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Summary.—The field of flow round a flat aerofoil at incidence can be regarded in linearised theory as the result of both bound and trailing vortices for supersonic as well as for low-speed flight. This leads to a convenient method, given the lift distribution over an aerofoil, for calculating the flow round it at supersonic speeds.

As an application of the results the downwash is calculated in the wake of a delta wing lying within the Mach cone emanating from its apex. The downwash is found to be least just aft the trailing edge and is everywhere less than the downflow at the aerofoil. It increases steadily to a limiting value which is attained virtually within two chord lengths of the trailing edge. The ratio of the downwash at any point in the wake to the downflow at the aerofoil decreases with increasing Mach number and apex angle.

1. *Introduction.*—In the first paper on three-dimensional supersonic aerofoil theory, written by Schlichting in 1936, the idea of a supersonic horseshoe vortex was used as an auxiliary concept. However, the Prandtl-Lanchester vortex approach, which is of such fundamental importance in low-speed aerofoil theory, has been almost entirely abandoned in the subsequent treatment of the matter. This, of course, is no accident, for it appears that the alternative methods of the supersonic theory lend themselves more readily to the solution of the main problem of finding the pressure distribution over an aerofoil of given shape and incidence ; furthermore there exists no supersonic counterpart to the lifting-line theory to which is due the remarkable success of Prandtl's approach. The purpose of this paper is to show that once the lift distribution is known, the vortex approach can still be of use in determining the flow round an aerofoil.

The general linearised theory of a field of flow due to an arbitrary distribution of vorticity under steady supersonic conditions is developed in the College of Aeronautics Report No. 9, 1947, and is applied in the present paper to aerofoil problems ; in particular the downwash along the continuation of the centre line of a delta wing is calculated for the quasi-subsonic case (apex semi-angle smaller than the Mach angle).

Other methods of determining the field of flow from the pressure distribution, such as first deriving the "acceleration potential" due to an equivalent doublet distribution, at least in some cases, lead to considerably more complicated calculations than those involved in the method adopted here.

*College of Aeronautics Report No. 10, received 20th February, 1948.

2. *Results*.—The downwash, w, along the continuation of the centre-line of a delta wing moving at a supersonic speed such that it lies entirely within the Mach cone emanating from its apex, is given by :—

when $d \leqslant \lambda c$,

$$\frac{w}{V\alpha} = -\frac{2}{\pi E'(\lambda)} \left\{ \frac{\lambda c}{d} \left[E\left(\frac{d}{\lambda c}\right) - \left\{1 - \left(\frac{d}{\lambda c}\right)^2\right\} K\left(\frac{d}{\lambda c}\right) \right] + \int_0^1 \frac{K(k) - E(k)}{k + \lambda}\, dk + \int_{d/\lambda c}^1 \frac{K(k) - E(k)}{k^2(1 + k\lambda)}\, dk \right\} . \quad (1, i)$$

When $d \geqslant \lambda c$,

$$\frac{w}{V\alpha} = -\frac{2}{\pi E'(\lambda)} \left\{ E\left(\frac{\lambda c}{d}\right) + \int_0^{\lambda c/d} \frac{(K)k - E(k)}{k + \lambda}\, dk \right\} , \quad (1, ii)$$

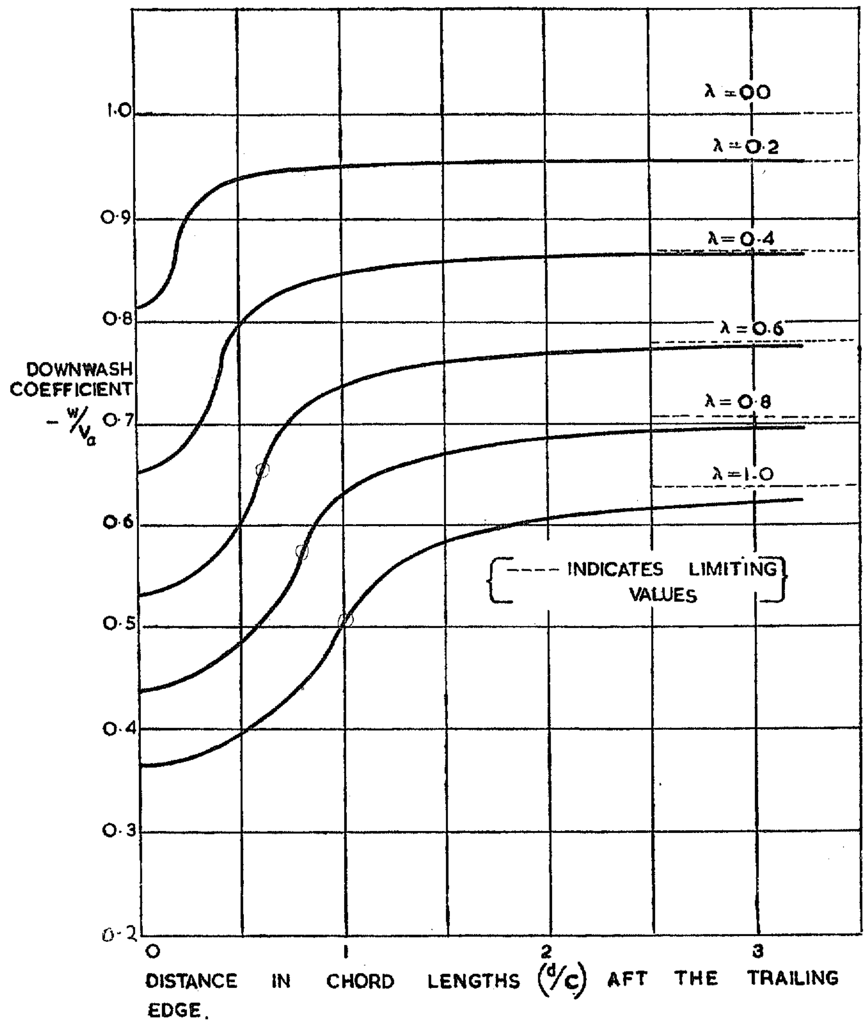

FIG. 1. Downwash distribution for different values of $\lambda = \cot \mu \tan \gamma$.

where K, E and E' are the well-known complete elliptic integrals

V	=	the velocity
α	=	the incidence
c	=	maximum wing chord
d	=	distance aft of the trailing edge
μ	=	Mach angle
γ	=	wing apex semi-angle
λ	=	$\cot \mu \tan \gamma$

$\cot\mu = \beta$ or β^{-1}

The condition $d < \lambda c$ indicates that the point in question is outside the Mach cones emanating from the wing tips, and vice versa (*see* Fig. 4).

The corresponding spanwise lift distribution over the aerofoil as given in R.A.E. Report No. Aero. 2151 (A.R.C. 10222)[3] is:—

$$l(y) = \frac{2\rho V^2 \alpha}{E'(\lambda)} \sqrt{(c^2 \tan^2 \gamma - y^2)}\,, \qquad \ldots \quad (2)$$

where ρ = air density $\qquad y$ = spanwise co-ordinate

As d tends to infinity in (1, *ii*), $w/V\alpha$ tends to $1/E'(\lambda)$, which is exactly the same result as obtained for the downwash in incompressible flow far behind the trailing edge of an aerofoil with spanwise lift given by (2). This is a special case of a more general result stated in Ref. 3, according to which, for a given spanwise lift distribution, the trailing vortex field tends in regions far behind the aerofoil, where the chordwise co-ordinate is large compared to the other co-ordinates, to the same limit in supersonic as in subsonic flow.

In Fig. 1 the downwash is plotted against the distance from the trailing edge for various values of the parameter λ. It will be noted that for $\lambda = 0$, that is for very small aspect ratios or at speeds very near that of sound, the downwash becomes equal to the donwflow at the aerofoil ($w/V\alpha = 1$).

Fig. 2 shows what the downwash would be if the entire lift were regarded as being concentrated at the trailing edge for the given value of 0·4 for λ.

To assist in applying the results given in Fig. 1 to particular cases, the values of λ for specified values of aspect ratio and Mach number can be found from Fig. 3.

3. *Vortex Plane Theory for Supersonic Conditions.*—Consider a flat aerofoil placed approximately in the xy-plane at a small incidence in an airstream of velocity V, greater than that of sound, in the positive x-direction. Then according to linearised theory we have:—

The equation of continuity—

$$-\beta^2 \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0\,, \qquad \ldots \quad (3)$$

where $\beta^2 = M^2 - 1$ $\qquad \Phi$ = velocity potential.

The Eulerian equations—

$$\left.\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial x} &= V\frac{\partial u}{\partial x} \\ -\frac{1}{\rho}\frac{\partial p}{\partial y} &= V\frac{\partial v}{\partial x} \\ -\frac{1}{\rho}\frac{\partial p}{\partial z} &= V\frac{\partial w}{\partial x} \end{aligned}\right\} \qquad \ldots \quad (4)$$

where p = pressure $\qquad u, v, w$ = velocity components.

It is assumed as in subsonic lifting plane theory that the kinematic boundary conditions must be fulfilled at the normal projection of the aerofoil on the xy-plane rather than at the aerofoil itself, and that Φ is continuous everywhere in the fluid except across the wake. The latter is taken to be the strip lying in the xy-plane subtended downstream by the aerofoil. Finally it is assumed that the pressure is continuous across the wake. The exact or approximate validity of these assumptions under supersonic conditions is, in the last resort, a matter for experimental verification.

Since the flow is assumed to be irrotational, $\partial v/\partial x$ and $\partial w/\partial x$ may be replaced in equations (4) by $\partial u/\partial y$ and $\partial u/\partial z$ respectively. Integrating these we obtain the linearised form of Bernoulli's Equation :—

$$Vu + \frac{p}{\rho} = \text{const.}, \qquad \ldots \qquad (5)$$

where the constant is the same throughout space, that is both sides of the wake. It follows that u, like the pressure p, is continuous across the wake.

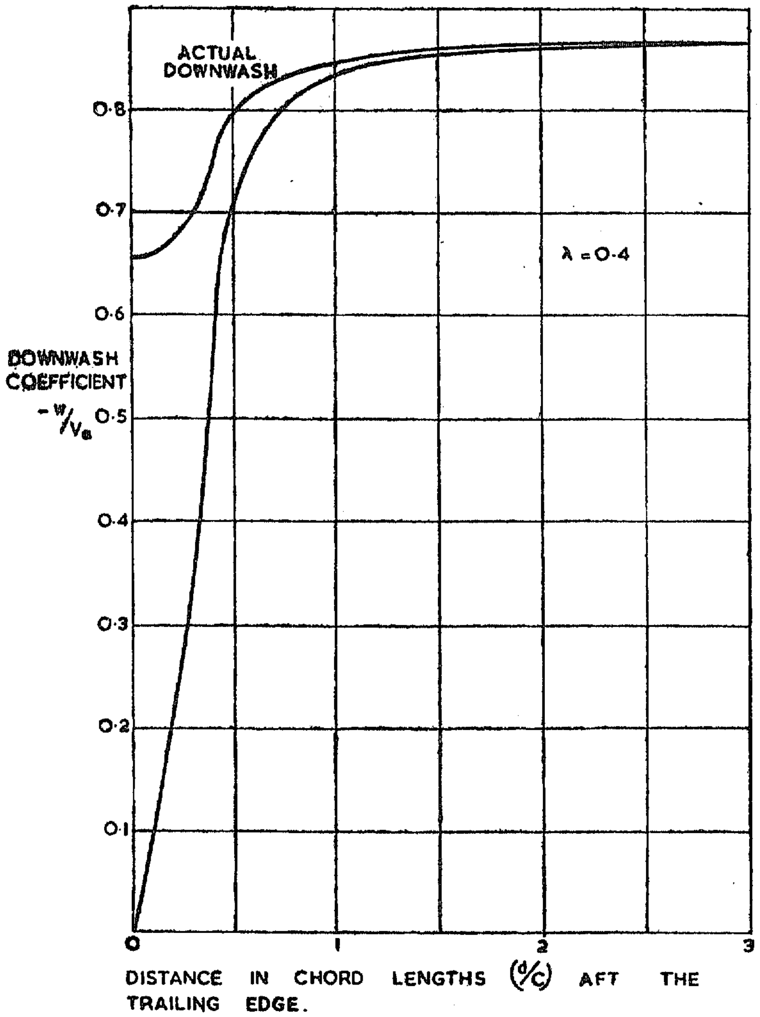

FIG. 2. Downwash on the assumption that lift is concentrated at the trailing edge.

Furthermore the normal velocity, w, is continuous at the aerofoil, because it is assumed to be flat, and similarly across the wake, since a discontinuity would indicate the presence of sources contrary to the condition of continuity.

This also follows from the boundary conditions which require $\Phi - Vx$ to be anti-symmetrical with respect to the xy-plane, so that $\Phi(x, y, -z) = -\Phi(x, y, -z)$.

Hence we have :—

At the aerofoil,

$$\begin{aligned} u(x, y, +0) - V &= -[u(x, y, -0) - V] \\ v(x, y, +0) &= -v(x, y, -0) \\ w(x, y, +0) &= +w(x, y, -0) \end{aligned} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

and at the wake,

$$\begin{aligned} u(x, y, +0) &= +u(x, y, -0) = +V \\ v(x, y, +0) &= -v(x, y, -0) \\ w(x, y, +0) &= +w(x, y, -0) \end{aligned} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

These equations show that we may regard the area comprising the aerofoil and its wake a a vortex sheet with a surface distribution $\underline{\omega}\,(x, y)$ of vortices given by

$$\underline{\omega} = \big(-(v_+ - v_-),\ (u_+ - u_-),\ 0\big) = \big(-2v,\ 2(u - V),\ 0\big) \ldots \quad \ldots \quad \ldots \quad (8)$$

and, in particular, in the wake,

$$\underline{\omega} = (-2\,v_+,\ 0,\ 0),\ \text{where } u_+ = u(x, y, +0) \text{ etc.} \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

Now, since the flow is irrotational, we have

$$\frac{\partial v_+}{\partial x} - \frac{\partial u_+}{\partial y} = 0\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

and

$$\frac{\partial v_-}{\partial x} - \frac{\partial u_-}{\partial y} = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Hence, at the aerofoil

$$\frac{\partial(v_+ - v_-)}{\partial x} - \frac{\partial(u_+ - u_-)}{\partial y} = 0\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

and at the wake, taking into account equation (7),

$$\frac{\partial(v_+ - v_-)}{\partial x} = 2\,\frac{\partial v_+}{\partial x} = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Equations (12) and (13) show that div $\underline{\omega} = 0$, as required for a vorticity vector.

To find the field of flow due to this vorticity distribution we apply formula (60) of Ref. 2, which states that the velocity vector (u, v, w) due to vorticity (ξ, η, ϱ) is given by :—

$$(u, v, w) = \operatorname{curlh} \underline{\Psi} + (V, 0, 0)$$

$$\Psi(x, y, z) = \frac{1}{2\pi} \,{}^*\!\!\int_{R'} (\xi, \eta, \varrho)\ \frac{dx\,dy\,dz}{s}\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where (a) $s = \sqrt{\{(x - x_0)^2 - \beta^2\,[(y - y_0)^2 + (z - z_0)^2]\}}$

(b) R' is the subdomain of the region concerned for which s is real and $x_0 < x$.

(c) curlh is the hyperbolic curl and ${}^*\!\!\int$ Hadamard's "finite part of an infinite integral" as defined in Ref. 2.

In our case the vortex layer becomes infinitely thin, while the product $(\xi, \eta, \varrho)dz_0$ remain finite and equal to $\underline{\omega}$. We obtain

$$\Psi = \frac{1}{2\pi} {}^{*}\int_{R'} \underline{\omega}(x_0, y_0) \frac{dx_0\, dy_0}{s}, \text{ where now } z_0 = 0 \text{ in } s. \quad .. \quad .. \quad .. \quad (15)$$

The vorticity distribution over the aerofoil will be called the bound vorticity and that in the wake the trailing vorticity. The latter consists of straight vortex lines of constant strength extending from the trailing edge to infinity in the negative x-direction.

If we write $u^* = u_+ - V$ and $v^* = v_+$, equation (15) becomes

$$\underline{\Psi} = \frac{1}{\pi} \int_{R'} (-v^*, u^*, 0) \frac{dx_0\, dy_0}{s}. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (16)$$

By equations (7) and (13) we have in the wake

$$u^* = 0, \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (17)$$

$$\frac{\partial v^*}{\partial x} = 0. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (18)$$

Also, by equation (5), u^* is connected with the pressure difference, $\Delta p = p_+ - p_-$, between the top and bottom aerofoil surfaces by the relation

$$u^* = \frac{\Delta p}{2V\rho}. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (19)$$

Hence, if the lift distribution is known, so is u^* and vice versa. Equation (18) shows that v^* is independent of x in the wake, so that to evaluate the integral (16) over the wake it is sufficient to know the variation of v^* along the trailing edge only. It will be shown that at the trailing edge, as well as elsewhere on the aerofoil,

$$v^* = \frac{\partial}{\partial y} \int u^* dx, \text{ where the integral is taken along a chord from the leading edge.}$$

Consider a circuit ABCDD′C′B′A′A, where AB,CD are parallel to the x-axis and AD,BC to the y-axis, so that AB and CD are separated by a small distance δy; A,B,C,D, are just above the xy-plane and A′,B′,C′,D′ form their image just below it; A,A′,D,D′ are points ahead of the aerofoil (*see* Fig. 4). Applying Stokes' theorem to the flow round this circuit we obtain :—

$$\int_A^B u_+ dx + \int_B^C v_+ dy + \int_C^D u_+ dx + \int_{D'}^{C'} u_- dx + \int_{C'}^{B'} v_- dy + \int_{B'}^{A'} u\ dx = 0. \quad .. \quad .. \quad (20)$$

This may be written

$$\int_A^B (u_+ - u_-)dx - \int_D^C (u_+ - u_-)\, dx + (v_+ - v_-)\, \delta y = 0. \quad .. \quad .. \quad .. \quad (21)$$

Hence, as δy tends to zero,

$$(v_+ - v_-) = \frac{\partial}{\partial y} \int_A^B (u_+ - u_-)dx, \text{ since } u_+ = u_-, \text{ as far as the aerofoil.} \quad .. \quad .. \quad (22)$$

Now $\left.\begin{aligned} u_+ - u_- &= 2u^* \\ v_+ - v_- &= 2v^* \end{aligned}\right\}$ at the aerofoil, so that

$$v^* = \frac{\partial}{\partial y}\int u^* dx, \text{ as asserted.} \qquad \ldots \qquad (23)$$

This relation might have been derived directly from equation (12), which can be written as $\dfrac{\partial v^*}{\partial x} - \dfrac{\partial u^*}{\partial y} = 0$, but for the possible irregularities of $\dfrac{\partial u^*}{\partial y}$ and $\dfrac{\partial v^*}{\partial x}$ at the leading edge and at the envelope of the Mach cones emanating from it. What about the proof of Stokes th. in this case?

Define $\bar{u} = \int u^* dx$ with the condition $\bar{u} = 0$ at the leading edge. Then $v^* = \dfrac{\partial \bar{u}}{\partial y}$. It will be seen from equation (19) that $\bar{u}$ is proportional to the excess pressure integral from the leading edge to the point in question.

Divide R' into two subdomains S' and W', belonging to the normal projection on to the xy-plane of the aerofoil, S, and the wake, W, respectively.

Then

$$\underline{\Psi} = \frac{1}{\pi}\int_{S'}\left\{-\frac{\partial \bar{u}}{\partial y}, \frac{\partial \bar{u}}{\partial x}, 0\right\}\frac{dx_0\, dy_0}{s} + \frac{1}{\pi}\int_{W'}\left\{-\left(\frac{\partial \bar{u}}{\partial y}\right)_t, 0, 0\right\}\frac{dx_0\, dy_0}{s}, \qquad \ldots \qquad (24)$$

where $\left(\dfrac{\partial \bar{u}}{\partial y}\right)_t$ is the value of $\dfrac{\partial \bar{u}}{\partial y}$ at the trailing edge for given y_0.

Now $$\int_{S'}\frac{\partial \bar{u}}{\partial x}\frac{dx_0\, dy_0}{s} = \int_{C'}\bar{u}\frac{dy_0}{s} - {}^*\!\!\int_{S'}\bar{u}\frac{(x - x_0)\, dx_0\, dy_0}{s^3},$$

by integration by parts, where C' is that segment of the trailing edge included in R'.

We can now represent $\underline{\Psi}$ as the sum of two vectors, $\underline{\Psi}' = (\Psi_1'\ \Psi_2'\ \Psi_3')$ and $\underline{\Psi}'' = (\Psi_1'', \Psi_2'', \Psi_3'')$, where

$$\left.\begin{aligned} \Psi_1' &= -\frac{1}{\pi}\int_{S'}\frac{\partial \bar{u}}{\partial y}\frac{dx_0\, dy_0}{s} \\ \Psi_2' &= -\frac{1}{\pi}{}^*\!\!\int_{S'}\frac{\bar{u}(x - x_0)\, dx_0\, dy_0}{s_3} \\ \Psi_3' &= \quad 0 \end{aligned}\right\}, \qquad \ldots \qquad (25)$$

and

$$\left.\begin{aligned} \Psi_1'' &= -\frac{1}{\pi}\int_{W'}\left(\frac{\partial \bar{u}}{\partial y}\right)_t\frac{dx_0\, dy_0}{s} \\ \Psi_2'' &= +\frac{1}{\pi}\int_{C'}\bar{u}\frac{dy_0}{s} \\ \Psi_3'' &= \quad 0. \end{aligned}\right\} \qquad \ldots \qquad (26)$$

It will be seen that $\underline{\Psi}$ conincides with $\underline{\Psi}''$ if we imagine that the whole bound vorticity, for any given span position, is concentrated at the trailing edge.

It will be observed that, if the aerofoil is assumed to be symmetrical with respect to the zx-plane and to have a straight trailing edge, $\underline{\Psi}''$ may be regarded as being due to the sum of a set of horse-shoe vortices of strength $-2\left(\frac{\partial \bar{u}}{\partial y}\right)_t$ whose spanwise segments extend from $-y_0$ to $+y_0$ (*see* Ref. 2, section 6. (64)). To find the velocity components (u'', v'', w'') due to this combination it is necessary to integrate the expression given in equation (68) of Ref. 2 from the midpoint of the trailing edge to the positive endpoint with respect to y_0, thus:—

$$u'' = -\frac{\beta^2}{\pi}\left\{\int \frac{\partial \bar{u}}{\partial y_0}\frac{(y-y_0)z\,.\,dy_0}{[(x-x_0)^2]-\beta^2z^2\sqrt{\{(x-x_0)^2-\beta^2[(y-y_0)^2+z^2]\}}}\right.$$

$$\left.-\int \frac{\partial \bar{u}}{\partial y_0}\frac{(y+y_0)z.\,dy_0}{[(x-x_0)^2]-\beta^2z^2\sqrt{\{(x-x_0)^2-\beta^2[(y+y_0)^2+z^2]\}}}\right\}, \quad \ldots \quad \ldots \quad (27)$$

where the integrals extend over those segments of the trailing edge for which y_0 is positive and the integrands are real. From the above assumption it follows that $v^* = \frac{\partial \bar{u}}{\partial y}$ is anti-symmetrical with respect to the zx-plane, so that the second integral in (27) is equal to but opposite in sign to the first taken over the remainder of the path of integration, C'.

Hence $$u'' = -\frac{\beta^2}{\pi}\int_{C'}\frac{\partial \bar{u}}{\partial y_0}\,\frac{(y-y_0)z\,.\,dy_0}{[(x-x_0)^2-\beta^2z^2]\,.\,s}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

Similarly $$v'' = -\frac{1}{\pi}\int_{C'}\frac{\partial \bar{u}}{\partial y_0}\,\frac{(x-x_0)z_0\,dy_0}{[(y-y_0)^2+z^2]\,.\,s}\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

and $$w'' = -\frac{1}{\pi}\int_{C'}\frac{\partial \bar{u}}{\partial y_0}\,\frac{(x-x_0)\,(y-y_0)\,\{(x-x_0)^2-\beta^2\,[(y-y_0)^2+2z^2]\,dy_0\}}{[(x-x_0)^2-\beta^2z^2]\,[(y-y_0)^2+z^2]\,.\,s}\,. \quad \ldots \quad (30)$$

In calculating (u'', v'', w''), x_0 will take the value of x at the trailing edge.

To calculate the field of flow due to $\underline{\Psi}'$ behind the aerofoil on the assumption that it is symmetrical with respect to the zx-plane and that it has a straight trailing edge, or more precisely one that is not so curved that it meets a line parallel to the y-axis in more than two points, we rewrite equation (25) in the form :—

$$\Psi_1' = \int X_1(x_0)dx_0; \quad \Psi_2' = \int X_2(x_0)dx_0; \quad \Psi_3' = 0, \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

where

$$X_1 = -\frac{1}{\pi}\int \frac{\partial \bar{u}}{\partial y_0}\,\frac{dy_0}{s}\,; \quad X_2 = -\frac{1}{\pi}{}^*\!\!\int \frac{\bar{u}\,.\,(x-x_0)dy_0}{s^3}\,. \quad \ldots \quad \ldots \quad (32)$$

It will be observed that, for a given x_0, X_1 is obtained from Ψ_1'' by differentiating it with respect to x and putting $\left(\frac{\partial \bar{u}}{\partial y}\right)_t = \frac{\partial \bar{u}}{\partial y_0}$, since

$$\frac{\partial \Psi_1''}{\partial x} = -\frac{1}{\pi}\frac{\partial}{\partial x}\int_{W'}\left(\frac{\partial \bar{u}}{\partial y}\right)_t \frac{dx_0\,dy_0}{s} = +\frac{1}{\pi}\int_{W'}\left(\frac{\partial \bar{u}}{\partial y}\right)_t \frac{(x-x_0)dx_0\,dy}{s^3}$$

$$= -\frac{1}{\pi}\int_{C'}\left(\frac{\partial \bar{u}}{\partial y}\right)_t \frac{dy_0}{s}\,, \quad \left(\frac{\partial \bar{u}}{\partial y}\right)_t \text{ being independent of } x_0.$$

Similarly we may derive X_2 from Ψ_2''. It will be noted that, though the limits of integration may be dependent on x, we are justified in differentiating under the integral sign of the finite part of an infinite integral as demonstrated in section 2 of Ref. 2.

Thus (u', v', w') is obtained from (u'', v'', w'') by differentiation with respect to x and subsequent integration with respect to x_0 across the aerofoil. Hence

$$u' = -\frac{\beta^2}{\pi}\,{}^*\!\!\int_{S'}\frac{\partial \bar{u}}{\partial y_0}\frac{\partial}{\partial x}\left\{\frac{(y-y_0)z}{[(x-x_0)^2-\beta^2z^2]\,.\,s}\right\}dx_0\,dy_0 \quad \ldots \quad (33)$$

$$v' = -\frac{1}{\pi}\,{}^*\!\!\int_{S'}\frac{\partial \bar{u}}{\partial y_0}\frac{\partial}{\partial x}\left\{\frac{(x-x_0)z}{[(y-y_0)^2+z^2]\,.\,s}\right\}dx_0\,dy_0$$

$$w' = -\frac{1}{\pi}\,{}^*\!\!\int_{S'}\frac{\partial \bar{u}}{\partial y_0}\frac{\partial}{\partial x}\left\{\frac{(x-x_0)(y-y_0)\{(x-x_0)^2-\beta^2[(y-y_0)^2+2z^2]\}}{[(x-x_0)^2-\beta^2z^2][(y-y_0)^2+z^2]\,.\,s}\right\}dx_0\,dy_0$$

It will be seen that these latter equations involve the representation of a vortex sheet by a system of line vortices. Hence, in accordance with a remark at the end of Ref. 2, they are not valid everywhere, but can be shown to be so inside the envelope of the Mach cones emanating from the trailing edge. In particular, the formulae are valid in the region of the wake. Thus for the downwash, $w = w' + w''$, we have in the wake, where $z = 0$, by equations (30) and (33):—

$$w = -\frac{1}{\pi}\left\{\int_{C'}\frac{\partial \bar{u}}{\partial y_0}\frac{\sqrt{\{(x-x_0)^2-\beta^2(y-y)^2\}}}{(x-x_0)(y-y_0)}dy_0\right.$$

$$\left.+\int_{S'}\frac{\partial \bar{u}}{\partial y_0}\frac{\partial}{\partial x}\left[\frac{\sqrt{\{(x-x_0)^2-\beta^2(y-y_0)^2\}}}{(x-x_0)(y-y_0)}\right]dx_0\,dy_0,\right\} \quad \ldots \quad (34)$$

i.e.

$$w = -\frac{1}{\pi}\left\{\int_{C'}\frac{\partial \bar{u}}{\partial y_0}\frac{\sqrt{\{(x-x_0)^2-\beta^2(y-y_0)^2\}}}{(x-x_0)(y-y_0)}dy_0\right.$$

$$\left.+\int_{S'}\frac{\partial \bar{u}}{\partial y_0}\frac{\beta^2(y-y_0)}{(x-x_0)^2\sqrt{\{(x-x_0)^2-\beta^2(y-y_0)^2\}}}dx_0\,dy_0\right\}. \quad \ldots \quad (35)$$

Before applying our results to calculating the downwash in the wake of a delta wing, it is instructive to consider the case of two-dimensional flow.

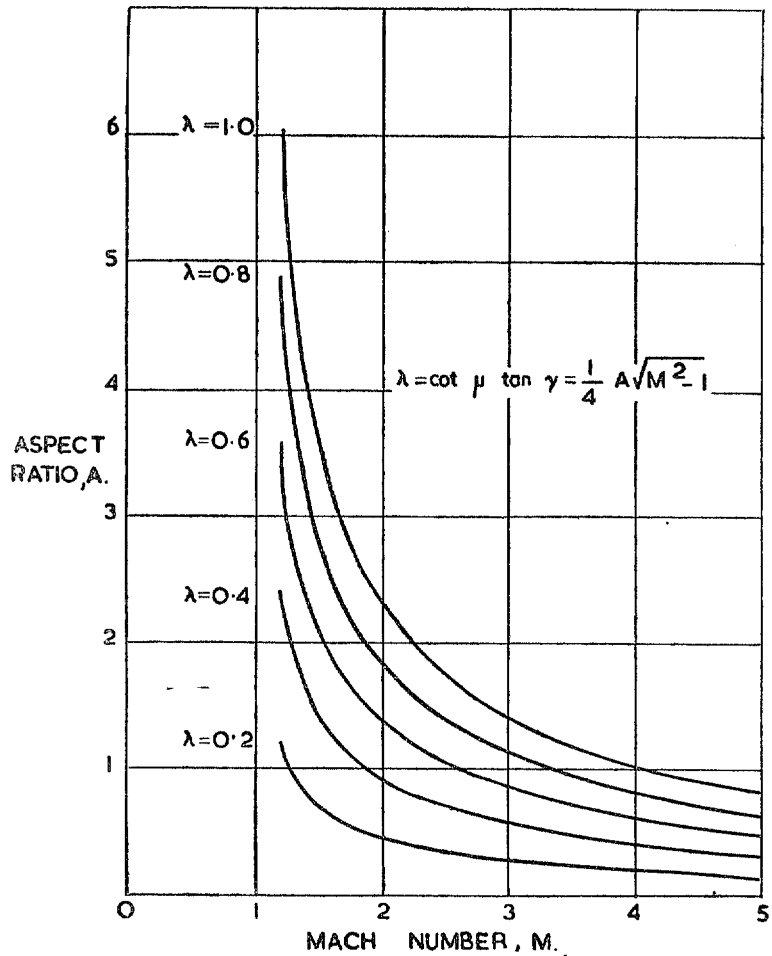

FIG. 3. Value of λ for varying aspect ratio and Mach number for delta wings.

In two-dimensional flow parallel to the zx-plane $v^* = 0$, so by equation (16) $\underline{\Psi}$ is given by $\Psi_1 = \Psi_3 = 0$, and

$$\Psi_2 = \frac{1}{\pi} \int_{S'} u^* \frac{dx_0\, dy_0}{s}, \qquad \text{(36)}$$

which we can integrate directly with respect to y_0 since u^* is independent of y_0. Hence

$$\Psi_2 = \frac{1}{\pi} \int u^* \left\{ \left[-\frac{1}{\beta} \sin^{-1} \frac{\beta(y - y_0)}{\sqrt{\{(x - x_0)^2 - \beta^2 z^2\}}} \right]_{y_1}^{y_2} \right\} dx, \qquad \text{(37)}$$

where y_1 and y_2 are the roots of $s^2 = 0$. Therefore

$$\Psi_2 = \frac{1}{\beta} \int u^* \, dx_0, \qquad \text{(38)}$$

where the integral is taken from the leading edge to $x_0 = x - \beta \mid z \mid$, or to the trailing edge, whichever is the smaller.

In particular, if u^* is constant and the leading edge coincides with $x_0 = 0$ and the trailing edge with $x_0 = c$, then :—

$$\Psi_2 = \frac{1}{\beta} u^* (x - \beta \mid z \mid), \text{ or } = \frac{1}{\beta} u^* c \quad \ldots \quad (39)$$

The components, $\bar{u} - V$ and w, are given by the hyperbolic curl of $\underline{\Psi}$, which in this case is $\left(-\frac{\partial \Psi_2}{\partial z}, 0, -\beta^2 \frac{\partial \Psi_2}{\partial x}\right)$. Hence :—

$$\left.\begin{aligned} u &= V \pm u^* \\ w &= -\beta u^* \end{aligned}\right\} \text{ if } x - \beta \mid z \mid < c \quad \ldots \quad (40)$$

and

$$\left.\begin{aligned} u &= V \\ w &= 0 \end{aligned}\right\} \text{ if } x - \beta \mid z \mid > c,$$

which is in agreement with Ackeret's theory. On the other hand formulae (33), while providing the right answer for $x - \beta \mid z \mid > c$, fail for $x - \beta \mid z \mid < c$ for the reasons mentioned previously.

4. *The Downwash in the Wake of a Delta Wing.*—Consider a delta wing at a small angle of incidence α, in a uniform airstream of supersonic velocity so that the apex semi-angle, γ, is less than the Mach angle. The apex is at the origin, and $x = c$ at the trailing edge, such that the wing is approximately in the xy-plane with its axis along the x-direction.

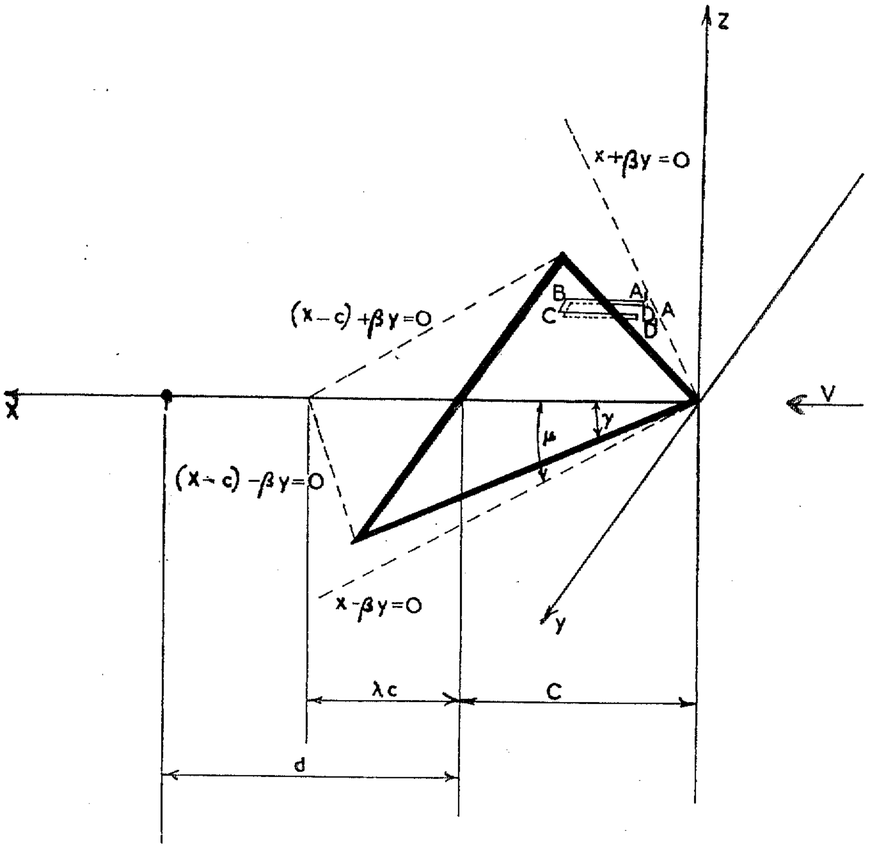

FIG. 4. Diagram of delta wing

Under these conditions according to Ref. 3 we have

$$u^* = \frac{V\alpha}{E'(\lambda)} \frac{x_0 \tan^2 \gamma}{\sqrt{(x_0^2 \tan^2 \gamma - y_0^2)}}, \text{ where } \lambda = \beta \tan \gamma \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

hence
$$\bar{u} = \frac{V\alpha}{E'(\lambda)} \sqrt{(x_0^2 \tan^2 \gamma - y_0^2)},$$

and
$$\frac{\partial \bar{u}}{\partial y} = -\frac{V\alpha}{E'(\lambda)} \frac{y_0}{\sqrt{(x_0^2 \tan^2 \gamma - y_0^2)}}.$$

Here $\bar{u}$ is in fact identical with the induced velocity potential of the aerofoil and can be obtained directly.

The downwash, $w = w' + w''$, at the centre line of the wake can now be found directly from equation (34) by substitution and integration.

We have:
$$w' = -\frac{V\alpha}{\pi E'(\lambda)} \int_{S'} \frac{1}{\sqrt{(x_0^2 \tan^2 \gamma - y_0^2)}} \frac{\partial}{\partial x} \frac{\sqrt{\{(x - x_0)^2 - \beta^2 y_0^2\}}}{x - x_0} \, dx_0 \, dy_0$$

$$= -\frac{V\alpha}{\pi E'(\lambda)} \int_{S'} \left\{ \frac{1}{\left\{1 - \sqrt{\left(\frac{\beta y_0}{x - x_0}\right)^2}\right\}} - \sqrt{\left\{1 - \left(\frac{\beta y_0}{(x - x_0)}\right)^2\right\}} \right\}$$

$$\frac{dx_0 \, dy_0}{(x - x_0 \sqrt{(x_0^2 \tan^2 \gamma - y_0^2)}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (42)$$

In S' the limits of integration with respect to y_0 have to be such that the integrand is real; they are $\pm x_0 \tan \gamma$ or $\pm \frac{1}{\beta}(x - x_0)$, whichever is numerically the less. The limits are $\pm x_0 \tan \gamma$ if $\lambda x_0/(x - x_0) < 1$, which is always the case if $d = x - c > \lambda c$.

Consider
$$\frac{\lambda x_0}{x - x_0} < 1, \text{ i.e. } x_0 < \frac{x}{1 + \lambda}.$$

Put $y_0 = t x_0 \tan \gamma$ and $k = \dfrac{\lambda x_0}{x - x_0}$, so that

$$w' = -\frac{V\alpha}{\pi E'(\lambda)} \int \int_{-1}^{+1} \left\{ \frac{1}{\sqrt{(1 - k^2 t^2)}} - \sqrt{(1 - k^2 t^2)} \right\} \frac{dx_0 \, dt}{(x - x_0)\sqrt{(1 - t^2)}} \quad \ldots \quad (43)$$

$$= -\frac{2V\alpha}{\pi E'(\lambda)} \int \frac{K(k) - E(k)}{x - x_0} \, dx_0,$$

$$= -\frac{2V\alpha}{\pi E'(\lambda)} \int \frac{K(k) - E(k)}{\lambda + k} \, dk. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

Consider now $\frac{\lambda x_0}{x - x_0} > 1$, i.e. $x_0 > \frac{x}{1 + \lambda}$

Put $\beta y_0 = t(x - x_0)$ and $\bar{k} = \frac{x - x_0}{\lambda x_0}$, so that

$$w' = -\frac{V\alpha}{\pi E'(\lambda)} \int \int_{-1}^{+1} \left\{ \frac{1}{\sqrt{(1 - t^2)}} - \sqrt{(1 - t^2)} \right\} \frac{dx_0 \, dt}{\lambda x_0 \sqrt{(1 - \bar{k}^2 t^2)}}, \quad \ldots \quad (45)$$

$$= -\frac{2V\alpha}{\pi E'(\lambda)} \int \left\{ K(\bar{k}) - \frac{E(\bar{k}) - (1 - \bar{k}^2) K(\bar{k})}{\bar{k}^2} \right\} \frac{dx_0}{\lambda x_0}$$

$$= +\frac{2V\alpha}{\pi E'(\lambda)} \int \frac{K(\bar{k}) - E(\bar{k})}{\bar{k}^2 (1 + \lambda \bar{k})} d\bar{k}, \quad \ldots \quad (46)$$

Now the range of integration with respect to x_0 is 0 to c, so that when $d < \lambda c$ we must split the range into two parts — 0 to $\frac{x}{1 + \lambda}$ and $\frac{x}{1 + \lambda}$ to c. For the first part, integral (42) reduces to (44), for which the range of integration with respect to k is 0 to 1, and for the second it reduces to (46), for which the range of integration with respect to $\bar{k}$ is 1 to $d/\lambda c$.

Therefore for $d < \lambda c$ we have

$$w' = -\frac{2V\alpha}{\pi E'(\lambda} \left\{ \int_0^1 \frac{K - E}{\lambda + k} dk + \int_{d/\lambda c}^1 \frac{K - E}{k^2 (1 + \lambda k)} dk \right\} \quad \ldots \quad (47, i)$$

When $d > \lambda c$ the full range of $x = 0$ to c is covered by $k = 0$ to $\lambda c/d$ in the integral (44).

Therefore for $d > \lambda c$ we have

$$w' = -\frac{2V\alpha}{\pi E'(\lambda)} \int_0^{\lambda c/d} \frac{K - E}{\lambda + k} dk. \quad \ldots \quad (47, ii)$$

We also derive from equation (34) the following expression for w'':—

$$w'' = -\frac{V\alpha}{\pi E'(\lambda)} \int_{C'} \frac{\sqrt{(x - c)^2 - \beta^2 y_0^2} \, dy_0}{(x - c) \sqrt{(c^2 \tan^2 \gamma - y)_0^2}} \quad \ldots \quad (48)$$

As before, the limits of integration have to be such that the integrand is real; they are $\pm\, c \tan \gamma$ if $d > \lambda c$, otherwise $\pm\, d/\beta$.

Consider $d > \lambda c$ and put $y_0 = tc \tan \gamma$ and $k = \frac{\lambda c}{x - c} = \frac{\lambda c}{d}$, so that

$$w'' = -\frac{V\alpha}{\pi E'(\lambda)} \int_{-1}^{+1} \sqrt{\left(\frac{1 - k^2 t^2}{1 - t^2} \right)} dt = -\frac{2V\alpha}{\pi E'(\lambda)} E(k) \quad \ldots \quad (49)$$

Consider now $d < \lambda c$ and put $\beta y_0 = t(x - c)$ and $\bar{k} = \frac{x - c}{\lambda c} = \frac{d}{\lambda c}$, so that

$$w'' = -\frac{V\alpha}{\pi E'(\lambda)} \int_{-1}^{+1} \sqrt{\left(\frac{1 - t^2}{1 - \bar{k}^2 t^2}\right)} \bar{k} dt = -\frac{2V}{\pi E'(\lambda)} \frac{E(\bar{k}) - (1 - \bar{k}^2) K(\bar{k})}{\bar{k}} \qquad (50)$$

Therefore for $d < \lambda c$ we have

$$w'' = -\frac{2V\alpha}{\pi E'(\lambda)} \frac{\lambda c}{d} \left\{ E\left(\frac{d}{\lambda c}\right) - \left[1 - \left(\frac{d}{\lambda c}\right)^2\right] K\left(\frac{d}{\lambda c}\right) \right\} \qquad (51, i)$$

And for $d > \lambda c$

$$w'' = -\frac{2V\alpha}{\pi E'(\lambda)} E\left(\frac{\lambda c}{d}\right). \qquad (51, ii)$$

It will be noted that w' and w'' are continuous at $d = \lambda c$. The gradient of w, however, has a logarithmic singularity at this point.

The component w'' represents the downwash that would be obtained if the entire lift were concentrated at the trailing edge for the same spanwise lift distribution ; $w''/V\alpha$ is plotted in Fig. 2 versus d/c for $\lambda = 0{\cdot}4$.

The total downwash, $w = w' + w''$, is therefore given by:—
where $d < \lambda c$,

$$\frac{w}{V\alpha} = -\frac{2}{\pi E'(\lambda)} \left\{ \frac{\lambda c}{d} \left[E\left(\frac{d}{\lambda c}\right) - \left[1 - \left(\frac{d}{\lambda c}\right)^2\right] K\left(\frac{d}{\lambda c}\right)\right] + \int_0^1 \frac{K(k) - E(k)}{k + \lambda} dk + \int_{d/\lambda c}^1 \frac{K(k) - E(k)}{k^2(1 + \lambda k)} dk \right\}, \qquad (52, i)$$

and where $d > \lambda c$,

$$\frac{w}{V\alpha} = -\frac{2}{\pi E'(\lambda)} \left\{ E\left(\frac{\lambda c}{d}\right) + \int_0^{\lambda c/d} \frac{K(k) - E(k)}{k + \lambda} dk \right\} \qquad (52, ii)$$

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