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On the Solution of Linear Simultaneous Differential Equations with Constant Coefficients by a Process of Isolation

By

J. MORRIS, B.A.

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On the Solution of Linear Simultaneous Differential Equations with Constant Coefficients by a Process of Isolation

By

J. MORRIS, B.A.

COMMUNICATED BY THE PRINCIPAL DIRECTOR OF SCIENTIFIC RESEARCH (AIR),
MINISTRY OF SUPPLY

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Summary.—In this report a process is given for the solution of linear differential equations with constant coefficients. The operative artifice is closely akin to Routh's method of Isolation by means of which the constants of integration are found separately for each root of the characteristic equation.

Introduction.—Heaviside (1850-1925) appears to have decided that the recognised conventional methods for solving linear differential equations with constant coefficients were not the most efficacious in application to the analysis of electric networks in practical problems; and thus it was in quest of a more direct process of solution that he devised his operational calculus. But because of Heaviside's unconventional procedure and obscurity of presentation his work did not receive favourable attention. Bromwich (1875-1930) did much to elucidate Heaviside's peculiar calculus by the agency of the theory of functions of a complex variable.

Another important deviation from the standard method has become known as the Laplace transformation method. An interesting account of the development of the Laplace artifice is given by Carslaw and Jaeger[1].

In this report a totally different process is proposed for the same problem. It is closely akin to Routh's method of Isolation[2]. An important feature of the process is the simplicity with which the constants of integration associated with the various roots of the equation for the complementary function are found by separation and isolation.

1. *Equations with One Dependent Variable*.—1.1. We first consider the equation

$$F(D)\,x = f(t), \qquad (1)$$

where

$$F(D) = a_{11} + b_{11}D + c_{11}D^2, \qquad (2)$$

a_{11}, b_{11}, c_{11}, are constants, $f(t)$ is a function of t only; and D represents the operator d/dt. Regarding D as a parameter let m_1, m_2, be the two roots (real or complex) of the equation $F(m) = 0$.

Thus

$$(a_{11} + b_{11}\,m_1 + c_{11}\,m_1^{\,2})\,x_1 = 0, \qquad (3)$$

$$(a_{11} + b_{11}\,m_2 + c_{11}\,m_2^{\,2})\,x_2 = 0, \qquad (4)$$

where x_1, x_2, are for the present quite arbitrary.

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We easily derive from equations (3) and (4) the relation

$$(m_1 - m_2)\left[(b_{11} + c_{11}\, m_1)\, x_1 \cdot x_2 + c_{11}\, x_1 \cdot m_2\, x_2\right] = 0, \qquad (5)$$

or

$$(m_2 - m_1)\left[(b_{11} + c_{11}\, m_2)\, x_2 \cdot x_1 + c_{11}\, x_2 \,.\, m_1\, x_1\right] = 0. \qquad (6)$$

Thus, assuming that m_1, m_2, are not equal, we must have

$$\xi_r\, x_s + \bar{\xi}_r\, m_s\, x_s = 0, \qquad (7)$$

where

$$\xi_r = b_{11}\, x_r + m_r \bar{\xi}_r, \qquad (8)$$

$$\bar{\xi}_r = c_{11}\, x_r, \qquad (9)$$

in which for the particular case $r = 1, 2;\ s = 1, 2$. Also since x_r is quite arbitrary we may arrange for

$$\xi_r\, x_r + \bar{\xi}_r\, m_r\, x_r = 1. \qquad (10)$$

We notice that

$$\xi_r\, x_r + \bar{\xi}_r\, m_r\, x_r = x_r^2\, F'(m_r); \qquad (11)$$

so that the relation (10) gives

$$x_r^2 = 1/F'(m_r). \qquad (12)$$

We call the 'mode' x_r, subject to condition (10) a rectified mode.

It follows from relations (7) and (10) that

$$\xi_1\, x_1 + \xi_2\, x_2 = 1, \qquad (13)$$

$$\xi_1\, m_1\, x_1 + \xi_2\, m_2\, x_2 = 0, \qquad (14)$$

$$\bar{\xi}_1\, x_1 + \bar{\xi}_2\, x_2 = 0, \qquad (15)$$

$$\bar{\xi}_1\, m_1\, x_1 + \bar{\xi}_2\, m_2\, x_2 = 1. \qquad (16)$$

We have at once from equation (15) that

$$\Sigma\, x_r^2 = x_1^2 + x_2^2 = 0, \qquad (17)$$

and thus

$$\Sigma\, 1/F'(m_r) = 1/F'(m_1) + 1/F'(m_2) = 0. \qquad (18)$$

Reverting now to equation (1), by multiplying both sides with the rectified value of x_1, we obtain

$$(D - m_1)\,(\xi_1\, x + \bar{\xi}_1\, Dx) = x_1\, f(t). \qquad (19)$$

Integrating we derive

$$\xi_1\, x + \bar{\xi}_1\, D\, x = A_1\, e^{m_1 t} + \frac{x_1\, f(t)}{(D - m_1)}, \qquad (20)$$

where A_1 is a constant of integration. Hence taking account of the initial conditions

$$\xi_1\, x_0 + \bar{\xi}_1\, \dot{x}_0 = A_1 + \left[\frac{x_1\, f(t)}{(D - m_1)}\right]_0, \qquad (21)$$

where the suffix zero represents $t = 0$. We thus obtain the value of A_1 direct.

Similarly for the other root m_2,

$$\xi_2\, x + \bar{\xi}_2\, D\, x = A_2\, e^{m_2 t} + \frac{x_2\, f(t)}{(D - m_2)}, \qquad (22)$$

$$\xi_2\, x_0 + \bar{\xi}_2\, \dot{x}_0 = A_2 + \left[\frac{x_2\, f(t)}{(D - m_2)}\right]_0. \qquad (23)$$

The complete solution of equation (1) is, therefore,

$$x = A_1 x_1 e^{m_1 t} + A_2 x_2 e^{m_2 t} + \frac{x_1^2 f(t)}{(D - m_1)} + \frac{x_2^2 f(t)}{(D - m_2)}, \quad \ldots \quad \ldots \quad (24)$$

or

$$\begin{aligned} x = {} & x_0(\xi_1 x_1 e^{m_1 t} + \xi_2 x_2 e^{2mt}) \\ & + \dot{x}_0(\bar{\xi}_1 x_1 e^{m_1 t} + \bar{\xi}_2 x_2 e^{m_2 t}) \\ & - x_1^2 \left\{ \left[\frac{f(t)}{(D - m_1)}\right]_0 e^{m_1 t} - \frac{f(t)}{(D - m_1)} \right\} \\ & - x_2^2 \left\{ \left[\frac{f(t)}{(D - m_2)}\right]_0 e^{m_2 t} - \frac{f(t)}{(D - m_2)} \right\}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25) \end{aligned}$$

If $\dot{x}_0 = x_0 = 0$, and $f(t) = 1$, we obtain, after a little algebra,

$$x = \frac{1}{F(0)} + \sum_1^2 \frac{e^{m_r t}}{m_r F'(m_r)}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

which is Heaviside's expansion theorem as applicable to the particular case of equation (1) with specified conditions.

1.2.—Suppose next we consider the case of an equation of the third order in D, *viz.*,

$$F(D)\, x = (a_{11} + b_{11} D + c_{11} D^2 + d_{11} D^3)\, x = f(t)\ldots \quad \ldots \quad \ldots \quad (1)$$

Let m_r $(r = 1,2,3)$ be a root of $F(D) = 0$, and let

$$\xi_r = b_{11} x_r + m_r \bar{\xi}_r, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\bar{\xi}_r = c_{11} x_r + m_r \bar{\bar{\xi}}_r, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$\bar{\bar{\xi}}_r = d_{11} x_r. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

We find as in the preceding case that if s differs from r

$$\xi_r x_s + \bar{\xi}_r m_s x_s + \bar{\bar{\xi}}_r m_s^2 x_s = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

and, as in that case, we may arrange for

$$\xi_r x_r + \bar{\xi}_r m_r x_r + \bar{\bar{\xi}}_r m_r^2 x_r = 1; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

so that again x_r^2 will be $1/F'(m_r)$. .. .. (7)

We now have the following orthogonal relations

$$\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 = 1, \quad \ldots \quad (8)$$

$$\xi_1 m_1 x_1 + \xi_2 m_2 x_2 + \xi_3 m_3 x_3 = 0, \quad \ldots \quad (9)$$

$$\xi_1 m_1^2 x_1 + \xi_2 m_2^2 x_2 + \xi_3 m_3^2 x_3 = 0, \quad \ldots \quad (10)$$

$$\bar{\xi}_1 m_1 x_1 + \bar{\xi}_2 m_2 x_2 \bar{\xi}_3 + m_3 x_3 = 1, \quad \ldots \quad (11)$$

$$\bar{\xi}_1 m_1^2 x_1 + \bar{\xi}_2 m_2^2 x_2 + \bar{\xi}_3 m_3^2 x_3 = 0, \quad \ldots \quad (12)$$

$$\bar{\bar{\xi}}_1 m_1^2 x_1 + \bar{\bar{\xi}}_2 m_2^2 x_2 + \bar{\bar{\xi}}_3 m_3^2 x_3 = 1, \quad \ldots \quad (13)$$

$$x_1 \bar{\xi}_1 + x_2 \bar{\xi}_2 + x_3 \bar{\xi}_3 = 0, \quad \ldots \quad (14)$$

$$x_1 \bar{\bar{\xi}}_1 + x_2 \bar{\bar{\xi}}_2 + x_3 \bar{\bar{\xi}}_3 = 0, \quad \ldots \quad (15)$$

$$m_1 x_1 \bar{\bar{\xi}}_1 + m_2 x_2 \bar{\bar{\xi}}_2 + m_3 x_3 \bar{\bar{\xi}}_3 = 0. \quad \ldots \quad (16)$$

Reverting now to the equation (1) we find,

$$\xi_r x + \bar{\xi}_r D x + \bar{\bar{\xi}}_r D^2 x = A_r e^{m_r t} + x_r \frac{f(t)}{(D - m_r)}, \quad \ldots \quad (17)$$

$$\xi_r x_0 + \bar{\xi}_r \dot{x}_0 + \bar{\bar{\xi}}_r \ddot{x}_0 = A_r + x_r \left[\frac{f(t)}{(D - m_r)}\right]_0; \quad \ldots \quad (18)$$

where $r = 1, 2, 3$. Thus

$$x = x_0 \ \Sigma \ \xi_r x_r e^{m_r t} + \dot{x}_0 \ \Sigma \ \bar{\xi}_r x_r e^{m_r t} + \ddot{x}_0 \ \Sigma \ \bar{\bar{\xi}}_r x_r e^{m_r t}$$

$$- \Sigma x_r^2 \left\{ \left[\frac{f(t)}{(D - m_r)}\right]_0 e^{m_r t} - \frac{f(t)}{(D - m_r)} \right\} \quad \ldots \quad (19)$$

We notice from equation (15) that $\Sigma x_r^2 = 0$ and from equation (16) that $\Sigma m_r x^2_r = 0$.

It is easy to see that although we have dealt only with second and third order equations, the method is quite general and of obvious extension to equations of any order.

Also equation (19) may be regarded as indicating the form of the generalised expression of Heaviside's expansion theorem.

2. *Simultaneous Linear Equations.*—We now consider the two symmetrical equations

$$(a_{11} + b_{11} D + c_{11} D^2) x + (a_{12} + b_{12} D + c_{12} D^2) y = f_1(t), \quad \ldots \quad (1)$$

$$(a_{12} + b_{12} D + c_{12} D^2) x + (a_{22} + b_{22} D + c_{22} D^2) y = f_2(t), \quad \ldots \quad (2)$$

where the a's, b's, c's, are constants, f_1, f_2, are functions of t only; and D, as before, represents the operator d/dt.

First we solve the equations (1), (2), with the right-hand sides put zero and with m regarded as a parameter, *i.e.*, the equations

$$(a_{11} + b_{11} m + c_{11} m^2) x + (a_{12} + b_{12} m + c_{12} m^2) y = 0, \quad \ldots \quad (3)$$

$$(a_{12} + b_{12} m + c_{12} m^2) x + (a_{22} + b_{22} m + c_{22} m^2) y = 0. \quad \ldots \quad (4)$$

Let m_r ($r = 1, 2, 3, 4$) be a root of these equations and let x_r, y_r, be its associated relative modes. Multiply equation (3) by x_r, equation (4) by y_r, and add.

We obtain

$$(D - m_r)\Big[x \ \xi_r + y \ \eta_r + D \ (x \ \bar{\xi}_r + y \ \bar{\eta}_r)\Big] = 0, \quad \ldots \quad (5)$$

where

$$\xi_r = b_{11} x_r + b_{12} y_r + m_r \bar{\xi}_r, \quad \ldots \quad (6)$$

$$\eta_r = b_{12} x_r + b_{22} y_r + m_r \bar{\eta}_r, \quad \ldots \quad (7)$$

$$\bar{\xi}_r = c_{11} x_r + c_{12} y_r, \quad \ldots \quad (8)$$

$$\bar{\eta}_r = c_{12} x_3 + c_{22} y_r. \quad \ldots \quad (9)$$

We notice that ξ_r may be written

$$- (a_{11} x_r + a_{12} y_r)/m_r,$$

and η_r as

$$- (a_{12} x_r + a_{22} y_r)/m_r.$$

Thus it follows from equation (5) that if s differs from r, we have the following modal relation, *viz.*,

$$x_s \xi_r + y_s \eta_r + m_s (x_s \bar{\xi}_r + y_s \bar{\eta}_r) = 0. \quad \ldots \quad (10)$$

Now since the ratios only of the relative modes are determinate from the equations (3), (4), we may without loss of generality, arrange for

$$x_r \xi_r + y_r \eta_r + m_r (x_r \bar{\xi}_r + y_r \bar{\eta}_r) = 1, \quad \ldots \quad (11)$$

and we call modes subject to such condition rectified modes.

In consequence of equations (10) and (11) we have the relations

$$\left.\begin{aligned} x_1 \xi_1 + y_1 \eta_1 + m_1 (x_1 \bar{\xi}_1 + y_1 \bar{\eta}_1) &= 1, \\ x_1 \xi_2 + y_1 \eta_2 + m_1 (x_1 \bar{\xi}_2 + y_1 \bar{\eta}_2) &= 0, \end{aligned}\right\} \quad \ldots \quad (12)$$

and so on. In addition

$$\left.\begin{aligned} x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_3 + x_4 \xi_4 &= 1, \\ x_1 \bar{\xi}_1 + x_2 \bar{\xi}_2 + x_3 \bar{\xi}_3 + x_4 \bar{\xi}_4 &= 0, \\ x_1 \eta_1 + x_2 \eta_2 + x_3 \eta_3 + x_4 \eta_4 &= 0, \\ x_1 \bar{\eta}_1 + x_2 \bar{\eta}_2 + x_3 \bar{\eta}_3 + x_4 \bar{\eta}_4 &= 0, \end{aligned}\right\} \quad \ldots \quad (13)$$

and so on;

$$\left.\begin{aligned} m_1 x_1 \bar{\xi}_1 + m_2 x_2 \bar{\xi}_2 + m_3 x_3 \bar{\xi}_3 + m_4 x_4 \bar{\xi}_4 &= 1, \\ m_1 x_1 \xi_1 + m_2 x_2 \xi_2 + m_3 x_3 \xi_3 + m_4 x_4 \xi_4 &= 0, \end{aligned}\right\} \quad \ldots \quad (14)$$

and so on.

It follows from equations (8) and (9), and having regard to the relations,

$$\left.\begin{aligned} \Sigma x_r \bar{\xi}_r = 0, \quad \Sigma x_r \bar{\eta}_r = 0, \\ \Sigma y_r \bar{\xi}_r = 0, \quad \Sigma y_r \bar{\eta}_r = 0, \end{aligned}\right\} \quad \ldots \quad (15)$$

that we must have

$$\left.\begin{aligned} \Sigma x_r^2 = 0, \quad \Sigma x_r y_r = 0, \quad \Sigma y_r^2 = 0; \\ \Sigma \bar{\xi}_r^2 = 0, \quad \Sigma \bar{\xi}_r \bar{\eta}_r = 0, \quad \Sigma \bar{\eta}_r^2 = 0. \end{aligned}\right\} \quad \ldots \quad (16)$$

Reverting now to the equations (1), (2), multiply the former by x_r, the latter by y_r, and add. We obtain

$$(D - m_r) [x \xi_r + y \eta_r + D (x \bar{\xi}_r + y \bar{\eta}_r)] = F_r, \quad \ldots \quad (17)$$

where $$F_r = x_r f_1(t) + y_r f_2(t). \quad \ldots \quad (18)$$

Integrating we derive

$$x \xi_r + y \eta_r + D (x \bar{\xi}_r + y \bar{\eta}_r) = A_r \, e^{m_r t} + F_r/(D - m_r). \quad \ldots \quad (19)$$

Making use of the initial conditions, we have

$$x_0 \xi_r + y_0 \eta_r + \dot{x}_0 \bar{\xi}_r + \dot{y}_0 \bar{\eta}_r = A_r + [F_r/(D - m_r)]_0. \quad \ldots \quad (20)$$

We thus obtain the constant of integration A_r direct.

Next in virtue of the properties of the rectified modes, we find that

$$x = \Sigma A_r x_r e^{m_r t} + \Sigma x_r \Big[F_r/(D - m_r) \Big], \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

$$y = \Sigma A_r y_r e^{m_r t} + \Sigma y_r \Big[F_r/(D - m_r) \Big]. \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

Although we have dealt only with second order equations of which the operators of the elements are of the second degree, it is fairly evident that the method is of a general nature and readily extended to equations of any order with operators of any degree.

For the corresponding procedure in the case of unsymmetrical equations see Ref. 3.

3. *Case of Equal and Repeated Roots.*—In this case no difficulty arises provided that corresponding to each and every root there is a distinct set of appropriate orthogonal modes (in this connexion see Ref. 4, p. 746 et seq.). Let us take as an example the equations:

$$(1 - D)\,x + 2y + z = 0,$$
$$2x + (1 - D)\,y + z = 0,$$
$$x + y + (-\tfrac{1}{2} - D)\,z = 0.$$

We obtain

$$m_{31} = -1;\, x_{31} = \sqrt{2}/6,\, y_{31} = \sqrt{2}/6,\, z_{31} = -4\sqrt{2}/6,$$
$$m_{32} = -1;\, x_{32} = 1/\sqrt{2},\, y_{32} = -1/\sqrt{2},\, z_{32} = 0,$$
$$m_{33} = 7/2;\, x_{33} = 2/3,\, y_{33} = 2/3,\, z_{33} = 1/3,$$

in which the modes are rectified, *i.e.*, (in the particular case)

$$x_{3r}^2 + y_{3r}^2 + z_{3r}^2 = 1.$$

We thus derive

$$(m_{31} - D)\,(x\,x_{31} + y\,y_{31} + z\,z_{31}) = 0,$$
$$(m_{32} - D)\,(x\,x_{32} + y\,y_{32} + z\,z_{32}) = 0,$$
$$(m_{33} - D)\,(x\,x_{33} + y\,y_{33} + z\,z_{33}) = 0;$$

and thereby

$$x\,x_{31} + y\,y_{31} + z\,z_{31} = A_{31}\,e^{m_{31}t},$$
$$x\,x_{32} + y\,y_{32} + z\,z_{32} = A_{32}\,e^{m_{32}t},$$
$$x\,x_{33} + y\,y_{33} + z\,z_{33} = A_{33}\,e^{m_{33}t}.$$

Hence

$$x_0\,x_{31} + y_0\,y_{31} + z_0\,z_{31} = A_{31},$$
$$x_0\,x_{32} + y_0\,y_{32} + z_0\,z_{32} = A_{32},$$
$$x_0\,x_{33} + y_0\,y_{33} + z_0\,z_{33} = A_{33}.$$

The complete solution is therefore

$$x = [x_0(x_{31}^2 + x_{32}^2) + y_0(x_{31}\,y_{31} + x_{32}\,y_{32}) + z_0(z_{31}\,x_{31} + x_{32}\,z_{32})]\,e^{m_{31}t}$$
$$+ (x_0\,x_{33}^2 + y_0\,x_{33}\,y_{33} + z_0\,x_{33}\,z_{33})\,e^{m_{33}t},$$

with similar expressions for y and z.

We notice that since

$$x_{31}^2 + x_{32}^2 + x_{33}^2 = 1, \qquad x_{31}\,y_{31} + x_{32}\,y_{32} + x_{33}\,y_{33} = 0.$$

and so on, we need only have determined the modes for the root m_{33}.

When, however, distinct sets of appropriate modes or their required aggregates cannot be found then the usual methods for dealing with equal and repeated roots may be applied.

Consider for instance the two unsymmetrical equations

$$(1 - D)\,x - y = 0,$$
$$x + (3 - D)\,y = 0,$$

and their transposed

$$(1 - D)\,x' + y' = 0,$$
$$-x' + (3 - D)\,y' = 0.$$

There are two equal roots in D, *viz.*, $m_{21} = 2$, $m_{22} = 2$; while the only ascertainable modal relations are

$$x_{21}/y_{21} = -1\,;\, x_{21}'/y_{21}' = 1.$$

Thus for purposes of rectification we put

$$(x_{21}\,x_{21}' + y_{21}\,y_{21}')/y_{21}\,y_{21}' = 1/y_{21}\,y_{21}',$$

we find that $y_{21}\,y_{21}'$ must be infinite since the left-hand side is zero.

In this case therefore, we adopt the usual procedure for dealing with equal roots, *viz.*,

$$x = (A_1 + B_1\,t)\,\mathrm{e}^{2t}, \quad y = (A_2 + B_2\,t)\,\mathrm{e}^{2t};$$

from which by insertion in the original equations leads to the solution

$$x = \left[x_0 - (x_0 + y_0)\,t\right]\mathrm{e}^{2t},$$
$$y = \left[y_0 + (x_0 + y_0)\,t\right]\mathrm{e}^{2t}.$$

Suppose next we consider the equation

$$F(D)\,x = (D - m_1)^2\,(D - m_3)\,x = f(t)$$

In such case we may proceed as follows

$$\Phi\,(D)\,\mathrm{X} = f(t)$$

where
$$\Phi\,(D) = (D - m_1)\,(D - m_3)$$
and
$$X = (D - m_1)\,x.$$

Having found X the appropriate solution for x will be of the form

$$x = A^1\,\mathrm{e}^{m_1 t} + X/(D - m_1)$$

in which the constant of integration A_1 is readily determined from the initial conditions.

REFERENCES

No.	*Author*	*Title, etc.*
1	Carslaw and Jaeger	*Operational Methods in Applied Mathematics.* Oxford University Press. 1941.
2	Routh	*Advanced Rigid Dynamics.* Macmillan. 1905.
3	Morris	The Escalator Process for the Solution of Damped Lagrangian Frequency Equations. *Phil. Mag.* Ser. 7, vol. 38, April, 1947.
4	Morris and Head	The Escalator Process for the Solution of Lagrangian Frequency Equations. *Phil. Mag.* Ser. 7, Vol. 35, November, 1944.
5	Frazer, Duncan, and Collar	*Elementary Matrices.* Cambridge University Press. 1938.

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