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Introduction and Summary.—Recent advances in electronic computing devices suggest that it may soon be feasible to attempt numerical solutions of problems involving three independent variables. In this paper, preliminary consideration is given to the extension of the numerical method of characteristics for hyperbolic equations to the case of three independent variables.

A general quasi-linear second order partial differential equation in three variables is first considered, and the characteristic surfaces and curves are derived, together with the differential relations which hold along them. It is shown that numerical integration should be possible along the faces or edges of a hexahedral grid.

The equations are developed in more detail for two special cases of compressible flow, namely steady isentropic supersonic flow in three-dimensional space, and unsteady flow in two dimensions.

1. *The General Quasi-linear Second-order Partial Differential Equation in Three Independent Variables.*—Consider first the general second-order partial differential equation, linear in the second derivatives, which may be written,

$$a_{11}p_{11} + a_{22}p_{22} + a_{33}p_{33} + 2a_{23}p_{23} + 2a_{31}p_{31} + 2a_{12}p_{12} - l = 0,$$

or, using the summation convention,

$$a_{ij}\,p_{ij} - l = 0, \qquad \begin{Bmatrix} i, j = 1, 2, 3 \\ a_{ij} = a_{ji} \end{Bmatrix}, \qquad \ldots \quad (1.01)$$

where, the independent variable ϕ being a function of x_1, x_2, x_3,

$$p_i = \frac{\partial \phi}{\partial x_i}, \quad p_{ij} = \frac{\partial^2 \phi}{\partial x_i\, \partial x_j} \text{etc.}, \qquad \ldots \quad (1.02)$$

and a_{ij}, l, are functions of ϕ, x_i, p_i, only.

Consider next any surface in (x_1, x_2, x_3)-space, which may be defined, in terms of two parameters α, β, by

$$x_i = x_i(\alpha, \beta)\ ; \qquad \ldots \quad (1.03)$$

and let it be supposed that the values of ϕ, p_1, p_2, p_3 are known at all points of this surface.

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Then, at all points on this surface, these known values must satisfy the six relations

$$\frac{\partial p_i}{\partial \alpha} = p_{ij} \frac{\partial x_j}{\partial \alpha} \qquad (1.04)$$

$$\frac{\partial p_i}{\partial \beta} = p_{ij} \frac{\partial x_j}{\partial \beta} \qquad (1.05)$$

of which only five are independent. (The summation convention is used in these and subsequent relations).

Thus any five of these equations, together with the original partial differential equation (1.01), may be solved to give values of the second derivatives p_{ij} at all points of the surface (1.03), in terms of known quantities.

The solution is best obtained as follows: Write,

$$L_i = \varepsilon_{ijk} \frac{\partial x_j}{\partial \alpha} \frac{\partial x_k}{\partial \beta} \qquad (1.06)$$

where ε_{ijk} is the alternating tensor, so that L_1, L_2, L_3 are thus proportional to the direction cosines of the normal at (x_1, x_2, x_3) to the surface (1.03). The six relations (1.04) and (1.05) may then be written,

$$\varepsilon_{jkl} \cdot p_{il} L_k = \frac{\partial p_i}{\partial \alpha} \frac{\partial x_j}{\partial \beta} - \frac{\partial p_i}{\partial \beta} \frac{\partial x_j}{\partial \alpha} = X_{ij} \text{ say} \qquad (1.07)$$

where, it should be noted,

$$X_{ij} \neq X_{ji}.$$

Of these nine relations, again only five are independent, by virtue of the four identities,

$$L_j X_{ij} = 0 \qquad (1.08)$$

and $$X_{ii} = 0. \qquad (1.09)$$

Thus, combining any independent five of the relations (1.07) with the original partial differential equation (1.01), the solution for p_{ij} is obtained in the form,

$$p_{ij} = \frac{\Pi_{ij}}{\varepsilon} \qquad (1.10)$$

where, except for sign, the six Π_{ij}'s and ε are the seven determinants of the sixth order of a matrix with six rows and seven columns. Using, for example, the relations in X_{11}, X_{12}, X_{21}, X_{23}, X_{32}, the matrix is,

$$\left\| \begin{matrix} 0 & 0 & 0 & 0 & L_2 & -L_3 & X_{11} \\ L_3 & 0 & 0 & 0 & -L_1 & 0 & X_{12} \\ 0 & -L_3 & 0 & L_2 & 0 & 0 & X_{21} \\ 0 & L_1 & 0 & 0 & 0 & -L_2 & X_{23} \\ 0 & 0 & -L_1 & 0 & L_3 & 0 & X_{32} \\ a_{11} & a_{22} & a_{33} & 2a_{23} & 2a_{31} & 2a_{12} & 1 \end{matrix} \right\| \qquad (1.11)$$

Thus, if ε does not vanish, the second derivatives p_{ij} at all points of the surface (1.03) are determined, and, proceeding in a similar manner, it may be shown that the third and higher order derivatives of ϕ are also determinable, provided the same quantity ε does not vanish. In this case Cauchy's problem is solved, and ϕ is determined uniquely, if it is holomorphic, or, as Kowalewski has shown, if the given values of ϕ, p_1, p_2, p_3 on the curve (1.03) are analytic and regular. (*cf.* Ref. 1).

Suppose, however, that at all points of the surface (1.03) the quantity ε vanishes. It follows from (1.10) that all the Π_{ij}'s must also vanish if a non-trivial solution exists, and this introduces just the one further condition necessary to make the matrix (1.11) of rank 5.

The condition, $\varepsilon = 0$, is, in full,

$$\varepsilon = a_{ij} \, L_i \, L_j = 0 \qquad (1.12)$$

i.e., the condition that the normal (L_1, L_2, L_3) to the surface (1.03) should lie on a cone of the second degree, which is real, unless the quadratic form (1.12) is either positive definite or negative definite.

Defining

$$\Delta_1 = a_{11},$$

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

and the Hessian,

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \qquad (1.13)$$

the quadratic form (1.12) is positive definite if, [*cf.*, for example, Ref. 2]

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \Delta_3 > 0, \qquad (1.14)$$

and negative definite if,

$$\Delta_1 < 0, \quad \Delta_2 < 0, \quad \Delta_3 < 0. \qquad (1.15)$$

Thus, if either of these sets of conditions is satisfied at any point by the coefficients a_{ij}, the cone (1.12) is imaginary and the original partial differential equation (1.01) is said to be *elliptic* at the point; if the quadratic form (1.12) is a perfect square at any point, the cone (1.12) reduces to two coincident planes, and the original equation (1.01) is said to be *parabolic* at the point; and otherwise, (1.12) gives a real cone of possible normals and the original equation (1.01) is said to be *hyperbolic* at the point in question.

Restricting consideration henceforth to the hyperbolic case when the cone given by (1.12) is a real cone, it follows that, if the normal (L_1, L_2, L_3), at any point (x_{10}, x_{20}, x_{30}), to the surface (1.03), lies on this cone, then the tangent plane at (x_{10}, x_{20}, x_{30}), namely

$$L_1 \, (x_1 - x_{10}) + L_2 \, (x_2 - x_{20}) + L_3 \, (x_3 - x_{30}) = 0 \qquad (1.16)$$

has an envelope, which may be shown to be the real quadratic cone,

$$A_{ij} \, (x_i - x_{i0}) \, (x_j - x_{j0}) = 0 \qquad (1.18)$$

where A_{ji} is the co-factor of a_{ij} in the Hessian,

$$\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The cone (1.18) is called the *characteristic cone* at the point (x_{10}, x_{20}, x_{30}), the coefficients A_{ij} being, of course, constants, with values appropriate to the point (x_{10}, x_{20}, x_{30}).

Similarly, any displacement (dx_1, dx_2, dx_3) along the surface (1.03), satisfies, in the limit,

$$L_1 dx_1 + L_2 dx_2 + L_3 dx_3 = 0; \qquad (1.19)$$

and it follows that, if (L_1, L_2, L_3) lies on the cone (1.12), there is also an envelope of such displacements, given by,

$$A_{ij}\, dx_i\, dx_j = 0 \qquad (1.20)$$

This differential relation (1.20), in which the coefficients are, in general, functions of x_1, x_2, x_3, defines a ' curvilinear cone ', whose curvilinear generators are tangent at the vertex to the generators of the characteristic cone (1.18) with the same vertex. The curvilinear cone (1.20) is called the *characteristic conoid* at the point.

The nature of the characteristic surfaces, in the case of three independent variables, now becomes evident. The general characteristic surface has the property that, at any point, its normal (L_1, L_2, L_3) lies on the local cone (1.12). The general characteristic surfaces which pass through any point all touch an envelope, the local characteristic conoid (1.20) which is thus a special type of characteristic surface associated with a point. The curves along which the general characteristic surfaces touch the characteristic conoid are called *bi-characteristic curves*. Thus the characteristic conoid may also be regarded as generated by the bi-characteristic curves through a point, its vertex.

Alternatively, there is a characteristic conoid (1.20) associated with each point in (x_1, x_2, x_3)-space. The characteristic conoids associated with the points of an arbitrary curve have an envelope, consisting in general, of two sheets. Thus two general characteristic surfaces pass through any arbitrary curve in (x_1, x_2, x_3)-space.

The relations between the general characteristic surfaces, the characteristic cones, conoids, and the bi-characteristic curves, are illustrated in Fig. 1.

The condition that all the Π_{ij}'s should vanish simultaneously with ε, is given by the vanishing of any one of these quantities, *e.g.*, by

$$\Pi_{12} \equiv a_{11}L_1X_{13} - a_{22}L_2X_{23} + a_{33}(L_2X_{32} - L_3X_{11}) + 2a_{23}L_2X_{22} - 2a_{31}L_1X_{11} + L_1L_2l = 0. \qquad (1.21)$$

This differential relation, which holds along any characteristic surface satisfying (1.12) at every point, is sufficient, with (1.12), in the hyperbolic case when the characteristic surfaces are real, to develop a numerical solution starting from a given open boundary surface along which the conditions of a problem are known.

In place of the characteristic grid, familiar in hyperbolic problems with only two independent variables, there are now two possible types of network along which numerical integration may be carried out—a hexahedral network of general characteristic surfaces, or a hexahedral network of bi-characteristic curves. In each case, the known solution of the problem, already determined at three points, P_1, P_2, P_3, leads to the solution at a fourth point P_4. The ' units ' of such networks are shown in Figs. 2 and 3.

In Fig. 2 the rear branch of the characteristic conoid at P_4 cuts the plane $P_1P_2P_3$ in a curve inscribed to the triangle $P_1P_2P_3$, and the three faces $P_4P_2P_3$, $P_4P_3P_1$, $P_4P_1P_2$ of the tetrahedron $P_1P_2P_3P_4$ are segments of general characteristic surfaces, which, in a sufficiently small ' unit ' may be taken as plane sections. Associated with each of these faces there are two relations, corresponding to (1.12) and (1.21), *i.e.*, six relations in all, and these six relations, together with the known conditions at P_1, P_2, P_3 are sufficient to determine the three unknown space co-ordinates of P_4, and the values of P_1, P_2, P_3 at P_4. There are three other ' units ', similar to that in Fig. 2, associated with the three points P_1, P_2, P_3, given by the points P_4 whose characteristic conoids intersect the plane $P_1P_2P_3$ in curves escribed to the triangle $P_1P_2P_3$.

In Fig. 3, the rear branch of the characteristic conoid at P_4 cuts the plane $P_1P_2P_3$ in a curve circumscribed to the triangle $P_1P_2P_3$, and the three edges P_4P_1, P_4P_2, P_4P_3 of the tetrahedron $P_1P_2P_3P_4$ are segments of bi-characteristic curves, which, in a sufficiently small ' unit ' may be taken as linear segments. Again, associated with each of these three edges there are two relations corresponding to (1.12) and (1.21), *i.e.*, six relations in all, and these are sufficient together with the information known at P_1, P_2, P_3, to determine the six essential unknown quantities at P_4.

In theory then, it should be possible to integrate numerically in a progressive manner, along a hexahedral grid of one of the above types, starting from the known conditions of a problem on an open boundary surface. The complexity of such a numerical process, hitherto beyond contemplation, may become tractable soon by the rapid computational facilities afforded by new electronic machines.

2. *Steady Supersonic Compressible-flow in Three-dimensional Space.*—With cartesian co-ordinates (x_1, x_2, x_3), let the components of fluid velocity be u_1a_s, u_2a_s, u_3a_s respectively, where a_s is a constant speed, and consider the case of adiabatic irrotational flow. Then a velocity potential ϕ may be defined so that,

$$u_i = p_i = \frac{\partial \phi}{\partial x_i}. \qquad (2.01)$$

The speed of sound, aa_s, is given by

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho} = a^2 a_s^2 \qquad (2.02)$$

and Bernoulli's equation, obtained by integrating the equations of steady motion of the fluid, gives

$$\tfrac{1}{2}a_s^2 (p_1^2 + p_2^2 + p_3^2) + a^2 a_s^2/(\gamma - 1) = \text{constant}$$

$$= \frac{\gamma + 1}{2(\gamma - 1)} a_s^2 \qquad (2.03)$$

if the constant speed a_s is chosen to be the local speed of sound in the fluid when the velocity is sonic; *i.e.*, $p_1^2 + p_2^2 + p_3^2 = 1$ when $a = 1$.

Thus $$p_1^2 + p_2^2 + p_3^2 - a^2 = (1 - a^2)/\lambda^2 \qquad (2.04)$$

where $$\lambda^2 = \frac{\gamma - 1}{\gamma + 1} < 1. \qquad (2.05)$$

The equation of continuity reduces to the potential equation

$$\Sigma(p_1^2 - a^2)\, p_{11} + 2\Sigma\, p_2 p_3 p_{23} = 0. \qquad (2.06)$$

Hence, by (1.12) the normal (L_1, L_2, L_3) to a characteristic surface satisfies

$$\Sigma(p_1^2 - a^2)\, L_1^2 + 2\Sigma\, p_2 p_3 L_2 L_3 = 0. \qquad (2.07)$$

The conditions (1.14) and 1.15) that this should be a definite quadratic form, and thus represent an imaginary cone, reduce to

$$p_1^2 + p_2^2 + p_3^2 < a^2,$$

and it thus follows that the equation (2.06) is elliptic at points where the flow is subsonic, parabolic where the flow is sonic, and hyperbolic where the flow is supersonic.

In the hyperbolic case of supersonic flow, the Mach angle μ may be defined by

$$\sin^2 \mu = \frac{a^2}{p_1^2 + p_2^2 + p_3^2} \qquad (2.08)$$

and it may easily be seen that the cone (2.07) is a circular cone whose axis is in the direction of the flow, (p_1, p_2, p_3), and whose generators are inclined at an angle $(\pi/2 + \mu)$ to the axis.

The characteristic cone (1.18) at the point (x_{10}, x_{20}, x_{30}) is

$$\Sigma(p_2^2 + p_3^2 - a^2)\,(x_1 - x_{10})^2 - 2\Sigma\, p_2 p_3\,(x_2 - x_{20})\,(x_3 - x_{30}) = 0 \qquad (2.09)$$

i.e., the circular cone with vertex (x_{10}, x_{20}, x_{30}), axis (p_1, p_2, p_3) and semi-angle μ, the Mach angle.

And the equation of the characteristic conoid is

$$\Sigma(p_2^2 + p_3^2 - a^2)\,(dx_1)^2 - 2\Sigma\, p_2 p_3\, dx_2 \,.\, dx_3 = 0. \qquad (2.10)$$

The differential relation $\Pi_{12} = 0$, (*cf.* (1.21)) is

$$\Pi_{12} \equiv (p_1^2 - a^2)\, L_1 X_{13} - (p_2^2 - a^2)\, L_2 X_{23} + (p_3^2 - a^2)\,(L_2 X_{32} - L_3 X_{11})$$
$$+ 2 p_2 p_3 L_2 X_{22} - 2 p_3 p_1 L_1 X_{11} = 0. \qquad (2.11)$$

Fig. 4 illustrates the characteristic geometry of this particular flow. Now from (2.07) the normal (L_1, L_2, L_3) to any characteristic surface satisfies

$$(u_1 L_1 + u_2 L_2 + u_3 L_3)^2 = a^2(L_1^2 + L_2^2 + L_3^2) \qquad (2.12)$$

Thus if (l_1, l_2, l_3) are the *actual* direction cosines of the normal direction which makes an angle $(\pi/2 + \mu)$ with the direction of flow, then

$$u_1 l_1 + u_2 l_2 + u_3 l_3 = -a. \qquad (2.13)$$

This merely expresses the physical fact that the component of fluid velocity, normal to the characteristic conoid, is equal in magnitude to the speed of sound.

The bi-characteristic direction corresponding to (l_1, l_2, l_3) is

$$(al_1 + u_1,\ al_2 + u_2,\ al_3 + u_3). \qquad (2.14)$$

Put now

$$u_1 = p_1 = V \sin\theta \cos\psi$$

$$u_2 = p_2 = V \sin\theta \sin\psi \qquad \ldots \qquad (2.15)$$

$$u_3 = p_3 = V \cos\theta$$

so that θ, ψ are the polar angles of the flow direction, and

$$u_1^2 + u_2^2 + u_3^2 = V^2 = \frac{a^2}{\sin^2\mu} \cdot \qquad \ldots \qquad (2.16)$$

Then, by (2.13)

$$l_1 \sin\theta \cos\psi + l_2 \sin\theta \sin\psi + l_3 \cos\theta = -\sin\mu \qquad \ldots \qquad (2.17)$$

and the angle δ between the diametral plane of the characteristic cone passing through the bi-characteristic, and the diametral plane parallel to the x_3-axis thus satisfies

$$\cos\delta = \frac{l_1 \cos\theta \cos\psi + l_2 \cos\theta \sin\psi - l_3 \sin\theta}{\cos\mu}, \qquad \ldots \qquad (2.18)$$

$$\sin\delta = \frac{l_2 \cos\psi - l_1 \sin\psi}{\cos\mu} \cdot \qquad \ldots \qquad (2.19)$$

The relations (2.18) and (2.19), together with

$$l_1^2 + l_2^2 + l_3^2 = 1$$

thus enable l_1, l_2, l_3 to be expressed in terms of the single parameter δ. Hence, if the bi-characteristic curves are chosen as the parametric curves β = constant, and their orthogonal trajectories as the parametric curves α = constant, the following relations may be taken for any characteristic surface $\delta = \delta(\alpha, \beta)$

$$\left.\begin{aligned} \partial x_1/\partial\alpha &= al_1 + u_1 \\ \partial x_2/\partial\alpha &= al_2 + u_2 \\ \partial x_3/\partial\alpha &= al_3 + u_3 \end{aligned}\right\} \qquad \ldots \qquad (2.20)$$

$$\left.\begin{aligned} \partial x_1/\partial\beta &= u_2 l_3 - u_3 l_2 \\ \partial x_2/\partial\beta &= u_3 l_1 - u_1 l_3 \\ \partial x_3/\partial\beta &= u_1 l_2 - u_2 l_1 \end{aligned}\right\} \qquad \ldots \qquad (2.12)$$

Substituting these values, the differential relation $\Pi_{12} = 0$, and the identity (1.09), namely

$$X_{11} + X_{22} + X_{33} = 0,$$

ultimately combine and reduce to the two relations,

$$\tan\mu \left\{\frac{\partial\theta}{\partial\alpha} + \sin\mu \sin\theta \frac{\partial\psi}{\partial\beta}\right\} - \left\{\frac{\cos\delta}{V}\frac{\partial V}{\partial\alpha} - \frac{\sin\mu \sin\delta}{V}\frac{\partial V}{\partial\beta}\right\} = 0 \qquad \ldots \qquad (2.22)$$

and

$$\tan \mu \left\{\sin \mu \frac{\partial \theta}{\partial \beta} - \sin \theta \frac{\partial \psi}{\partial \alpha}\right\} + \left\{\frac{\sin \mu \cos \delta}{V} \frac{\partial V}{\partial \beta} + \frac{\sin \delta}{V} \frac{\partial V}{\partial \alpha}\right\} = 0. \quad .. \quad (2.23)$$

For $\delta = 0$, $\psi = 0$, these reduce to the familiar two-dimensional relations

$$\frac{dV}{V} = \pm \tan \mu \, d\theta \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.24)$$

respectively, along the characteristics

$$\frac{dx_1}{dx_2} = \tan (\theta \pm \mu) \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.25)$$

and the same reduction is obtained in the cases,

$$\delta = 0, \quad \psi = \pi/2$$

and $\delta = \pi/2, \quad \theta = \pi/2.$

Further, (2.04) and (2.16) combine to give

$$V^2(\lambda^2 \cos^2 \mu + \sin^2 \mu) = 1 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.26)$$

whence

$$\frac{dV}{V} + \frac{(1 - \lambda^2) \sin \mu \cos \mu \, d\mu}{\lambda^2 \cos^2 \mu + \sin^2 \mu} = 0 \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.27)$$

so that

$$\cot \mu \frac{dV}{V} = - \frac{(1 - \lambda^2) \cos^2 \mu \, d\mu}{(\lambda^2 \cos^2 \mu + \sin^2 \mu)} \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.28)$$

$$= - d(\nu/\lambda - \mu)$$

where ν is an angle defined by

$$\lambda \tan \nu = \tan \mu. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.29)$$

The relations (2.22) and 2.23) may then be written,

$$\left\{\frac{\partial \theta}{\partial \alpha} + \sin \mu \sin \theta \frac{\partial \psi}{\partial \beta}\right\} + \left\{\cos \delta \frac{\partial(\nu/\lambda - \mu)}{\partial \alpha} - \sin \mu \sin \delta \frac{\partial(\nu/\lambda - \mu)}{\partial \beta}\right\} = 0 \quad .. \quad (2.30)$$

and

$$\left\{\sin \mu \frac{\partial \theta}{\partial \beta} - \sin \theta \frac{\partial \psi}{\partial \alpha}\right\} - \left\{\sin \mu \cos \delta \frac{\partial(\nu/\lambda - \mu)}{\partial \beta} + \sin \delta \frac{\partial(\nu/\lambda - \mu)}{\partial \alpha}\right\} = 0. \quad .. \quad (2.31)$$

For $\delta = 0$, $\psi = 0$, these reduce to the familiar integrals of steady motion in two dimensions,

$$\nu/\lambda - \mu \pm \theta = \text{constant}, \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (2.32)$$

respectively, along the characteristics

$$\frac{dx_1}{dx_2} = \tan (\theta \pm \mu)$$

and the same reduction is obtained in the cases,

$$\delta = 0, \quad \psi = \pi/2$$
$$\delta = \pi/2, \quad \theta = \pi/2.$$

The relations (2.30) and (2.31) or (2.22) and (2.23) would form the basis of numerical computation along a hexahedral grid, as suggested in Section 1.

3. *Unsteady Compressible-flow in Two Dimensions.*—With cartesian co-ordinates (x, y) and time t, let the components of fluid velocity be u, v, respectively, and again consider the case of adiabatic irrotational flow. Then a velocity potential ϕ may be defined so that,

$$p_1 = \frac{\partial \phi}{\partial x} = u$$

$$p_2 = \frac{\partial \phi}{\partial y} = v$$

$$p_3 = \frac{\partial \phi}{\partial t} = -\left\{\frac{u^2 + v^2}{2} + \int \frac{dp}{\rho}\right\} = -\left\{\frac{u^2 + v^2}{2} + \frac{a^2}{\gamma - 1}\right\} \quad \ldots \quad \ldots \quad (3.01)$$

where the speed of sound a satisfies

$$\frac{dp}{d\rho} = \frac{\gamma p}{\rho} = a^2. \quad \ldots \quad \ldots \quad (3.02)$$

The equations of motion are then automatically satisfied, and the equation of continuity reduces to the potential equation,

$$(p_1^2 - a^2)\, p_{11} + (p_2^2 - a^2)\, p_{22} + p_{33} + 2p_2p_{32} + 2p_1p_{13} + 2p_1p_2\, p_{12} = 0. \quad \ldots \quad (3.03)$$

Hence, by (1.12) the normal (L_1, L_2, L_3) to a characteristic surface satisfies

$$(u^2 - a^2)\, L_1^2 + (v^2 - a^2)\, L_2^2 + L_3^2 + 2vL_2L_3 + 2uL_3L_1 + 2uvL_1L_2 = 0. \quad \ldots \quad (3.04)$$

The conditions (1.14) and (1.15) for this to be a definite quadratic form are never satisfied, and it reduces to a perfect square only when $a = 0$. Thus the potential equation (3.03) is always hyperbolic at every point.

The characteristic cone (1.18) at the point (x_0, y_0, t_0) is,

$$(x - x_0)^2 + (y - y_0)^2 + (u^2 + v^2 - a^2)(t - t_0)^2$$
$$- 2u(x - x_0)(t - t_0) - 2v(y - y_0)(t - t_0) = 0 \quad \ldots \quad \ldots \quad (3.05)$$

which has the property that it cuts the plane

$$t = t_0 + \tau$$

in the circle,

$$(x - x_0 - u\tau)^2 + (y - y_0 - v\tau)^2 = (a\tau)^2$$

whose centre is the point $(x_0 + u\tau,\ y_0 + v\tau,\ t_0 + \tau)$ and whose radius is $a\tau$.

The equation of the characteristic conoid is,

$$(dx - u\, dt)^2 + (dy - v\, dt)^2 = (a\, dt)^2. \quad \ldots \quad \ldots \quad (3.06)$$

and the differential relation $\Pi_{12} = 0$ is,

$$\Pi_{12} \equiv (u^2 - a^2)\, L_1 X_{13} - (v^2 - a^2)\, L_2 X_{23} + L_2 X_{32} - L_3 X_{11} + 2vL_2\, X_{22} - 2uL_1 X_{11} = 0. \quad \ldots \quad (3.07)$$

Fig. 5 illustrates the characteristic geometry of this particular flow. Now, from (3.04) the normal (L_1, L_2, L_3) to a characteristic surface staisfies

$$(uL_1 + vL_2 + L_3)^2 = a^2(L_1^2 + L_2^2) \quad \ldots \quad (3.08)$$

and may, therefore, be expressed parametrically by

$$\frac{L_1}{\sin \delta} = \frac{L_2}{\cos \delta} = \frac{-L_3}{u \sin \delta + v \cos \delta + a}. \quad \ldots \quad (3.09)$$

The corresponding bi-characteristic direction is,

$$(u + a \sin \delta,\ v + a \cos \delta,\ 1). \quad \ldots \quad (3.10)$$

Hence if the bi-characteristic curves are chosen as the parametric curves β = constant, and the curves of intersection with the planes t = constant, are chosen as the parametric curves α = constant, the following relations may be taken, for any characteristic surface $\delta = \delta(\alpha, \beta)$

$$\left.\begin{aligned} \partial x/\partial\alpha &= u/a + \sin \delta \\ \partial y/\partial\alpha &= v/a + \cos \delta \\ \partial t/\partial\alpha &= 1/a \end{aligned}\right\} \quad \ldots \quad (3.11)$$

and

$$\left.\begin{aligned} \partial x/\partial\beta &= \cos \delta \\ \partial y/\partial\beta &= -\sin \delta \\ \partial t/\partial\beta &= 0\,. \end{aligned}\right\} \quad \ldots \quad (3.12)$$

Then

$$\left.\begin{aligned} L_1 &= \sin \delta/a \\ L_2 &= \cos \delta/a \\ L_3 &= -(u/a \sin \delta + v/a \cos \delta + 1). \end{aligned}\right\} \quad \ldots \quad (3.13)$$

Substituting these values, the differential relation $\Pi_{12} = 0$ ultimately reduces, for this choice of the parameters, α, β, to

$$(a + v \cos \delta)(\partial u/\partial\alpha - \partial v/\partial\beta) - v \sin \delta\, (\partial v/\partial\alpha + \partial u/\partial\beta) + 2a/(\gamma - 1).\ \sin \delta\ \partial a/\partial\alpha + 2/(\gamma - 1).\ (v + a \cos \delta)\ \partial a/\partial\beta = 0, \quad \ldots \quad (3.14)$$

whilst the identity (1.09)

$$X_{11} + X_{22} + X_{33} = 0$$

reduces to,

$$\cos \delta\ (\partial u/\partial\alpha - \partial v/\partial\beta) - \sin \delta(\partial v/\partial\alpha + \partial u/\partial\beta) + 2/(\gamma - 1)\ \frac{\partial a}{\partial \beta} = 0. \quad \ldots \quad (3.15)$$

Relations (3.14) and (3.15) combine to give, finally

$$\left\{\frac{\partial u}{\partial \alpha} - \frac{\partial v}{\partial \beta}\right\} + \frac{2}{\gamma - 1}\left\{\sin \delta \frac{\partial a}{\partial \alpha} + \cos \delta \frac{\partial a}{\partial \beta}\right\} = 0 \quad \ldots \qquad (3.16)$$

$$\left\{\frac{\partial u}{\partial \beta} + \frac{\partial v}{\partial \alpha}\right\} + \frac{2}{\gamma - 1}\left\{\cos \delta \frac{\partial a}{\partial \alpha} - \sin \delta \frac{\partial a}{\partial \beta}\right\} = 0. \quad \ldots \qquad (3.17)$$

These relations would form the basis of numerical computation along a hexahedral grid, as suggested in Section 1.

For $v = 0, \quad \delta = \pi/2,$

or $u = 0, \quad \delta = 0,$ respectively,

relations (3.16) and (3.17) reduce to the familiar integrals for unsteady flow in one dimension,

$$u \pm \frac{2a}{\gamma - 1} = \text{constant, along the characteristics } \frac{dx}{dt} = (u \pm a)$$

or

$$v \pm \frac{2a}{\gamma - 1} = \text{constant, along the characteristics } \frac{dy}{dt} = (v \pm a).$$

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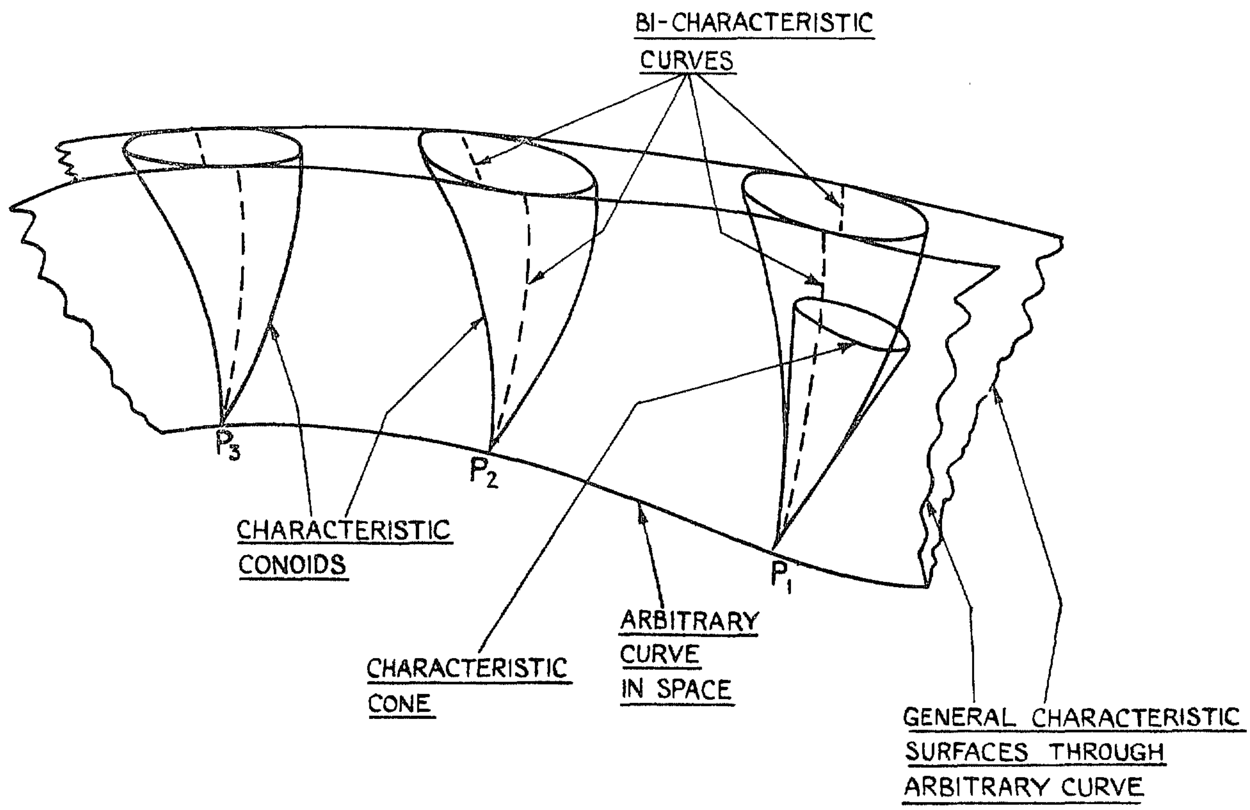

FIG. 1.

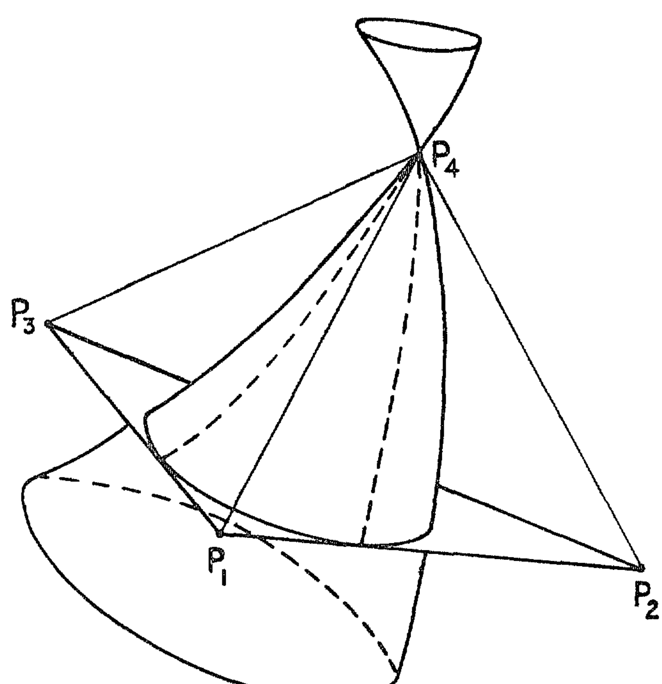

FIG. 2.

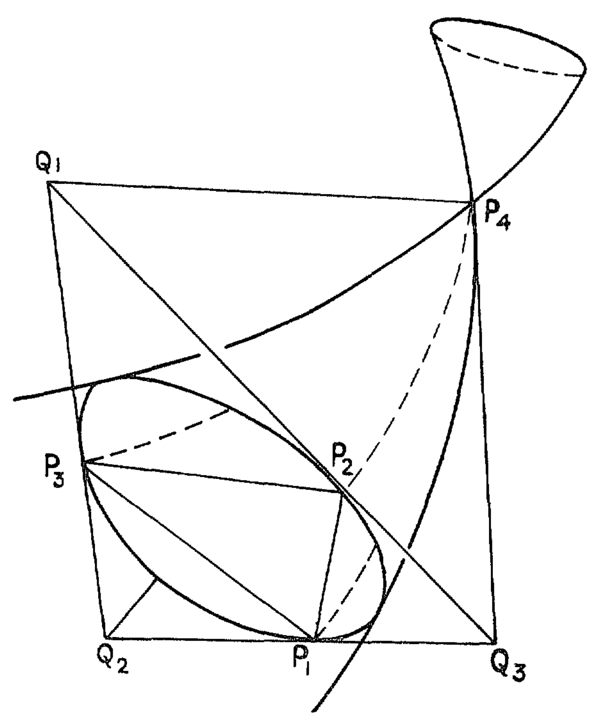

FIG. 3.

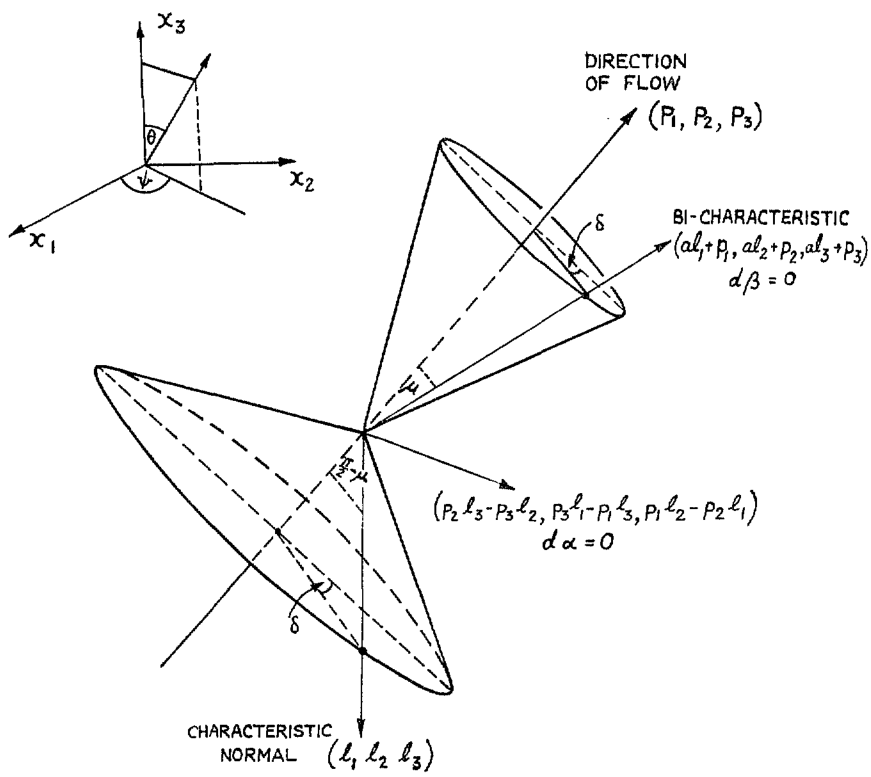

FIG. 4.

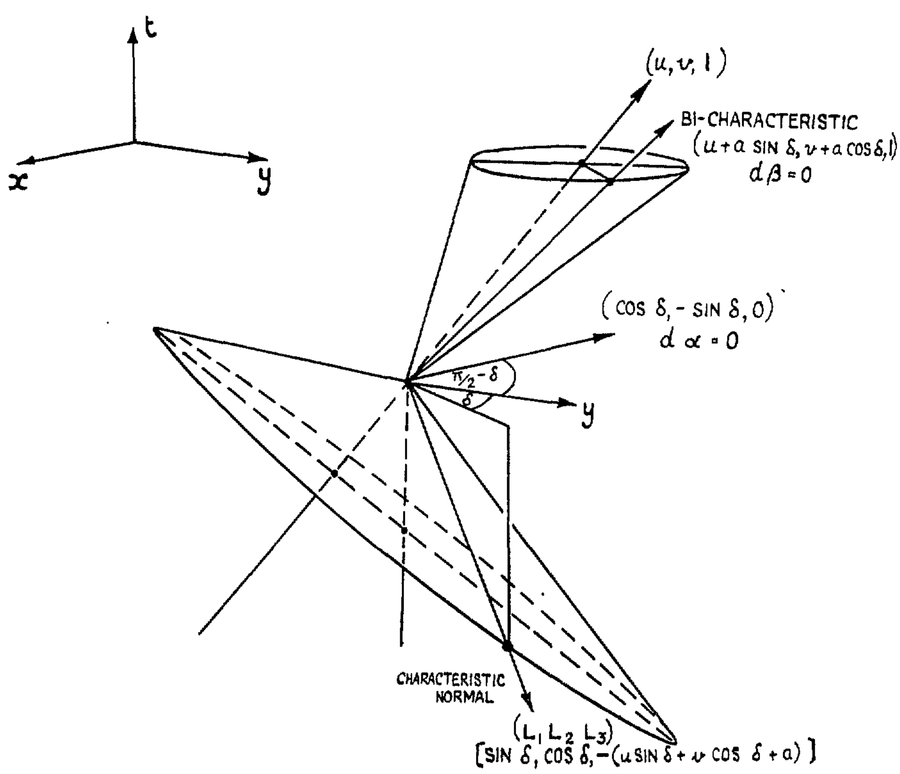

FIG. 5.

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