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1. *Summary and Introduction.*—The method presented here for obtaining an approximate solution of the laminar boundary-layer equations is based on the iteration process of Piercy and Preston[1]. It leads to a simple analytical approximation of good accuracy for Blasius' solution of the boundary-layer flow past a flat plate. The main purpose of this paper is, however, the application of the method to a generalisation of Blasius' problem, namely the case of a flat plate in a uniform stream when there is a suction velocity normal to the plate proportional to $x^{-1/2}$ where x is the distance along the plate from its leading edge. This generalisation was first given by Schlichting and Bussmann[2], and has also been considered by Thwaites[3] and Watson[4].

For the simpler problem of the flat plate in a uniform stream it is well known that by means of Blasius' transformation the solution is obtained from that of a third-order non-linear differential equation. The iteration method of Piercy and Preston for the solution of this consists in replacing the velocity where it occurs in the equation by an inferior approximation and solving the resultant linear equation to obtain a superior approximation. To start the process the velocity was assumed to be that of the stream, giving Oseen's solution as the next approximation. Here the start is made in a different manner. We take as the initial approximation to the velocity one of two choices—(i) a constant value or (ii) a linear function—and in either case have a parameter at our disposal. The iteration is performed, giving a second approximation containing this parameter, which we then determine by substituting the second approximation in the momentum equation. The necessary integrations can be performed analytically, and the quantities τ_0, δ^*, θ and H which characterise the boundary layer are readily determined. The following table shows the results achieved.

TABLE 1

Theory	$\frac{\tau_0}{\rho U^2}\left(\frac{Ux}{\nu}\right)^{1/2}$	$\theta\left(\frac{U}{\nu x}\right)^{1/2}$	$\delta^*\left(\frac{U}{\nu x}\right)^{1/2}$	$H = \frac{\delta^*}{\theta}$
Method (i)	0·363	0·726	1·753	2·414
Method (ii)	0·329	0·658	1·721	2·617
Exact	0·33206	0·66412	1·7208	2·5911

In the case of the more general problem Blasius' transformation gives the same non-linear differential equation, but with a different boundary condition. We treat this by exactly the same method, and since unfortunately the second choice of our initial approximation does not

(93014) A

yield a simple answer, the first choice only is considered here. The results obtained for the quantities τ_0, δ^*, θ and H are given in the tables and graphs. We find that when the suction velocity is large, the velocity distribution in the boundary layer approximates to the asymptotic suction profile, a result which has been proved more rigorously in R. & M. 2298[4]. The present method has the advantage over the differential analyser and asymptotic solutions of giving a simple analytical formula of fair accuracy for the whole range $0 \leqslant k < \infty$ of the suction, and the solutions can be improved numerically by further applications of the iteration process.

2. *Flat Plate Without Suction.*—The equations of the laminar boundary layer are, in the usual notation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} , \qquad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \qquad (2)$$

Equation (2) implies the existence of a stream function ψ such that

$$\left. \begin{aligned} u &= \frac{\partial \psi}{\partial y} \\ v &= -\frac{\partial \psi}{\partial x} \end{aligned} \right\} . \qquad (3)$$

For the flat plate in a uniform stream U is constant, and it is well known that the assumption

$$\psi = (\nu U x)^{1/2} f(\eta) , \qquad (4)$$

where

$$\eta = \left(\frac{U}{\nu x} \right)^{1/2} y , \qquad (5)$$

satisfies the equation of motion (1) provided that

$$f'''(\eta) + \tfrac{1}{2} f(\eta) f''(\eta) = 0 . \qquad (6)$$

In this case

$$\left. \begin{aligned} u &= U f'(\eta) , \\ v &= \tfrac{1}{2} \left(\frac{U\nu}{x} \right)^{1/2} (\eta f'(\eta) - f(\eta)) , \end{aligned} \right\} \qquad (7)$$

and hence we obtain the boundary conditions for equation (6),

$$\left. \begin{aligned} \eta = 0: &\quad f = f' = 0 \\ \eta = \infty: &\quad f' = 1 \end{aligned} \right\} . \qquad (8)$$

To apply the iterative method of Piercy and Preston[1], equation (6) is replaced by

$$f_n''' + \tfrac{1}{2} f_{n-1} f_n'' = 0 , \qquad (9)$$

where f_{n-1} is the known inferior, and f_n the required superior, approximation to f. The solution of this equation with the boundary conditions (8) gives

$$f_n'(\eta) = A_n \int_0^\eta \exp \{ - \tfrac{1}{2} \int_0^\eta f_{n-1}(\eta)\, d\eta \}\, d\eta , \qquad (10)$$

where

$$\frac{1}{A_n} = \int_0^\infty \exp \{ - \tfrac{1}{2} \int_0^\eta f_{n-1}(\eta)\, d\eta \}\, d\eta . \qquad (11)$$

In Ref. 1 the iteration was started by taking $f_1'(\eta) = 1$, corresponding to the inviscid case, but here $f_1'(\eta)$ will be chosen so as to contain a parameter—which from (10) will occur also in $f_2'(\eta)$—and this parameter will be determined by the momentum equation, which in the present case has the form

$$\frac{\tau_0}{\rho U^2} = \frac{d\theta}{dx} , \quad \ldots \quad (12)$$

where

$$\frac{\tau_0}{\rho U^2} = \frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left(\frac{\nu}{Ux}\right)^{1/2} f''(0) . \quad \ldots \quad (13)$$

and

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy = \int_0^\infty f'(\eta)\,(1 - f'(\eta))\,d\eta \left(\frac{\nu x}{U}\right)^{1/2} . \quad \ldots \quad (14)$$

Thus (12) reduces to

$$f''(0) = \tfrac{1}{2}\int_0^\infty f'(\eta)\,(1 - f'(\eta))\,d\eta . \quad \ldots \quad (15)$$

It will appear that a close approximation is produced by the combination of a single iteration with the momentum equation for either of the two choices of $f_1'(\eta)$ considered.

(i) $f_1'(\eta) = K$.—For the first choice we take $f_1'(\eta)$ to be an arbitrary constant K instead of 1. This gives, since $f_1(0) = 0$,

$$f_1(\eta) = K\eta ,$$

and

$$\int_0^\eta f_1(\eta)\,d\eta = \tfrac{1}{2}K\eta^2 .$$

Hence from (10)

$$f_2'(\eta) = A_2 \int_0^\eta e^{-\frac{1}{4}K\eta^2}\,d\eta , \quad \ldots \quad (16)$$

$$= A_2 \frac{2}{\sqrt{K}} \int_0^\xi e^{-\xi^2}\,d\xi ,$$

where

$$\xi = \tfrac{1}{2}\sqrt{K}\eta . \quad \ldots \quad (17)$$

Also, by (11),

$$\frac{1}{A_2} = \frac{2}{\sqrt{K}} \cdot \frac{\sqrt{\pi}}{2} ,$$

and therefore

$$f_2'(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-\xi^2}\,d\xi = \operatorname{erf} \xi . \quad \ldots \quad (18)$$

For equation (15) we have

$$f_2''(\eta) = \frac{2}{\sqrt{\pi}} \cdot \tfrac{1}{2}\sqrt{K} \cdot e^{-\xi^2} ,$$

and thus

$$f_2''(0) = \sqrt{\left(\frac{K}{\pi}\right)} \quad \ldots \quad (19)$$

Also it is easily shown that

Hence from (15)

$$\int_0^\infty \text{erf}\,\xi\,(1-\text{erf}\,\xi)\,d\xi = \frac{\sqrt{2}-1}{\sqrt{\pi}}\,. \qquad \ldots \quad (20)$$

$$\sqrt{\left(\frac{K}{\pi}\right)} = \tfrac{1}{2}\int_0^\infty f_2'(\eta)\,(1-f_2'(\eta))\,d\eta\,,$$

$$= \tfrac{1}{2}\frac{2}{\sqrt{K}}\int_0^\infty \text{erf}\,\xi\,(1-\text{erf}\,\xi)\,d\xi\,,$$

$$= \frac{\sqrt{2}-1}{\sqrt{(K\pi)}}\,,$$

and so

$$K = \sqrt{2}-1\,. \qquad \ldots \quad (21)$$

Therefore

$$f_2'(\eta) = \text{erf}\,(\tfrac{1}{2}\eta\sqrt{(\sqrt{2}-1)})\,. \qquad \ldots \quad (22)$$

This gives for the characteristics τ_0, θ, δ^* and H of the boundary layer, the values

$$\frac{\tau_0}{\rho U^2} = \frac{K}{\sqrt{\pi}}\left(\frac{\nu}{Ux}\right)^{1/2} = \left(\frac{\sqrt{2}-1}{\pi}\cdot\frac{\nu}{Ux}\right)^{1/2} = 0\cdot363\left(\frac{\nu}{Ux}\right)^{1/2}, \qquad \ldots \quad (23)$$

$$\theta = 2\left(\frac{\sqrt{2}-1}{\pi}\cdot\frac{\nu}{Ux}\right)^{1/2} = 0\cdot726\left(\frac{\nu x}{U}\right)^{1/2}, \qquad \ldots \quad (24)$$

$$\delta^* = \int_0^\infty \left(1-\frac{u}{U}\right)dy\,,$$

$$= \frac{2}{\sqrt{K}}\left(\frac{\nu x}{U}\right)^{1/2}\int_0^\infty (1-\text{erf}\,\xi)\,d\xi\,,$$

$$= 2\left(\frac{1}{\pi\,(\sqrt{2}-1)}\cdot\frac{\nu x}{U}\right)^{1/2} = 1\cdot753\left(\frac{\nu x}{U}\right)^{1/2}, \qquad \ldots \quad (25)$$

$$H = \frac{\delta^*}{\theta} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1 = 2\cdot414\,. \qquad \ldots \quad (26)$$

These results compare quite well with the accurate values, but an improvement can be made by taking a different choice of $f_1'(\eta)$.

(ii) $f_1(\eta) = K\eta$.—It is the behaviour of $f'(\eta)$ when η is fairly small that is of greatest importance for the determination of τ_0, θ and δ^*, and so it is desirable to make $f_1(\eta)$ as close as possible in this range to the accurate values. This suggests that a linear function of η may give a better result than a constant for the choice of $f_1'(\eta)$, and it will be seen that this is in fact the case. We have as before

$$f_1(\eta) = \tfrac{1}{2}K\eta^2\,,$$

$$\int_0^\eta f_1(\eta)\,d\eta = \tfrac{1}{6}\,K\eta^3\,,$$

$$f_2'(\eta) = A_2\int_0^\eta e^{-\frac{1}{12}K\eta^3}d\eta\,,$$

$$= A_2\left(\frac{12}{K}\right)^{1/3}\int_0^\xi e^{-\xi^3}\,d\xi\,, \qquad \ldots \quad (27)$$

where

$$\xi = \left(\frac{K}{12}\right)^{1/3}\eta\,. \qquad \ldots \quad (28)$$

Let
$$F(\xi) = \int_0^{\xi} e^{-z^3}\, dz \quad \ldots \quad (29)$$
$$= \tfrac{1}{3}\,\Gamma\,(\tfrac{1}{3}, \xi^3)\,, \quad \ldots \quad (30)$$
an incomplete gamma function. Then
$$F(\infty) = \tfrac{1}{3}\Gamma\,(\tfrac{1}{3}) = \Gamma\,(\tfrac{4}{3})\,, \quad \ldots \quad (31)$$
and therefore
$$f_2'(\eta) = \frac{1}{\Gamma(4/3)}\, F(\xi)\,. \quad \ldots \quad (32)$$
Hence
$$f_2''(0) = \frac{1}{\Gamma(4/3)} \left(\frac{K}{12}\right)^{1/3}, \quad \ldots \quad (33)$$
and
$$\int_0^{\infty} f_2'(\eta)\,(1 - f_2'(\eta))\, d\eta = \left(\frac{12}{K}\right)^{1/3} \int_0^{\infty} \frac{F(\xi)}{\Gamma(4/3)} \left(1 - \frac{F(\xi)}{\Gamma(4/3)}\right) d\xi\,,$$
$$= \left(\frac{12}{K}\right)^{1/3} \frac{1}{\{\Gamma(4/3)\}^2}\, I\,, \quad \ldots \quad (34)$$
where
$$I = \int_0^{\infty} F(\xi)\,\{\Gamma(4/3) - F(\xi)\}\, d\xi\,. \quad \ldots \quad (35)$$
Thus (15) becomes
$$\frac{1}{\Gamma(4/3)} \left(\frac{K}{12}\right)^{1/3} = \tfrac{1}{2} \left(\frac{12}{K}\right)^{1/3} \frac{1}{\{\Gamma(4/3)\}^2}\, I\,,$$
and therefore
$$\left(\frac{K}{12}\right)^{2/3} = \frac{I}{2\Gamma(4/3)}\,. \quad \ldots \quad (36)$$
Now
$$I = \int_0^{\infty} F(\xi)\,\{\Gamma(4/3) - F(\xi)\}\, d\xi\,,$$
$$= \Big[F(\xi)\,\{\Gamma(4/3) - F(\xi)\}\,\xi\Big]_0^{\infty} - \int_0^{\infty} \{\Gamma(4/3) - 2F(\xi)\}\, e^{-\xi^3}\,\xi d\xi\,,$$
$$= -\,\Gamma(4/3) \int_0^{\infty} \xi e^{-\xi^3}\, d\xi + 2 \int_0^{\infty} \xi e^{-\xi^3}\, d\xi \int_0^{\xi} e^{-z^3}\, dz\,.$$

The first integral is expressible as a gamma function by writing $\xi = t^{1/3}$, and the second is evaluated by putting
$$z = \xi t \quad \ldots \quad (37)$$
and inverting the order of integration. We obtain
$$I = -\,\Gamma(4/3)\,.\,\tfrac{1}{3}\,\Gamma\,(\tfrac{2}{3}) + 2 \int_0^{1} dt \int_0^{\infty} \xi^2 \exp\{-\,\xi^3\,(1 + t^3)\}\, d\xi\,,$$
$$= -\,\tfrac{1}{9}\,\frac{\pi}{\sin\frac{1}{3}\pi} + \tfrac{2}{3} \int_0^{1} \frac{dt}{1 + t^3}$$
$$= \tfrac{2}{9}\,\log_e\,2\,. \quad \ldots \quad (38)$$
therefore
$$\left(\frac{K}{12}\right)^{2/3} = \frac{1}{9}\,\frac{\log_e 2}{\Gamma(4/3)}\,. \quad \ldots \quad (39)$$

Calculating τ_0, θ, δ^* and H as for case (i), we find

$$\frac{\tau_0}{\rho U^2} = \frac{1}{3}\frac{\{\log_e 2\}^{1/2}}{\{\Gamma(4/3)\}^{3/2}}\left(\frac{\nu}{Ux}\right)^{1/2} = 0\cdot 329\left(\frac{\nu}{Ux}\right)^{1/2}; \quad \ldots \quad (40)$$

$$\theta = \frac{2}{3}\frac{\{\log_e 2\}^{1/2}}{\{\Gamma(4/3)\}^{3/2}}\left(\frac{\nu x}{U}\right)^{1/2} = 0\cdot 658\left(\frac{\nu x}{U}\right)^{1/2}; \quad \ldots \quad (41)$$

$$\delta^* = \frac{2\pi}{3\sqrt{3}\{\log_e 2\}^{1/2}\{\Gamma(4/3)\}^{3/2}}\left(\frac{\nu x}{U}\right)^{1/2} = 1\cdot 721\left(\frac{\nu x}{U}\right)^{1/2}; \quad \ldots \quad (42)$$

$$H = \frac{\pi}{\sqrt{3}\log_e 2} = 2\cdot 617. \quad \ldots \quad (43)$$

These values differ from the accurate results by 1 per cent. at most.

The velocity distributions (22) and (32) are tabulated in Table 2 and are illustrated in Fig. 1, with the corresponding first approximations $f_1'(\eta)$. The approximation by the second method is good, for the difference in u/U between it and the accurate solution is never greater than $0\cdot 004$. To make the next step of the iteration by inserting (32) into the equation of motion would require numerical integration, but the result would be very close to the exact solution, and would involve much less labour than the direct numerical solution of the differential equation by Adams' method.

3. *Flat Plate with Suction Proportional to* $x^{-1/2}$.—The problem of the flow over a flat plate when there is a suction flow normal to the plate of strength proportional to $x^{-1/2}$ has been solved by Schlichting and Bussmann[2], and further solutions were given by Thwaites[3]. An approximate solution can be found by the methods of this paper in the following manner.

It was shown above (equation (7)) that

$$v = \tfrac{1}{2}\left(\frac{U\nu}{x}\right)^{1/2}\left(\eta f'(\eta) - f(\eta)\right). \quad \ldots \quad (44)$$

Consequently, if

$$f(0) = 2k, \quad \ldots \quad (45)$$

the velocity of suction is

$$v_0 = -v(x, 0) = k\left(\frac{U\nu}{x}\right)^{1/2}. \quad \ldots \quad (46)$$

The momentum equation has the form (for constant U)

$$\frac{d\theta}{dx} = -\frac{v_0}{U} + \frac{\tau_0}{\rho U^2}, \quad \ldots \quad (47)$$

which for the purpose of application is

$$\tfrac{1}{2}\int_0^\infty f'(1 - f')\, d\eta = f''(0) - k. \quad \ldots \quad (48)$$

The second choice of $f'_1(\eta)$ leads to rather intractable results involving the Airy-Hardy integral

$$\int_0^\infty e^{-t^3 - 3at}\, dt,$$

but the first method is easily applicable.

Let

$$f_1'(\eta) = K. \quad \ldots \quad (49)$$

Then
$$f_1(\eta) = K\eta + 2k\,,$$
$$\int_0^\eta f_1(\eta)\,d\eta = \tfrac{1}{2}K\eta^2 + 2k\eta\,.$$

Hence from (10)
$$f_2'(\eta) = A_2 \int_0^\eta e^{-\frac{1}{2}K\eta^2 - k\eta}\,d\eta\,. \qquad (50)$$

Put
$$\xi = \tfrac{1}{2}\sqrt{K}\eta\,, \qquad (51)$$
$$a = \frac{k}{\sqrt{K}}\cdot \qquad (52)$$

Then
$$f_2'(\eta) = A_2 \frac{2}{\sqrt{K}} \int_0^\xi e^{-\xi^2 - a\xi}\,d\xi\,,$$
and so, from (11),
$$f_2'(\eta) = \frac{\operatorname{erf}(\xi + a) - \operatorname{erf} a}{1 - \operatorname{erf} a}\,. \qquad (53)$$

Hence
$$f_2''(0) = \frac{1}{1 - \operatorname{erf} a} \frac{2}{\sqrt{\pi}} e^{-a^2} \frac{\sqrt{K}}{2}$$
$$= \sqrt{\left(\frac{K}{\pi}\right)} \cdot \frac{e^{-a^2}}{1 - \operatorname{erf} a}\,. \qquad (54)$$

Also
$$\int_0^\infty f_2'\,(1 - f_2')\,d\eta = \frac{2}{\sqrt{K}} \frac{I}{(1 - \operatorname{erf} a)^2}\,, \qquad (55)$$

where
$$I = \int_0^\infty [\operatorname{erf}(\xi + a) - \operatorname{erf} a]\,[1 - \operatorname{erf}(\xi + a)]\,d\xi\,,$$
$$= \sqrt{\left(\frac{2}{\pi}\right)} (1 - \operatorname{erf} a\sqrt{2}) - \frac{1}{\sqrt{\pi}} e^{-a^2} (1 - \operatorname{erf} a)\,, \qquad (56)$$

on integrating by parts in a manner similar to that employed for (20).

Hence we obtain by use of the momentum equation
$$K = \frac{\sqrt{2}\,(1 - \operatorname{erf} a\sqrt{2}) - e^{-a^2}(1 - \operatorname{erf} a)}{(1 - \operatorname{erf} a)\,(e^{-a^2} - a\sqrt{\pi}\,(1 - \operatorname{erf} a))}\,. \qquad (57)$$

Then with K given by (57) we find that
$$\frac{\tau_0}{\rho U^2} = \frac{e^{-a^2}}{1 - \operatorname{erf} a} \sqrt{\left(\frac{K}{\pi}\right)} \left(\frac{\nu}{Ux}\right)^{1/2}, \qquad (58)$$
$$\theta = \frac{2}{\sqrt{(K\pi)}} \cdot \frac{\sqrt{2}\,(1 - \operatorname{erf} a\sqrt{2}) - e^{-a^2}(1 - \operatorname{erf} a)}{(1 - \operatorname{erf} a)^2} \left(\frac{\nu x}{U}\right)^{1/2}, \qquad (59)$$
$$\delta^* = \frac{2}{\sqrt{(K\pi)}} \frac{e^{-a^2} - a\sqrt{\pi}\,(1 - \operatorname{erf} a)}{1 - \operatorname{erf} a} \left(\frac{\nu x}{U}\right)^{1/2}, \qquad (60)$$
$$H = \frac{(1 - \operatorname{erf} a)\,\{e^{-a^2} - a\sqrt{\pi}\,(1 - \operatorname{erf} a)\}}{\sqrt{2}\,(1 - \operatorname{erf} a\sqrt{2}) - e^{-a^2}(1 - \operatorname{erf} a)} = \frac{1}{K}\,. \qquad (61)$$

When k, the rate of suction, is large, the velocity profile (53) approximates to the asymptotic suction profile.

For when a is large

$$1 - \operatorname{erf} a \simeq \frac{2}{\sqrt{\pi}} e^{-a^2}\left(\frac{1}{2a} - \frac{1}{4a^3}\right), \qquad (62)$$

and consequently we find that $\quad K \to \frac{1}{2}. \qquad (63)$

Hence

$$k \simeq \frac{a}{\sqrt{2}}, \qquad (64)$$

and

$$\frac{\tau_0}{\rho U^2} \simeq k\left(\frac{\nu}{Ux}\right)^{1/2} \qquad (65)$$

$$\theta \simeq \frac{1}{2k}\left(\frac{\nu x}{U}\right)^{1/2}, \qquad (66)$$

$$\delta^* \simeq \frac{1}{k}\left(\frac{\nu x}{U}\right)^{1/2}, \qquad (67)$$

$$H \to 2. \qquad (68)$$

For the velocity distribution we have

$$f_2' = \frac{\operatorname{erf}(\frac{1}{2}\sqrt{K}\eta + a) - \operatorname{erf} a}{1 - \operatorname{erf} a}$$

$$\simeq \frac{\frac{2}{\sqrt{\pi}} e^{-(\frac{1}{2}\sqrt{K}\eta + a)^2} \cdot \frac{1}{2(\frac{1}{2}\sqrt{K}\eta + a)} - \frac{2}{\sqrt{\pi}} e^{-a^2} \frac{1}{2a}}{\frac{2}{\sqrt{\pi}} e^{-a^2} \cdot \frac{1}{2a}},$$

$$\simeq 1 - \exp(-\tfrac{1}{4}K\eta^2 - a\sqrt{K}\eta),$$

$$\simeq 1 - \exp\left(-\tfrac{1}{8}\eta^2 - \frac{a}{\sqrt{2}}\eta\right). \qquad (69)$$

Now when a is large the range of η for which f_2' lies between 0 and $1 - \varepsilon$ (for a fixed positive ε) is $O(1/a)$ and in this range we may write

$$f_2' \simeq 1 - \exp\left(-\frac{a}{\sqrt{2}}\eta\right).$$

Hence, provided that $1 - f_2'$ is not small,

$$f_2' \simeq 1 - e^{-k\eta}, \qquad (70)$$

$$\simeq 1 - 2^{-y/\delta^*}, \qquad (71)$$

which is the equation of the asymptotic suction profile. This result has been established by a more accurate investigation in which an asymptotic series was found for u/U and also series for the results (65) to (68)[4].

In Fig. 2 some of the velocity profiles are shown plotted against y/δ^*. Table 3 gives values of K, k, τ_0, θ, δ^* and H for a wide range of a. For small values of a these were obtained from tables of the error function and for large values from its asymptotic expansion. The variation of τ_0. θ, δ^*, and H with k is shown in Figs. 3 and 4, which are derived from Table 3, and the accurate results are shown also for comparison.

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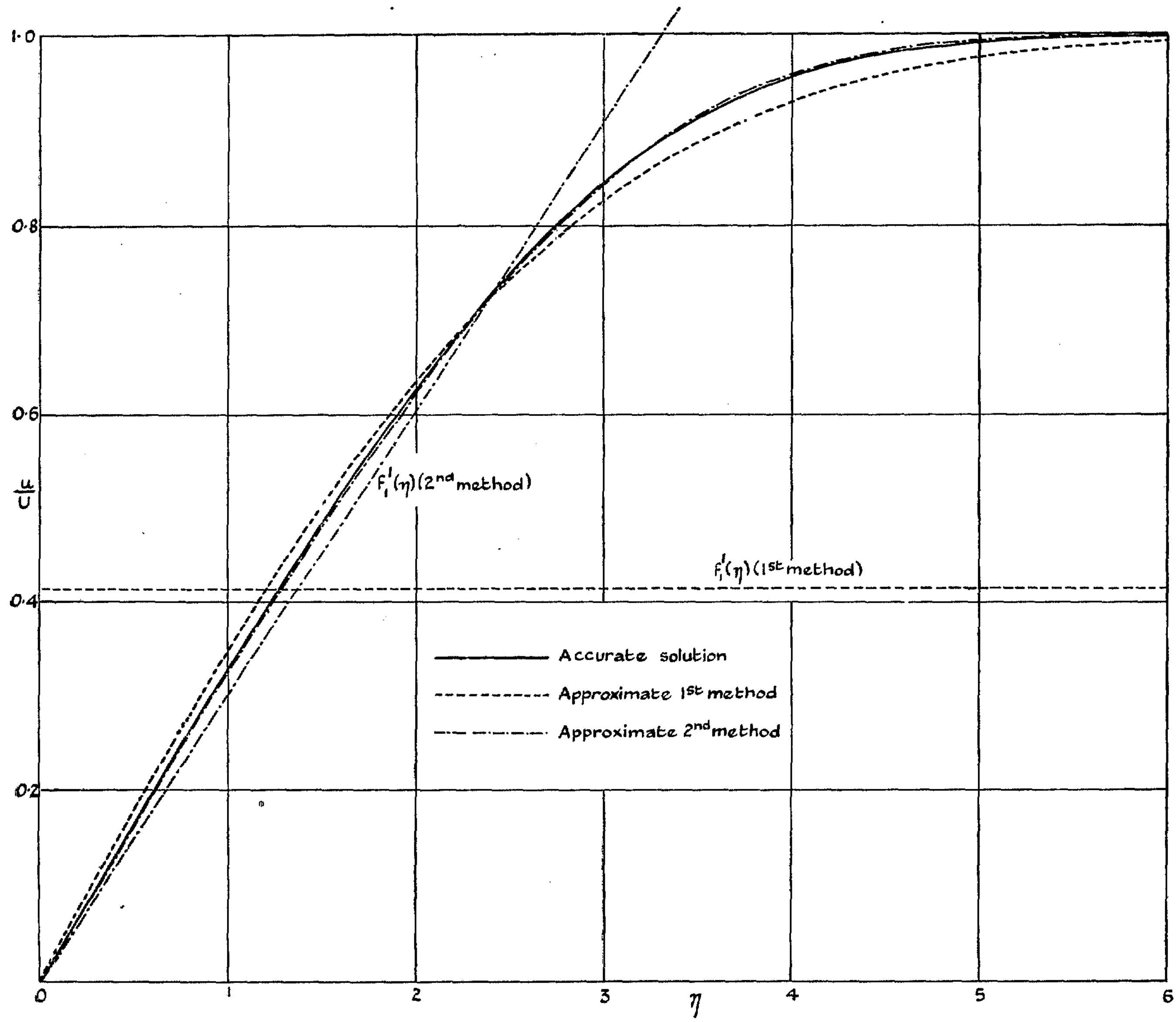
TABLE 2

Method (i)			*Method* (ii)		
ξ	erf ξ	η	ξ	$\frac{1}{\Gamma(4/3)}\int_0^{\xi} e^{-t^3}\,dt$	η
0	0	0	0	0	0
0·1	0·11246	0·31075	0·1	0·11196	0·34051
0·2	0·22270	0·62151	0·2	0·22352	0·68102
0·3	0·32863	0·93226	0·3	0·33370	1·02153
0·4	0·42839	1·24302	0·4	0·44090	1·36204
0·5	0·52050	1·55377	0·5	0·54303	1·70255
0·6	0·60386	1·86453	0·6	0·63776	2·04306
0·7	0·67780	2·17528	0·7	0·72277	2·38357
0·8	0·74210	2·48604	0·8	0·79616	2·72408
0·9	0·79691	2·79679	0·9	0·85674	3·06459
1	0·84270	3·10755	1	0·90429	3·40509
1·1	0·88021	3·41830	1·1	0·93955	3·74560
1·2	0·91031	3·72906	1·2	0·96411	4·08611
1·3	0·93401	4·03981	1·3	0·98008	4·42662
1·4	0·95229	4·35057	1·4	0·98974	4·76713
1·5	0·96611	4·66132	1·5	0·99511	5·10764
1·6	0·97635	4·97208	1·6	0·99787	5·44815
1·7	0·98379	5·28283	1·7	0·99915	5·78866
1·8	0·98909	5·59359	1·8	0·99969	6·12917
1·9	0·99279	5·90434	1·9	0·99990	6·46968
2	0·99532	6·21510	2	0·99997	6·81019

TABLE 3

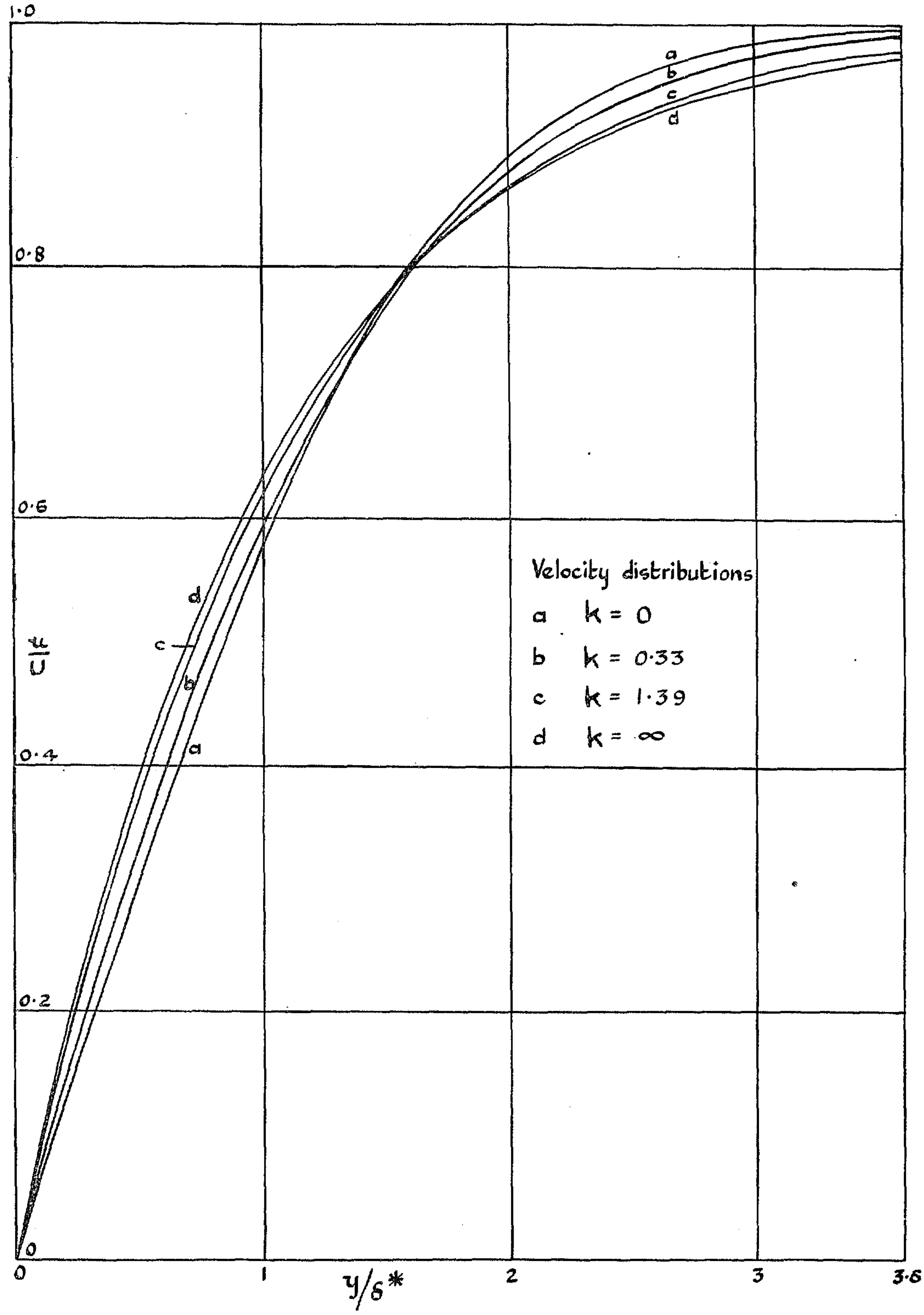
Method (i)

a	K	k	$\frac{\tau_0}{\rho U^2}\left(\frac{Ux}{\nu}\right)^{1/2}$	$\theta\left(\frac{U}{\nu x}\right)^{1/2}$	$\delta^*\left(\frac{U}{\nu x}\right)^{1/2}$	H
0	0·41421	0	0·36311	0·72622	1·75325	2·41421
0·5	0·44385	0·33311	0·61049	0·55476	1·24988	2·25301
1	0·46249	0·68007	0·89734	0·43453	0·93954	2·16221
1·5	0·47409	1·03281	1·20797	0·35033	0·73895	2·10930
2	0·48146	1·38775	1·53282	0·29015	0·60264	2·07702
2·5	0·48628	1·74334	1·86631	0·24595	0·50577	2·05643
3	0·48953	2·09899	2·20526	0·21253	0·43414	2·04278
3·5	0·49180	2·45451	2·54781	0·18661	0·37944	2·03333
4	0·49343	2·80980	2·89282	0·16604	0·33650	2·02661
4·5	0·49465	3·16490	3·23959	0·14938	0·30200	2·02164
5	0·49555	3·51978	3·58761	0·13565	0·27374	2·01794
6	0·49681	4·22907	4·28628	0·11442	0·23030	2·01285
7	0·49760	4·93787	4·98728	0·09881	0·19858	2·00963
8	0·49814	5·64632	5·68977	0·08690	0·17444	2·00747
9	0·49852	6·35451	6·39326	0·07751	0·15548	2·00595
10	0·49879	7·06250	7·09747	0·06994	0·14021	2·00486
15	0·49945	10·60079	10·62424	0·04691	0·09392	2·00219
20	0·49969	14·13775	14·15538	0·03526	0·07056	2·00124
25	0·49980	17·67415	17·68827	0·02823	0·05649	2·00080
30	0·49986	21·21027	21·23381	0·02354	0·04709	2·00055
40	0·49992	28·28207	28·29090	0·01767	0·03534	2·00031
50	0·49995	35·35357	35·36064	0·01414	0·02827	2·00020



1·0
0·8
0·6
0·4
0·2
u/U
0
1
2
3
4
5
6
η
F₁'(η) (2nd method)
F₁'(η) (1st method)
Accurate solution
Approximate 1st method
Approximate 2nd method

FIG. 1.



1·0
0·8
0·6
0·4
0·2
0
u/U
0
1
2
3
3·6
y/δ*
a
b
c
d
Velocity distributions
a k = 0
b k = 0·33
c k = 1·39
d k = ∞

FIG. 2.

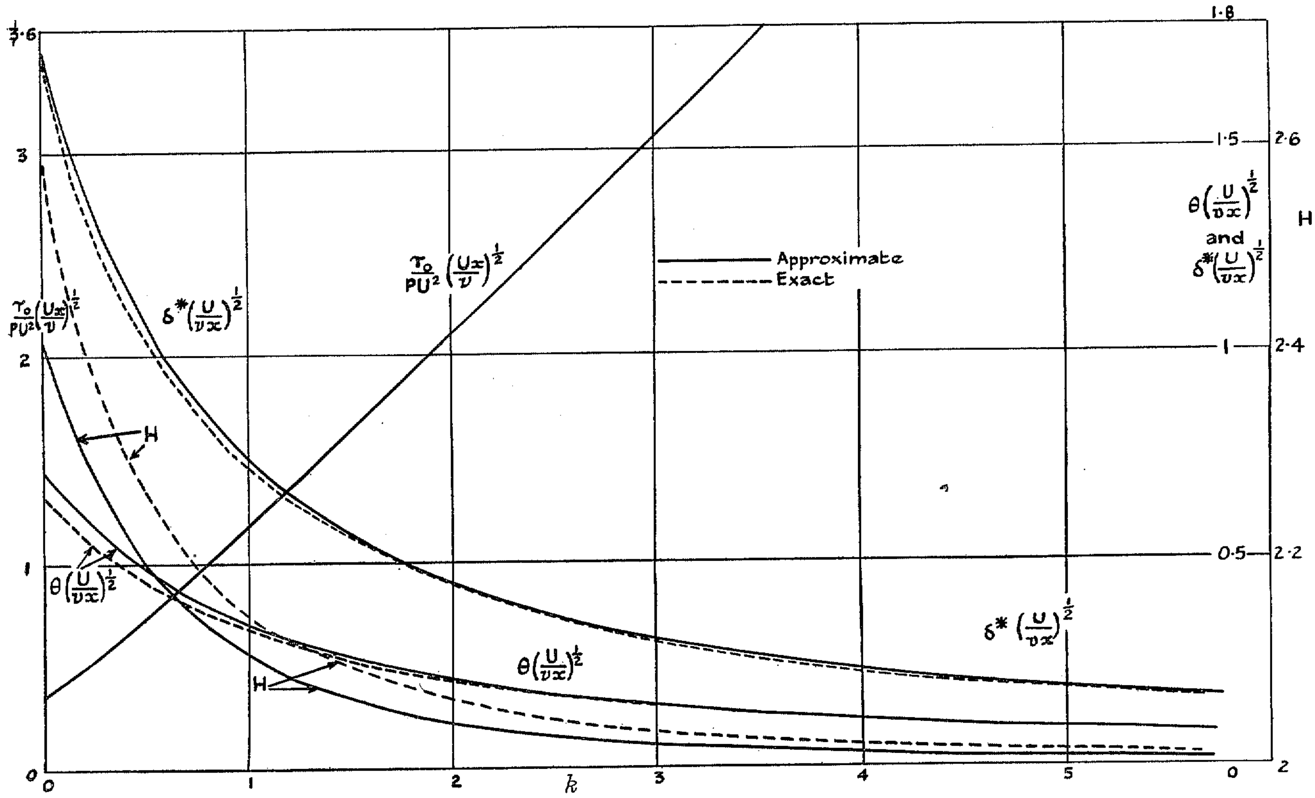


FIG. 3.

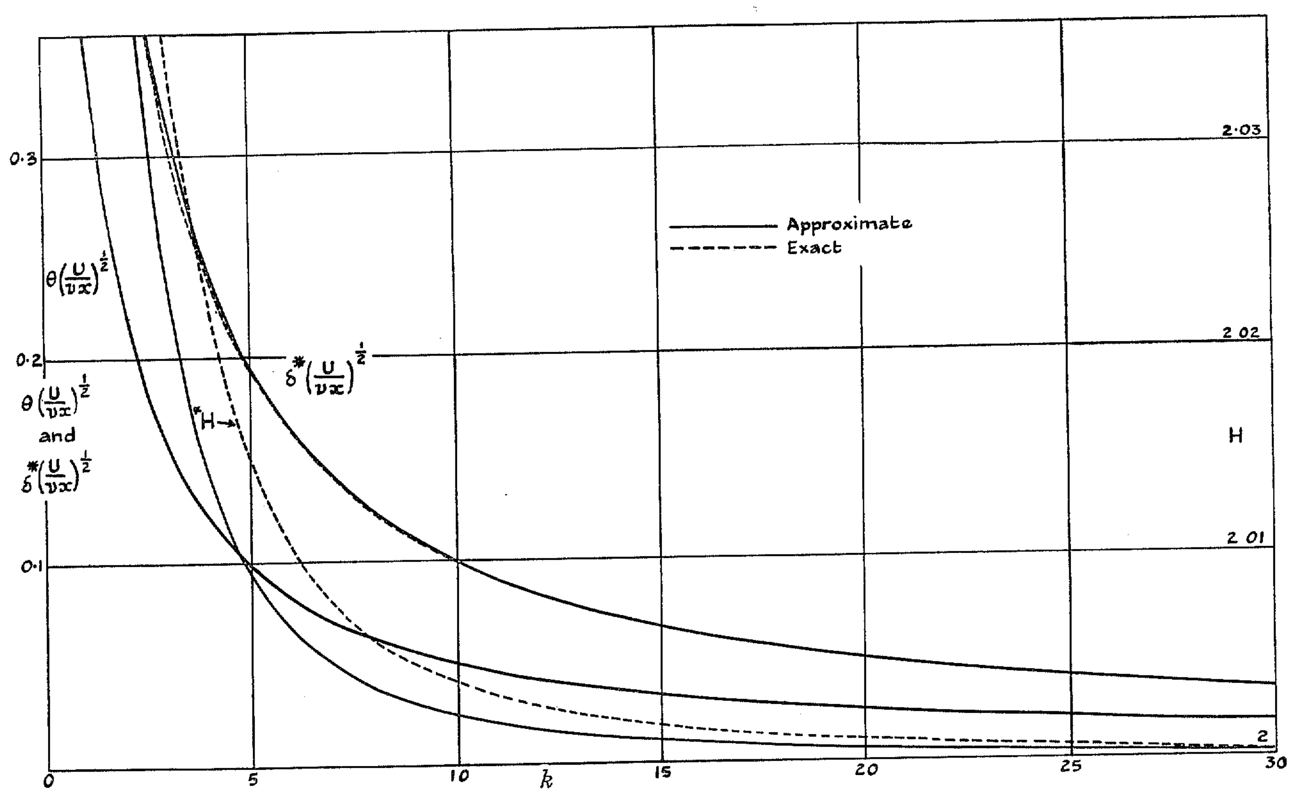


FIG. 4.

(93014) Wt. 13,806 K.5 10/51 Hw.

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