figs. 1, 2. Now on the wave front, $\Phi = 0$, and it can also be shown that $\partial\Phi/\partial\nu = 0$. This simplifies equation (9) and indicates that further simplification might be possible by the use of a particular solution Ψ which vanishes on Γ. In such a case $\partial\Psi/\partial\nu = 0$ also, since the co-normal at any point on Γ coincides with a generator. This procedure was used by Volterra[7] to obtain unique solutions of problems connected with the propagation of electric waves in a conducting medium, and is to a certain extent followed in this report.

4. *Particular Solutions.*—Since Γ is defined by $x^2 - r^2 = 0$, the required particular solutions which vanish on the cone are assumed to be functions of x and r only. They must therefore satisfy

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1}{r}\frac{\partial\Psi}{\partial r} - \frac{\partial^2\Psi}{\partial x^2} - k^2\Psi = 0 . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

Assume a solution of the form

$$\Psi = x^n f(q) g(\theta) , \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where $q^2 \equiv k^2(x^2 - r^2)$; $\theta \equiv x/r$.

It is readily verified that equation (11) will satisfy equation (10) provided

$$\frac{\partial^2 f}{\partial q^2} + \frac{2(n+1)}{q}\frac{\partial f}{\partial q} + f = 0 , \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$(\theta^2 - 1)\frac{\partial^2 g}{\partial \theta^2} + \left(\theta - \frac{2n}{\theta}\right)\frac{\partial g}{\partial \theta} - \frac{n(n-1)}{\theta^2} g = 0 . \qquad \ldots \quad \ldots \quad \ldots \quad (13)$$

Solutions of equations (12) and (13) for particular values of n are then obtained. When $n = 0$, equation (12) is satisfied by

$$f = \frac{A \cos q + B \sin q}{q} , \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where A and B are arbitrary constants. When $\theta = 1$, however, $q = 0$ and the first term in equation (14) becomes infinite. To avoid singularities, therefore, the particular solution is chosen to be

$$f_1 = \frac{\sin q}{q} . \qquad (15)$$

Similarly, when $n = 1$, a suitable solution of equation (12) is

$$f_2 = \frac{1}{q^3} (\sin q - q \cos q) . \qquad (16)$$

The required particular solutions of equation (13) must vanish when $\theta = 1$, and they are found to be

$$g_1 = \log_e [\theta + \sqrt{(\theta^2 - 1)}], \qquad (17)$$

and

$$g_2 = \log_e [\theta + \sqrt{(\theta^2 - 1)}] - \frac{\sqrt{(\theta^2 - 1)}}{\theta} \qquad (18)$$

for $n = 0$ and $n = 1$ respectively. Hence, particular solutions of equation (10) (and equation (3)) which vanish on the characteristic cone can be taken to be

$$\Psi_1 = \frac{\sin q}{q} \log_e [\theta + \sqrt{(\theta^2 - 1)}] , \qquad (19)$$

and

$$\Psi_2 = -\frac{x}{q}\left[\frac{\partial}{\partial q}\left(\frac{\sin q}{q}\right)\right]\left[\log_e [\theta + \sqrt{(\theta^2 - 1)}] - \frac{\sqrt{(\theta^2 - 1)}}{\theta}\right] . \qquad (20)$$

When $r = 0$, $\theta \to \infty$, and both Ψ_1 and Ψ_2 become infinite along a line through X_0, Y_0, Z_0 parallel to the X-axis. These singularities are excluded from the field of integration by a narrow cylinder as shown in Figs. 1, 2.

With the aid of Green's Theorem and the use of equations (19) and (20) it is then possible to obtain formal solutions of equation (3) for wings of any plan form in oscillatory or steady motion.

5. *Wing of Infinite Span*.—First, consider the relatively simple case of the oscillating flat plate of infinite span with leading edge along the OY-axis at right angles to the direction of flow. The characteristic cone Γ with vertex at X_0, Y_0, Z_0 will intersect the wave front represented in this case by the planes $Z = \pm X$ as shown in the diagram. It should be noted that the enclosed volume is cut in two by the wing, which is assumed to lie in the plane $Z = 0$. The singularities within the volume which arise from the particular solutions Ψ_1 and Ψ_2 are enclosed in the thin cylinder marked C. If Green's Theorem is now applied to the volume PADEB, the relation

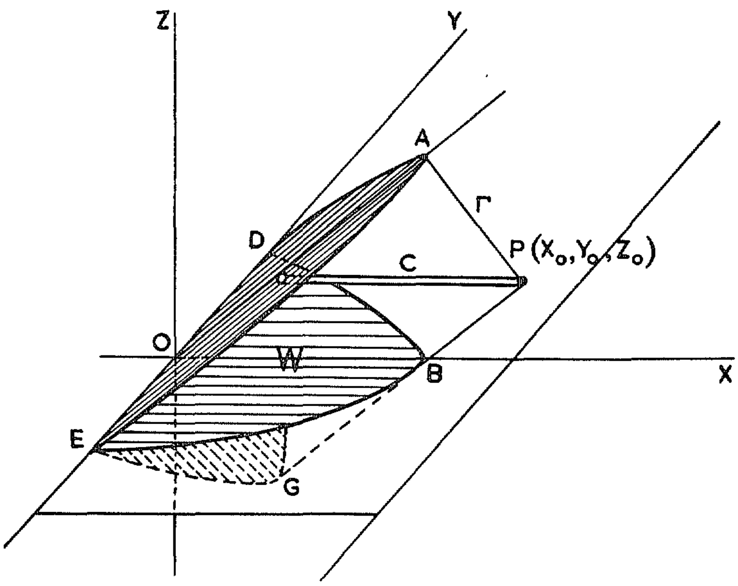

FIG. 1.

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial \nu} - \Psi \frac{\partial \Phi}{\partial \nu} \right) dS + \int_C \left(\Phi \frac{\partial \Psi}{\partial \nu} - \Psi \frac{\partial \Phi}{\partial \nu} \right) dS = 0 \quad \ldots \quad (21)$$

is obtained, where the first integral is taken over the external surface of the volume, and the second integral is taken over the surface of the thin cylinder. Over the area ADE of the wave front $\Phi = 0$, and $\partial \Phi / \partial \nu = 0$ since the co-normal lies in the plane $Z = X$. Over the cone, $\Psi = 0$, and $\partial \Psi / \partial \nu = 0$ since the co-normal is a generator. It then follows, since $\partial / \partial \nu = \partial / \partial Z$ at the wing surface, that

$$\int_S \left(\Phi \frac{\partial \Psi}{\partial \nu} - \Psi \frac{\partial \Phi}{\partial \nu} \right) dS = \int_W \left(\Phi \frac{\partial \Psi}{\partial Z} - \Psi \frac{\partial \Phi}{\partial Z} \right) dX\, dY \quad \ldots \quad (22)$$

taken over the area W representing the part BDE of the wing cut off by Γ. The remaining integral depends on the limiting form of the particular solution chosen as $r \to 0$. Over the surface of the cylinder $\partial / \partial \nu = \partial / \partial r$, and it can be shown that, as $r \to 0$, formulae (19) and (20) yield

$$\Psi_1 \to - \frac{\sin kx}{kx} \log_e r \, ,$$

$$\frac{\partial \Psi_1}{\partial r} \to - \frac{\sin kx}{kxr} \, ,$$

$$\Psi_2 \to - \frac{[\sin kx - kx \cos kx]}{k^3 x^2} \log_e r \, ,$$

$$\frac{\partial \Psi_2}{\partial r} \to - \frac{[\sin kx - kx \cos kx]}{k^3 x^2 r} = \frac{1}{k^2 r} \frac{\partial}{\partial x} \left(\frac{\sin kx}{kx} \right) . \quad \ldots \quad (23)$$

For the first particular solution

$$\int_C \left[\Phi \frac{\partial \Psi_1}{\partial \nu} - \Psi_1 \frac{\partial \Phi}{\partial \nu} \right] dS \to \int_{X_s}^{X_0} \int_0^{2\pi} \left(\Phi \frac{\partial \Psi_1}{\partial r} - \Psi_1 \frac{\partial \Phi_1}{\partial r} \right) r\, d\theta\, dX$$

$$\to - 2\pi \int_{X_s}^{X_0} \Phi \frac{\sin k(X_0 - X)}{k(X_0 - X)} dX \, , \quad \ldots \quad (24)$$

where X_s is the co-ordinate of the base of the cylinder. In this case, therefore, equation (21) yields

$$2\pi \int_{X_s}^{X_0} \Phi \frac{\sin k(X_0 - X)}{k(X_0 - X)} dX = \int_W \left(\Phi_a \frac{\partial \Psi_1}{\partial Z} - \Psi_1 \frac{\partial \Phi_a}{\partial Z} \right) dX\, dY \, . \quad \ldots \quad (25)$$

The suffix 'a' is introduced to indicate that the value of Φ immediately above the surface is used.

Similarly, if Green's Theorem is applied to the volume GBDE below the wing which contains no singularities, equation (21) yields

$$\int_W \left(\Phi_b \frac{\partial \Psi_1}{\partial Z} - \Psi_1 \frac{\partial \Phi_b}{\partial Z} \right) dX\, dY = 0 \, , \quad \ldots \quad (26)$$

where Φ_b refers to the value of Φ immediately below the wing. Integration over the image of the volume GBDE in the plane $Z = 0$ enclosed by the characteristic cone with vertex at X_0, Y_0, $-Z_0$ gives

$$\int_W \left(\Phi_a \frac{\partial \Psi_1}{\partial Z} + \Psi_1 \frac{\partial \Phi_a}{\partial Z} \right) dX\, dY = 0 \,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

since Ψ_1 involves terms in $(Z_0 - Z)^2$ only. For the particular problems considered in this paper $\partial \Phi_a / \partial Z = \partial \Phi_b / \partial Z$, and addition of equations (26) and (27) leads to the relation $\Phi_a = - \Phi_b$.

It then follows from equations (25) and (27) that

$$\pi \int_{X_s}^{X_0} \Phi \frac{\sin k(X_0 - X)}{k(X_0 - X)} dX = \int_W \Phi_a \frac{\partial \Psi_1}{\partial Z} dX\, dY$$

$$= - \int_W \Psi_1 \frac{\partial \Phi_a}{\partial Z} dX\, dY \,. \qquad \ldots \quad \ldots \quad \ldots \quad (28)$$

Equations (26) and (27), and hence (28), are only valid when the wing divides the volume enclosed by the characteristic cone and the wave front into separate parts. In the case of the second particular solution Ψ_2, the required solution Φ is given similarly by

$$\frac{\pi}{k^2} \int_{X_s}^{X_0} \Phi \frac{\partial}{\partial X_0} \frac{\sin k(X_0 - X)}{k(X_0 - X)} dX = - \int_W \Phi_a \frac{\partial \Psi_2}{\partial Z} dX\, dY$$

$$= \int_W \Psi_2 \frac{\partial \Phi_a}{\partial Z} dX\, dY \,. \qquad \ldots \quad \ldots \quad \ldots \quad (29)$$

From equation (28), by differentiation with respect to X_0, it follows that

$$\pi \Phi(X_0, Y_0, Z_0) + \pi \int_{X_s}^{X_0} \Phi \frac{\partial}{\partial X_0} \frac{\sin k(X_0 - X)}{k(X_0 - X)} dX = - \frac{\partial}{\partial X_0} \int_W \Psi_1 \frac{\partial \Phi_a}{\partial Z} dX\, dY \,. \quad \ldots \quad (30)$$

Then, by the use of equation (29), the above formula reduces to

$$\pi \Phi = - k^2 \int_W \Psi_2 \frac{\partial \Phi_a}{\partial Z} dX\, dY - \frac{\partial}{\partial X_0} \int_W \Psi_1 \frac{\partial \Phi_a}{\partial Z} dX\, dY. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

It can also be shown that

$$\frac{\partial \Psi_1}{\partial X_0} + k^2 \Psi_2 = \frac{\cos k \sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - (Z_0 - Z)^2]}}{\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - (Z_0 - Z)^2]}} \,. \qquad \ldots \quad \ldots \quad (32)$$

Since, if $\alpha = \sqrt{[(X_0 - X)^2 - Z_0^2]}$, it can be proved that

$$\frac{\partial}{\partial X_0} \int_W \Psi_1 \frac{\partial \Phi_a}{\partial Z} dX\, dY = \frac{\partial}{\partial X_0} \int_0^{X_0 - Z_0} \int_{Y_0 - \alpha}^{Y_0 + \alpha} \Psi_1 \frac{\partial \Phi_a}{\partial Z} dX\, dY$$

$$= \int_0^{X_0 - Z_0} \int_{Y_0 - \alpha}^{Y_0 + \alpha} \frac{\partial \Psi_1}{\partial X_0} \frac{\partial \Phi_a}{\partial Z} dX\, dY \,, \qquad \ldots \quad \ldots \quad (33)$$

it follows from equations (31) and (32) that

$$\pi\Phi(X_0, Y_0, Z_0) = -\int_0^{X_0 - Z_0}\int_{Y_0 - a}^{Y_0 + a} \frac{\partial\Phi_a}{\partial Z}\frac{\cos k\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}{\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}\, dX\, dY . \quad (34)$$

Without loss of generality, it can be assumed that $Y_0 = 0$, and if the substitution

$$Y = \sqrt{[(X_0 - X)^2 - Z_0^2]}\,\sin\beta , \quad \ldots \quad (35)$$

is made, formula (34) reduces to

$$\pi\Phi = -2\int_0^{X_0 - Z_0}\int_0^{\pi/2}\frac{\partial\Phi_a}{\partial Z}\cos(k\alpha\cos\beta)\, d\beta\, dX . \quad \ldots \quad (36)$$

Now, in terms of Bessel Functions,

$$\cos(k\alpha\cos\beta) = J_0(k\alpha) - 2J_2(k\alpha)\cos 2\beta + 2J_4(k\alpha)\cos 4\beta - \text{, etc.,} \quad \ldots \quad (37)$$

and $\partial\Phi_a/\partial Z$ is a function of X only. Hence, on integration, equation (36) yields

$$\Phi(X_0, Z_0) = -\int_0^{X_0 - Z_0}\frac{\partial\Phi_a}{\partial Z} J_0(k\sqrt{[(X_0 - X)^2 - Z_0^2)]}\, dX . \quad \ldots \quad (38)$$

When $Z_0 = 0$, equation (38) reduces to the formula

$$\Phi(X_0) = -\int_0^{X_0}\frac{\partial\Phi_a}{\partial Z} J_0[k(X_0 - X)]\, dX , \quad \ldots \quad (39)$$

which, when expressed in terms of the original co-ordinates, leads to the Temple and Jahn[2] solution, namely,

$$\Phi(x_0) = -\tan\mu\int_0^{x_0}\frac{\partial\Phi_a}{\partial z_1} J_0[\lambda\sec^2\mu\,.\,\sin\mu\,.\,(x_0 - x_1)/c]\, dx_1 . \quad \ldots \quad (40)$$

6. *Wings of Finite Aspect Ratio*.—For the purposes of this report, wings of finite aspect ratio are classified into two types A and B.

A plan form is of type A if the tangent at any point of its outer edge (leading edge wing tips and trailing edge) is inclined to the direction of flow at an angle γ greater than the Mach angle μ. Such wings will cut the volume enclosed by the wave front and Γ into two separate parts such as PABDE and BDEG in the infinite wing case. Then, as in the previous section, it follows that Φ is given by formula (34), namely

$$\pi\Phi(X_0, Y_0, Z_0) = -\int_W \frac{\partial\Phi_a}{\partial Z}\left(\frac{k\cos q}{q}\right) dS , \quad \ldots \quad (41)$$

where $q \equiv k\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}$ and W is the part of the wing cut off by Γ. By the use of equations (41), (2) and (5), the aerodynamic forces on any fluttering wing of type A for which $\partial\Phi_a/\partial Z$ is known can be calculated. Triangular wings with vertex angles greater than μ, and trapezoidal wings with wing tips inclined to the stream at angles greater than μ are of this class.

A plan form is of type B when $\gamma < \mu$ at one or more positions along the leading edge and wing tips. To avoid consideration of flow conditions in the wake at this stage, it is assumed that $\gamma > \mu$ for all points along the trailing edge as for wings of type A. Particular examples of type B plan forms are triangular wings with vertex angles less than μ, and rectangular wings. For the rectangular wing, the flows above and below are not everywhere independent since for points near the tip the wing does not divide the volume enclosed by the wave front and the characteristic cone with vertex at a particular point into two separate parts. For the triangular wing with $\gamma < \mu$, the wave front is taken to be the Mach cone with vertex at O as indicated in Fig. 2.

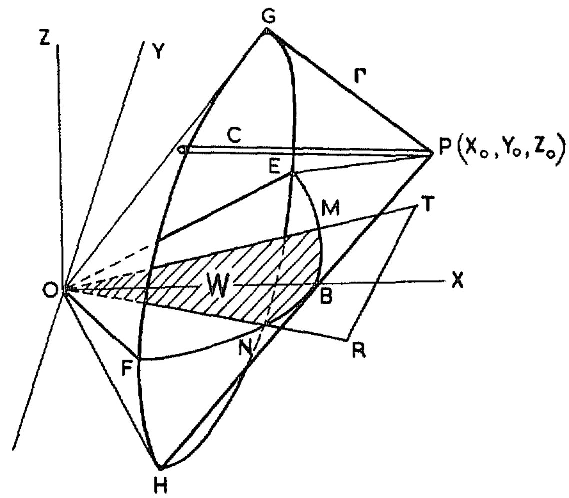

FIG. 2.

In this case, the volume PGEHFO is also not divided into two separate parts by the wing OTR. However, Green's Theorem can be applied to the volume as a whole to give

$$\int_S \left(\Phi \frac{\partial \Psi_1}{\partial \nu} - \Psi_1 \frac{\partial \Phi}{\partial \nu}\right) dS = 0\,, \qquad \ldots \qquad (42)$$

where the integral is taken over the surfaces of the Mach cone with vertex at O, the characteristic cone Γ with vertex at P, the upper and lower surfaces W of the part OMBN of the wing cut off by Γ, and over the cylinder O enclosing the singularities which arise from the chosen particular solutions. Over the Mach cone, $\Phi = 0 = \partial\Phi/\partial\nu$, while over Γ, $\Psi_1 = 0 = \partial\Psi_1/\partial\nu$, and the corresponding integrals vanish. At the wing surface, $\partial/\partial\nu = \pm\, \partial/\partial Z$ according as points on the upper or lower surfaces are considered. It should also be remembered that $\partial\Phi_a/\partial Z = \partial\Phi_b/\partial Z$. Integration over the wing surface W then gives

$$\int_W \left[\Phi \frac{\partial \Psi_1}{\partial \nu} - \Psi_1 \frac{\partial \Phi}{\partial \nu}\right] dS = \int \left(\Phi_a \frac{\partial \Psi_1}{\partial Z} - \Psi_1 \frac{\partial \Phi_a}{\partial Z}\right) dS - \int \left(\Phi_b \frac{\partial \Psi_1}{\partial Z} - \Psi_1 \frac{\partial \Phi_b}{\partial Z}\right) dS$$

$$= \int_W (\Phi_a - \Phi_b) \frac{\partial \Psi_1}{\partial Z}\, dS\,. \qquad \ldots \qquad (43)$$

The contribution due to the surface C has already been given in section 5. It then follows by comparison with equations (28), (29) and (30) that

$$2\pi\Phi(X_0, Y_0, Z_0) = \frac{\partial}{\partial X_0} \int_W K \frac{\partial \Psi_1}{\partial Z}\, dS + k^2 \int_W K \frac{\partial \Psi_2}{\partial Z}\, dS \qquad \ldots \qquad (44)$$

where $K = \Phi_a - \Phi_b$. Along OM and ON, $K = 0$, and along MN, $\Psi_1 = \Psi_2 = 0$. Then, since K is independent of Z_0 and $\partial\Psi/\partial Z_0 = -\partial\Psi/\partial Z$, equation (44) can be re-written in the form

$$2\pi\Phi = -\frac{\partial}{\partial Z_0} \int_W K \left(\frac{\partial \Psi_1}{\partial X_0} + k^2 \Psi_2\right) dS\,,$$

$$= -\frac{\partial}{\partial Z_0} \int_W K \frac{\cos k \sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}{\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}\, dX\, dY\,. \qquad \ldots \qquad (45)$$

Finally, $\partial\Phi/\partial Z$ is known over the wing surface, and by differentiating equation (45) the following integral equation for Φ_a is derived, namely,

$$\pi \frac{\partial \Phi}{\partial Z_0} = -\frac{\partial^2}{\partial Z_0^2}\int_W \Phi_a \frac{\cos k\sqrt{[(X_0-X)^2-(Y_0-Y)^2-Z_0^2]}}{\sqrt{[(X_0-X)^2-(Y_0-Y)^2-Z_0^2]}}\,dX\,dY\,, \qquad (46)$$

where $Z_0 \rightarrow 0$. It should be remembered that $\Phi_a = -\Phi_b$, and that, by equations (28) and (29), equation (44) is equivalent to equation (31) for wings of type A.

The form of equations (41) and (45) indicates that the particular solution which is infinite everywhere on the characteristic cone, namely,

$$\Psi = \frac{k\cos q}{q} \qquad (47)$$

could have been used in the preceding analysis. Formulae (41) and (45) then follow from (7), if the integral over the surface of the characteristic cone is neglected, and if the finite parts only of certain integrals are considered, as suggested by Hadamard. Such a method is used by Olga Todd[6] to solve the problem of a two-dimensional aerofoil in non-uniform motion. In the finite wing case, however, the justification of such a procedure would probably be rather complicated.

For wings of type A, the required solutions are given directly by equation (41). In general however, both equations (41) and (46) will have to be used. For a wing of type B as shown below,

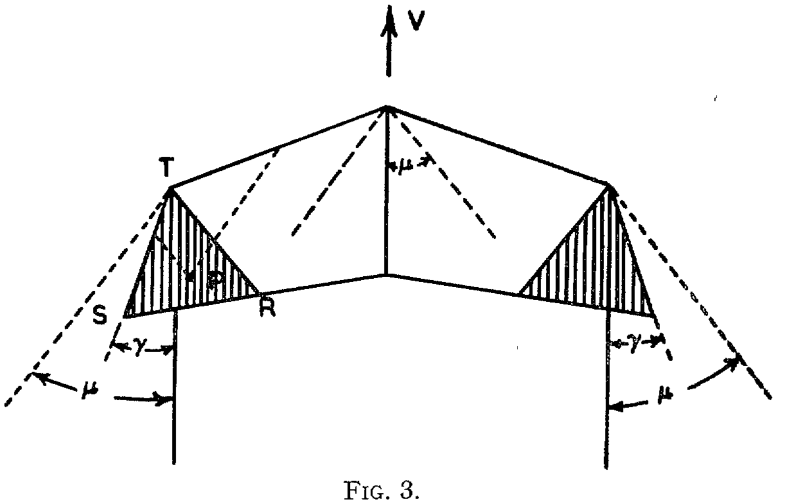

FIG. 3.

the velocity potential at any point of the unshaded part is given by equation (41), but for a point P in a shaded part equation (46) must be used. Over triangle TSR, for instance, the function Φ_a must satisfy equation (46), and it must also vanish along the wing tip TS. Along TR, Φ_a is given directly by equation (41). Solutions have already been given for problems of this type in the steady motion case[5,8]. With the aid of such solutions, the calculation of derivatives for oscillating wings of type B should not be very difficult, at least for low values of k. The numerical application of the theory does, however, require further consideration.

7. *Steady Motion*.—In this case, $k = 0$, $q = 0$, and the particular solution Ψ_1 reduces to $\Psi_1(0)$, where

$$\Psi_1(0) = \log_e \frac{X_0 - X + \sqrt{[(X_0-X)^2-(Y_0-Y)^2-(Z_0-Z)^2]}}{\sqrt{[(Y_0-Y)^2+(Z_0-Z)^2]}}\,. \qquad (48)$$

Substitution in equation (31) yields the following formulae

$$\pi\Phi = -\frac{\partial}{\partial X_0}\int_W \frac{\partial \Phi_a}{\partial Z} \Psi_1(0)\, dX\, dY\,, \qquad (49)$$

$$= -\int_W \frac{\partial \Phi_a}{\partial Z} \frac{\partial \Psi_1(0)}{\partial X_0}\, dX\, dY\,, \qquad (50)$$

$$= -\int_W \frac{\partial \Phi_a}{\partial Z} \frac{dX\, dY}{\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}\,. \qquad (51)$$

Equation (51) is also given directly by (41), when $k = 0$, and this is the form generally used for wings of type A in the steady case. It gives the potential due to a distribution of sources of strength $\partial\Phi_a/\partial Z$ over the part W of the wing cut off by the characteristic cone.

For wings of type B, equation (44) gives

$$2\pi\Phi(X_0, Y_0, Z_0) = \frac{\partial}{\partial X_0}\int_W (\Phi_a - \Phi_b) \frac{\partial \Psi_1(0)}{\partial Z}\, dS\,. \qquad (52)$$

This can be written alternatively in the forms

$$2\pi\Phi = -\frac{\partial^2}{\partial Z_0 \partial X_0}\int_W (\Phi_a - \Phi_b)\, \Psi_1(0)\, dS \qquad (53)$$

$$= -\frac{\partial}{\partial Z_0}\int_W (\Phi_a - \Phi_b) \frac{\partial \Psi_1(0)}{\partial X_0}\, dS \qquad (54)$$

$$= -\frac{\partial}{\partial Z_0}\int_W \frac{(\Phi_a - \Phi_b)\, dS}{\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}\,, \qquad (55)$$

where, as usual, the integral is taken over the part W of the wing cut off by the characteristic cone with vertex at X_0, Y_0, Z_0. Formula (55) may be regarded as giving the potential due to a distribution of doublets of strength $\Phi_a - \Phi_b$ equal to the discontinuity in the velocity potential at the wing surface. Another alternative form can be derived from equation (54) if use is made of the fact that $\Psi_1(0)[\Phi_a - \Phi_b] = 0$ round the edges of the area W. Then, since $\partial\Psi_1(0)/\partial X_0 = -\partial\Psi_1(0)/\partial X$, integration by parts gives

$$2\pi\Phi = -\frac{\partial}{\partial Z_0}\int_W \Psi_1(0) \frac{\partial}{\partial X}(\Phi_a - \Phi_b)\, dX\, dY\,, \qquad (56)$$

$$= -\int_W \frac{\partial \Psi_1(0)}{\partial Z_0} \frac{\partial}{\partial X}(\Phi_a - \Phi_b)\, dX\, dY\,, \qquad (57)$$

where differentiation under the integral sign does not introduce extra terms, since $\Psi_1(0) = 0$ along the boundary whenever the corresponding limits of the integral are not independent of Z_0. By the use of equation (48), it follows that, when $Z = 0$,

$$-\frac{\partial \Psi_1(0)}{\partial Z_0} = \frac{Z_0(X_0 - X)}{[(Y_0 - Y)^2 + Z_0^2]\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}} \qquad (58)$$

$$= \frac{\partial}{\partial Y}\tan^{-1}\left(\frac{Z_0\sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}}{(X_0 - X)(Y_0 - Y)}\right). \qquad (59)$$

The integral equation to be solved for wings of type B can then be written in the form

$$2\pi \frac{\partial \Phi}{\partial Z_0} = \frac{\partial}{\partial Z_0} \int_W \frac{\partial}{\partial X} (\Phi_a - \Phi_b) \frac{Z_0(X_0 - X)\, dX\, dY}{[(Y_0 - Y)^2 + Z_0^2] \sqrt{[(X_0 - X)^2 - (Y_0 - Y)^2 - Z_0^2]}} \quad .. \quad (60)$$

where $\partial\Phi/\partial Z_0$ is known over the wing, and $Z_0 \rightarrow 0$. When $\Phi_a - \Phi_b$ is independent of Y, equation (60) can be reduced to a single integral with respect to X by the use of equation (59).

8. *Concluding Remarks.*—The formulae given in this report provide a basis for the calculation of the aerodynamic forces acting on oscillating wings of any plan form for any mode of distortion. In the first place, however, it would be interesting to see whether the aerodynamic damping for pure pitching oscillations about certain axis positions is negative as in two-dimensional theory[2]. Calculations for triangular wings could be carried out and experiments on half-wing rigid models could possibly be made for comparison. The models however, would have to be thin as otherwise the theory might not apply[3].

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