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by

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DESIGN OF A SYMMETRICAL SECTION WITH SPECIFIED PRESSURE DISTRIBUTION

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SUMMARY

A method for the calculation of the section thickness distribution to produce a given symmetrical pressure distribution near the critical Mach number is presented. By linearisation about a known solution, obtained by Sells' program, a Neumann problem with the same field equations but a different boundary condition is established, and the computer solution of this problem forms one link in an iteration sequence. In the symmetrical case studied, the sequence converged to a slightly different solution from that sought. Attempts at solution of the lifting problem have been unsuccessful.

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CONTENTS

1 INTRODUCTION

The aerodynamic section design problem is the counterpart of the direct subcritical flow problem for a given section, and presents its own special rewards for successful solution: for instance the pressure distribution can be prescribed so that the turbulent boundary layer does not separate, and so that favourable drag-rise conditions obtain. No exact analytic method exists for either problem, but two computer programs now exist[1,2] which together provide the exact numerical solution to the direct problem, and it was felt worthwhile to see whether the design problem was also amenable to computer solution.

By linearisation about a known solution (section), an iterative technique is developed to generate a sequence of sections, which would ideally converge to the desired section. At each stage, the calculation of the next section is cast as a Neumann problem, with the same field equations as the direct problem but a different boundary condition on the section, so that very little reprogramming is needed to solve the equations. The sequence indeed converged, for the symmetrical case studied, but to a slightly different answer from that sought. When a lifting case was attempted, the sequence failed to converge at all, and so this attempt to solve the most interesting problem of section design must be counted a failure. However, it is thought worthwhile to place the method and its one concrete result on record.

2 LINEARISATION OF THE PROBLEM

The method depends fundamentally on perturbations from a known section, S_0 say. First we establish the coordinate system. If z, ζ, σ are complex variables, the contour S_0 and its exterior in the z plane (Fig.1) can be mapped conformally onto the unit circle and its exterior in the ζ plane (Catherall[1]); the mapping function $\sigma = 1/\zeta$ then transforms this region into the interior of the unit circle in the σ plane, and polar coordinates r, θ are fixed at the centre of this circle so that $0 \leq r \leq 1$ is the image in the σ plane of the exterior of S_0 in the z plane (Fig.2). Using these coordinates, for subcritical Mach numbers the flow round S_0 can be computed (Sells[2]).

Let the solution for some incidence α and Mach number M be

$$\psi = \psi_0 (r, \theta) \tag{2.1}$$

where ψ is the stream function, introduced to satisfy the equation of continuity.

Now keep the mapping function $z = z(\sigma)$ fixed, and consider a section S_1 in the z plane which differs slightly from S_o. Let the image of S_1 in the σ plane be (Figs.3 and 4)

$$r = 1 - R(\theta) \tag{2.2}$$

where $|R| \ll 1$, and suppose that the corresponding stream function representing the solution for S_1 at the same M and given C_p is

$$\psi = \psi_1 (r,\theta) = \psi_o + \Psi (r,\theta) \tag{2.3}$$

where Ψ represents a small perturbation.

Suppose that the desired pressure distribution is $C_p = C_{p_1}$. Then from Bernoulli's equation and the equation of state with $\gamma = 1.4$, the speed q_1 and density ρ_1 (scaled with respect to free stream values) on the body are given by

$$q_1^2 = 1 + \frac{5}{M^2}\left[1 - (1 + 0.7\, M^2\, C_{p_1})^{2/7}\right]$$

and

$$\rho_1 = (1 + 0.7\, M^2\, C_{p_1})^{5/7} \quad .$$

By definition of ψ, on S_1

$$|q_1| = \frac{1}{\rho_1}\left|\frac{\partial\psi_1}{\partial n}\right|$$

where n is the outward normal to S_1. Now in the z plane $|dn| = |dz|$ and in the σ plane $|dr| = |d\sigma|$, so

$$\left|\frac{\partial\psi_1}{\partial r}\right| = \rho_1 \left|\frac{dz}{d\sigma}\right| |q_1| = \rho_1\, B\, |q_1| \qquad \text{approximately} \tag{2.4}$$

where $B = |dz/d\sigma|$ on $r = 1$; linearisation allows us to transfer the evaluation of B from S_1 (where it is at present unknown) to S_o.

The actual sign of $\partial\psi_1/\partial r$ is most easily determined from the analogous incompressible solution without circulation, $\psi = (r - 1/r)\sin\theta$; we see that $\partial\psi_1/\partial r$ is positive on the lower surface of S_1, which corresponds to $0 < \theta < \theta_L$ where θ_L gives the position of the leading edge stagnation point, and $\partial\psi_1/\partial r$ is negative on the upper surface. (For the symmetrical section at zero lift the stagnation point is at the leading edge and the sign determination is trivial.) Hence $\partial\psi_1/\partial r$ is determined from the given pressure distribution C_{p_1}, in magnitude and sign.

Now

$$\psi_1\ (1 - R, \theta) = \psi_1\ (1, \theta) - R\left(\frac{\partial\psi_1}{\partial r}\right)(1, \theta) + 0\ (R^2)$$

$$= \psi_o\ (1, \theta) + \Psi\ (1, \theta) - R\left(\frac{\partial\psi_1}{\partial r}\right)(1, \theta) + 0\ (R^2) \quad . \qquad (2.5)$$

But the boundary conditions in the direct problems for S_o, S_1 are:

$$\psi_o = 0 \text{ on } S_o : r = 1$$

$$\psi_1 = 0 \text{ on } S_1 : r = 1 - R \quad .$$

Hence, to order R^2

$$0 = \Psi - R\,\partial\psi_1/\partial r$$

$$R = \frac{\Psi}{\partial\psi_1/\partial r} \quad . \qquad (2.6)$$

The evaluation of Ψ will be considered in the next section. The distribution of $R(\theta)$ follows from (2.6), and we can then determine the corresponding distribution of dz in the physical plane. We have

$$|dz| = \left|\frac{dz}{d\sigma}\right| |d\sigma| = B\ |R| \qquad (2.7)$$

and a displacement dz outwards from the surface S_o (Fig.4) corresponds to a displacement $d\sigma$ into the circle $r \leq 1$ from its boundary (Fig.3), i.e. to R positive, from (2.2); the displacement in the physical plane will be inwards when R is negative. In this way we obtain a linear approximation to S_1.

The process outlined must be part of an iteration cycle, because the non-linear effects have been neglected in the analysis so that the answer found is not necessarily the answer required. In the iteration cycle, having computed S_1, we can find the transformation that maps S_1 onto a circle, so that a new set of B's is known, and we can also compute the flow round S_1; if this is a better approximation to the desired flow than the original flow round S_o, then we may tentatively hope that the iteration procedure converges. S_1 then takes the place of S_o in the cycle, a new section (S_2 say) is found and the process continues.

3 COMPUTATION DETAILS

3.1 The stream function perturbations

The major problem is to compute the small perturbation Ψ. To do this we extend the method of Sells[2] for the direct problem. When the velocity $\underline{u}$ and density ρ are scaled with respect to free stream values, the field equations are:

Continuity: $$\nabla.(\rho\, \underline{u}) = 0 \qquad (3.1)$$

Irrotational flow: $$\nabla \times \underline{u} = 0 \qquad (3.2)$$

Bernoulli: $$\frac{\rho^{\gamma-1}}{M^2(\gamma - 1)} + \tfrac{1}{2}u^2 = \frac{1}{M^2(\gamma - 1)} + \tfrac{1}{2} \quad . \qquad (3.3)$$

Equation (3.1) is satisfied by introducing the stream function ψ_o, and then in the direct problem for S_o the boundary condition on ψ_o at $r = 1$ is

$$\psi_o = 0 \quad . \qquad (3.4)$$

There is also a complicated boundary condition as $r \to 0$. Equations (3.2), (3.3) and (3.4) are solved iteratively for ψ_o and ρ to give the flow field outside S_o. Because (3.4) does not involve the normal derivative $\partial\psi_o/\partial r$, the problem is of Dirichlet type.

Here the problem is to find an approximation to the stream function ψ_1 for the flow past S_1 in the S_o working plane. The condition $\psi_1 = 0$ now holds on the image of S_1, $r = 1 - R(\theta)$, and not on S_o; but because we are seeking a prescribed pressure distribution C_{p_1}, we can instead use (2.4) to prescribe $\partial\psi_1/\partial r$ on S_1 (within the linear approximation for B), and since R is assumed small we transfer the whole condition to S_o within the linear approximation. So the boundary condition on $r = 1$ becomes

$$\frac{\partial\psi_1}{\partial r} = \text{function of } C_{p_1} \qquad (3.5)$$

and the problem posed by (3.2), (3.3) and (3.5) is now of Neumann type. Unless the new problem is supercritical, or some other factor intervenes, the same iterative technique can be used as before, values of ψ_1 on $r = 1$ now being determined within the iteration with the aid of (3.5), and then since $\psi_o = 0$ on $r = 1$, the small perturbations Ψ will be just the converged values of ψ_1.

The reprogramming needed to solve (3.5) rather than (3.4) is almost trivial. Finally we remark that, when the symmetrical aerofoil at zero incidence is studied, the Kutta condition at the trailing edge is automatically satisfied; when a lifting case at given M is studied, C_N is prescribed and so the circulation is determined; the Kutta condition then becomes, in principle, an iterative equation for the incidence, just as in the corresponding Dirichlet problem for ψ_o.

3.2 The new points on the new section

We now have enough information to compute the displacements $|dz|$ from S_o to S_1, by (2.7) at each grid point.

We now mention briefly how the normal is computed. Since the grid points are equally spaced on $r = 1$, the tangent to S_o at the point $z_K = x_K + i\,y_K$ will be in the direction of the line joining the points z_{K+1}, z_{K-1}, with a second-order error (Fig.4); if the z_K go clockwise round S_o as K increases, the unit outward normal will be on the left following the curve, and so will be given by the complex number

$$n_K = i\,\frac{z_{K+1} - z_{K-1}}{|z_{K+1} - z_{K-1}|} \quad . \tag{3.6}$$

Then from (2.6) and (2.7), the point on S_1 corresponding to z_K on S_o is

$$z_K + n_K\,\frac{B\Psi}{\partial\psi_1/\partial r} \quad . \tag{3.7}$$

Now (Catherall[1]) the transformation $z \to \zeta \to \sigma$ is such that $|dz/d\zeta| \to 1$ in the free stream; then for a map onto the unit circle in the σ plane the chord c of the aerofoil is fixed (and can be calculated). In general we work with section ordinates x/c, y/c, and so here we rewrite the formula (3.7) as

$$\frac{z_K}{c} + \frac{n_K}{c}\,\frac{B\Psi}{\partial\psi_1/\partial r} \quad . \tag{3.8}$$

3.3 Under-relaxation round the leading-edge

When the program was run on a symmetrical section, it was found that the pressure distribution was very sensitive to the near-leading-edge corrections and that considerable supersonic peaks were introduced at first. So the corrections were arranged to decrease in proportion as the leading-edge was approached; satisfactory results were found by leaving all the corrections alone eight or more grid points from the leading-edge (sixty points were taken all round the section) and decreasing the corrections at the nearer points in proportion, the nth point away receiving n-eighths of the value in (3.8). This conveniently sets the correction to zero at the leading-edge itself; for the symmetrical section Ψ and $\partial\psi_1/\partial r$ vanish together there, so the correction is indeterminate. This system is a compromise, as it partly abandons the results of the linear theory in favour of the programmer's judgment, but it happens to produce a result for the symmetrical section.

4 RESULTS

4.1 The symmetrical case

The symmetrical 10% thick RAE 101 section was used to start the iterations. At zero incidence the critical Mach number is about 0.75 and there is a triangular peak at about $x/c = 0.3$. The goal sought was a rooftop pressure distribution $C_p = C_p/\text{max}$ from $x/c = 0.05$ to $x/c = 0.4$, as near critical as possible at $M = 0.75$, with a steep linear rise region near the leading-edge and a linear fall region from $x/c = 0.4$ back to near the trailing edge. The input value of C_p/max was -0.5300, compared with the critical value $C_p/\text{crit.} = -0.5912$.

The pressure distribution attained after nine iterations is shown in Fig.5; the RAE 101 pressure distribution is given also, for comparison. We shall refer to this section as the N10111; this is the author's file number for this particular section and has nothing to do with any external nomenclature. These two sections are shown in Fig.6, and the upper surface co-ordinates are listed in Fig.6a. We see that the N10111 is thicker (10.6% at 35% chord) and has lower leading-edge curvature than the RAE 101 (for the N10111 $(\rho/2c)^{\frac{1}{2}} = 0.0816$, for the RAE 101 it is 0.061781). The corresponding incompressible pressure distributions are shown in Fig.7. That for the RAE 101 resembles a rooftop up to $x/c = 0.3$, by design; that for the N10111 has a leading-edge peak, which is flattened by compressibility effects as M tends to critical. This demonstrates anew the vanity of incompressible theory in the design of high Mach number aerofoils.

On examination of Fig.5, we notice that the rooftop is a trifle bumpy. This is a legacy from the previous section shapes obtained during the iteration, small peaks at the front of the intended rooftop were left behind even with the under-relaxation technique described in sub-section 3.3, and these oscillated as the iteration proceeded from section to section before gradually dying out to leave only the marginal bumps of N10111. Fig.8 shows these peaks in the pressure distribution of an intermediate section iterate (N1016 in the author's file). The rear of the rooftop here is actually less bumpy than the N10111. Next, it is observed that although the input $C_p/\max$ was -0.5300, the effective value of $C_p/\max$ for the N10111 is only about -0.51; and there was no substantial change from the last section iterate in this regard. Thus the linear scheme has converged to the wrong answer. As mentioned at the end of section 2, this and the peaky behaviour of the previous section iterates may be due to neglected non-linear effects, especially round the leading-edge where the pressures are so sensitive; no difficulty has been experienced behind the rooftop, except that on a few section iterates the trailing edge had to be straightened out to eliminate unwanted concavity or convexity, before continuing with the conformal mapping program[1].

4.2 The lifting case

No worthwhile results have been obtained at all for the lifting case, which is unfortunate because this is the case of interest in section design. What happens here is that at the first iteration peaks appear at the front and rear of the intended rooftop, and to get rid of these a very lucky guess for the under-relaxation scheme would be needed (if no under-relaxation is used at all, the points convolute and do not lie on any sensible section); failing that, as iteration proceeds the peaks go supercritical and cannot be reduced. One reason in theory why things go wrong is that near the leading-edge stagnation point $\partial\psi_1/\partial r$ is small but Ψ (as computed by the Neumann program) is not, so that the linearisation process based on (2.6) breaks down. The author has tried to correct this by taking the Taylor series (2.5) to one more term, order R^2, and using the values of $\partial^2\psi_1/\partial r^2$, but this quantity (as computed) fluctuates around zero near the leading-edge (this is to be expected from the proximity of the streamline $\psi_1 = 0$), and no discernible improvement was found, even with severe under-relaxation. This could mean, for instance, that the picture of Fig.3, where normal displacements in the two planes correspond, no longer holds. It was concluded that the leading-edge modifications predicted by this simple theory bore little relation to those required in practice, and so this line of attack was abandoned.

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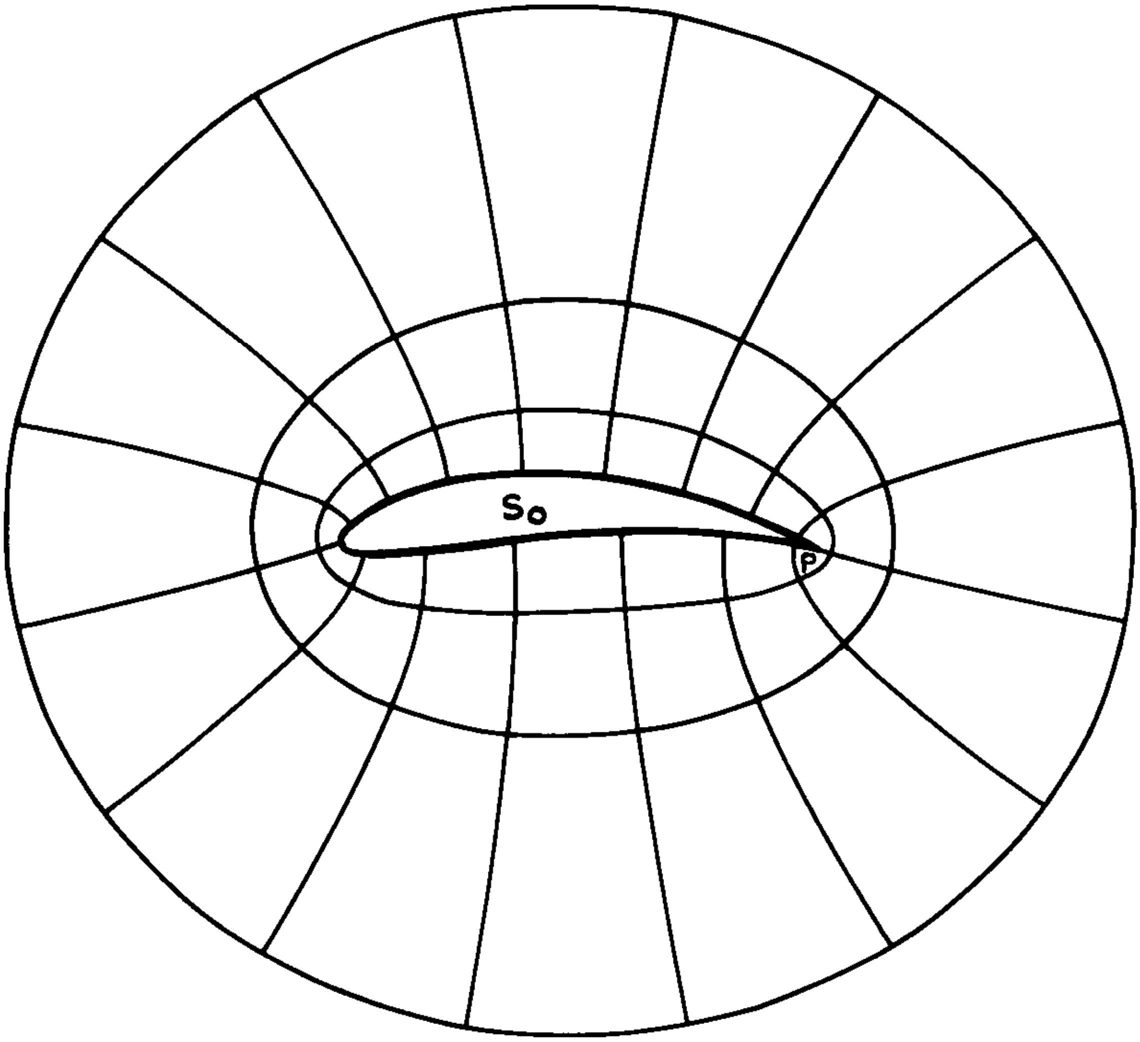

Fig.1 Section S_o and grid in physical plane

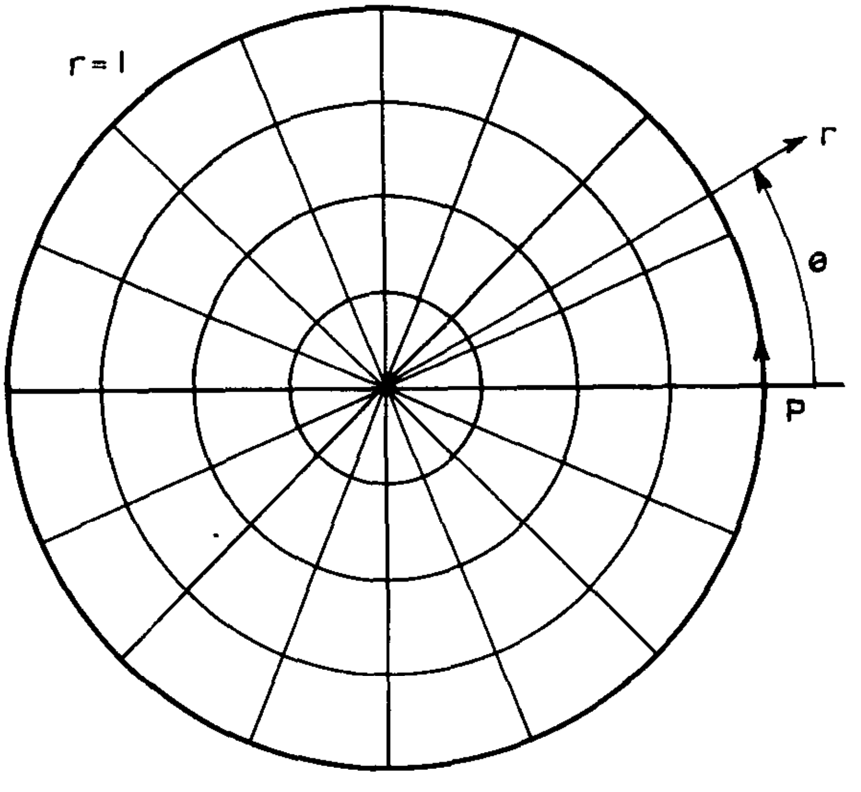

Fig. 2 The regular grid in the working (σ) plane

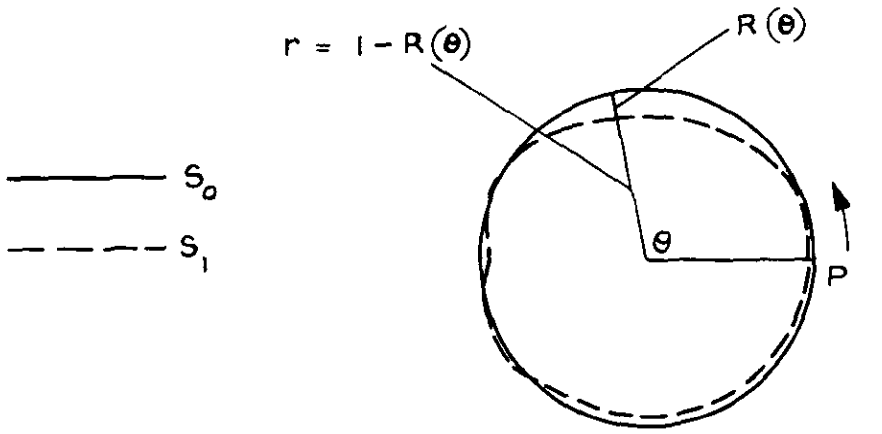

Fig. 3 Images of S_0 and S_1 in the σ plane

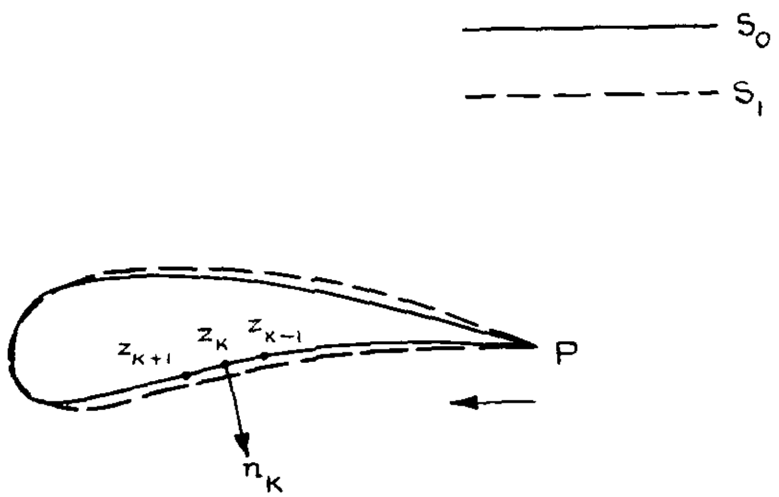

Fig. 4 The perturbation section S_1

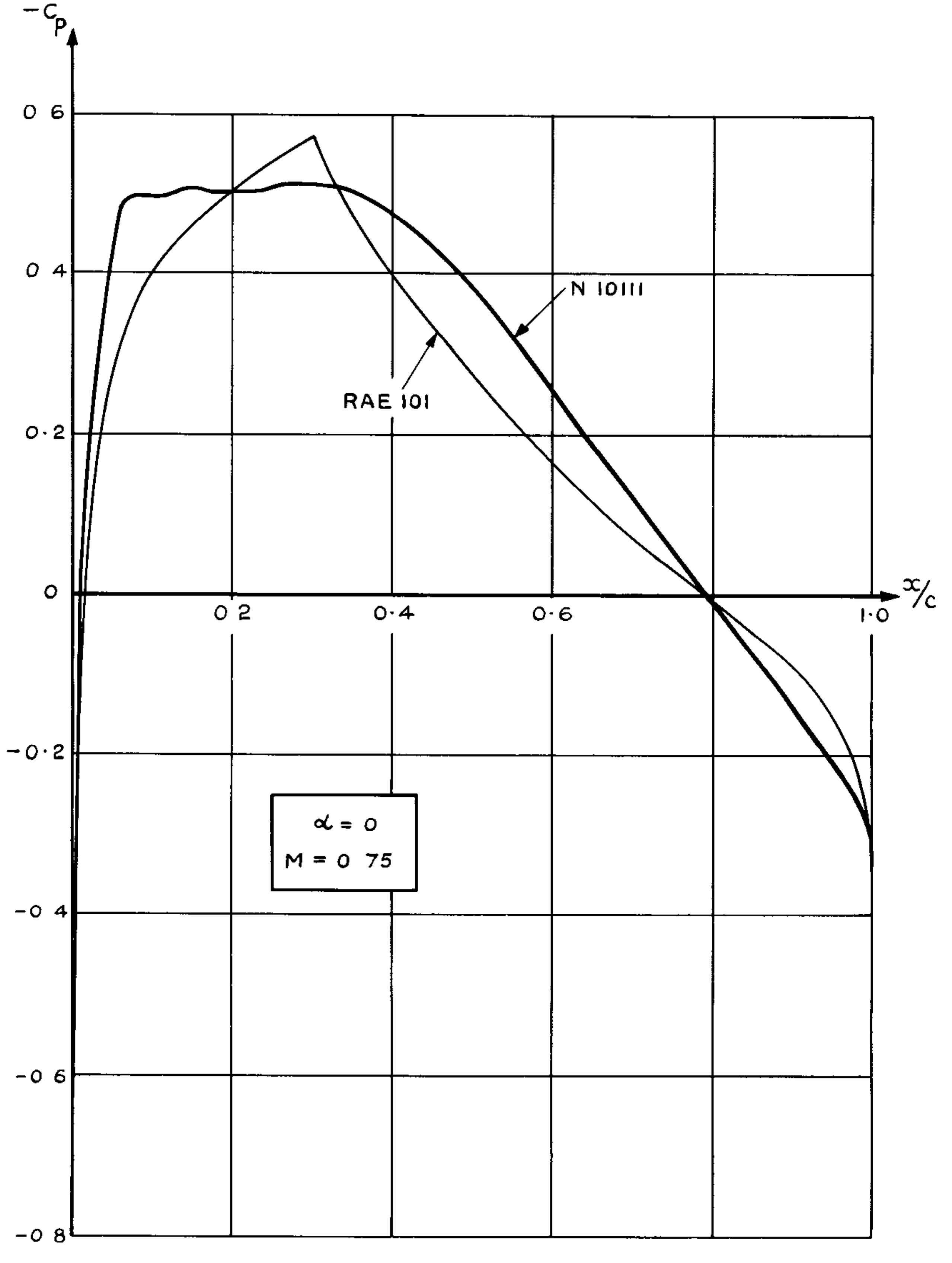

Fig. 5 Pressure distributions for N 10111 and RAE 101

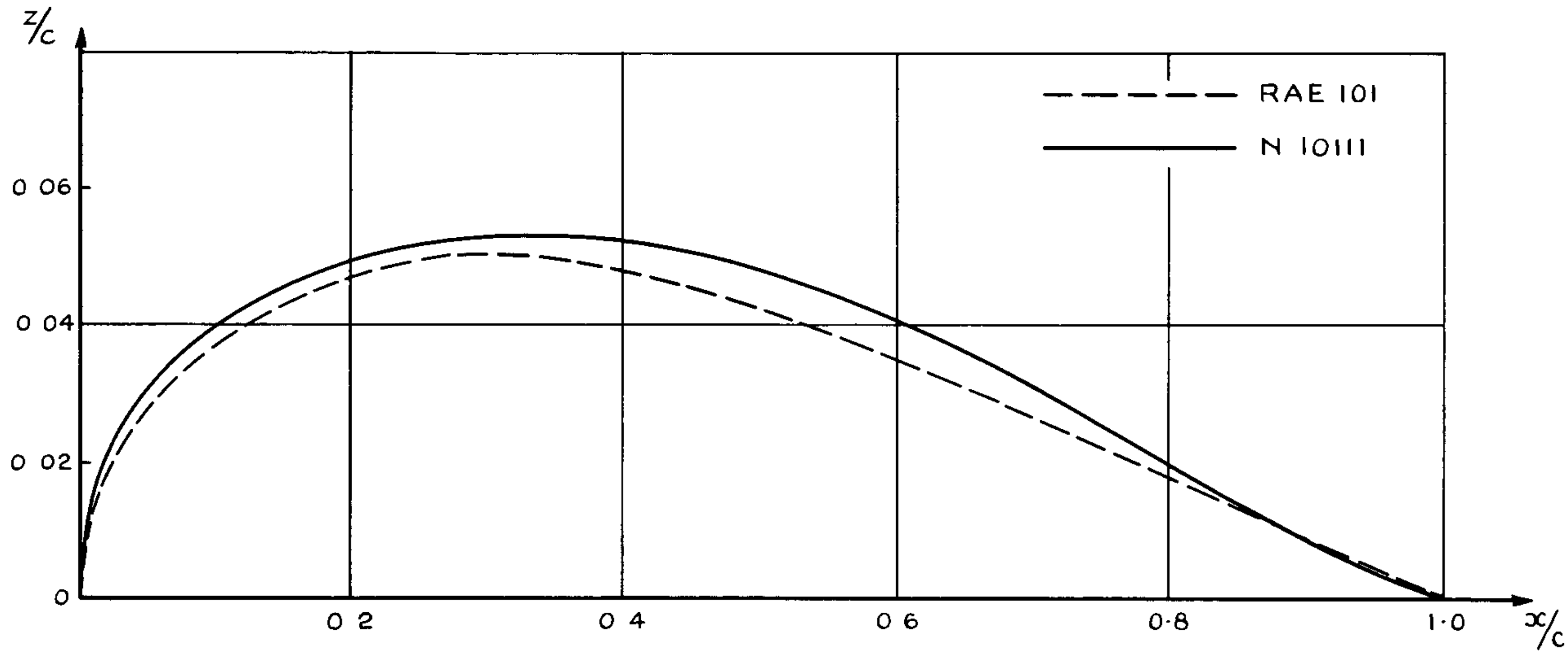

Fig. 6 N 10111 and RAE 101 sections

x/c	z/c
1.0000000	0.0000000
0.9965716	0.0001756
0.9866285	0.0008089
0.9704922	0.0019851
0.9484894	0.0038119
0.9210549	0.0064019
0.8887155	0.0098321
0.8520540	0.0138789
0.8116458	0.0184911
0.7680751	0.0234834
0.7219258	0.0286558
0.6737643	0.0337875
0.6241409	0.0386697
0.5735536	0.0431188
0.5225031	0.0469303
0.4715396	0.0499220
0.4211108	0.0520072
0.3715284	0.0530002
0.3231802	0.0529881
0.2765120	0.0521512
0.2320270	0.0505584
0.1902654	0.0482775
0.1517113	0.0453648
0.1168369	0.0418140
0.0860482	0.0376710
0.0597211	0.0328759
0.0380304	0.0273080
0.0209558	0.0211971
0.0090177	0.0149708
0.0022716	0.0077588
0.0000000	0.0000000

Fig.6a Ordinates of N10111

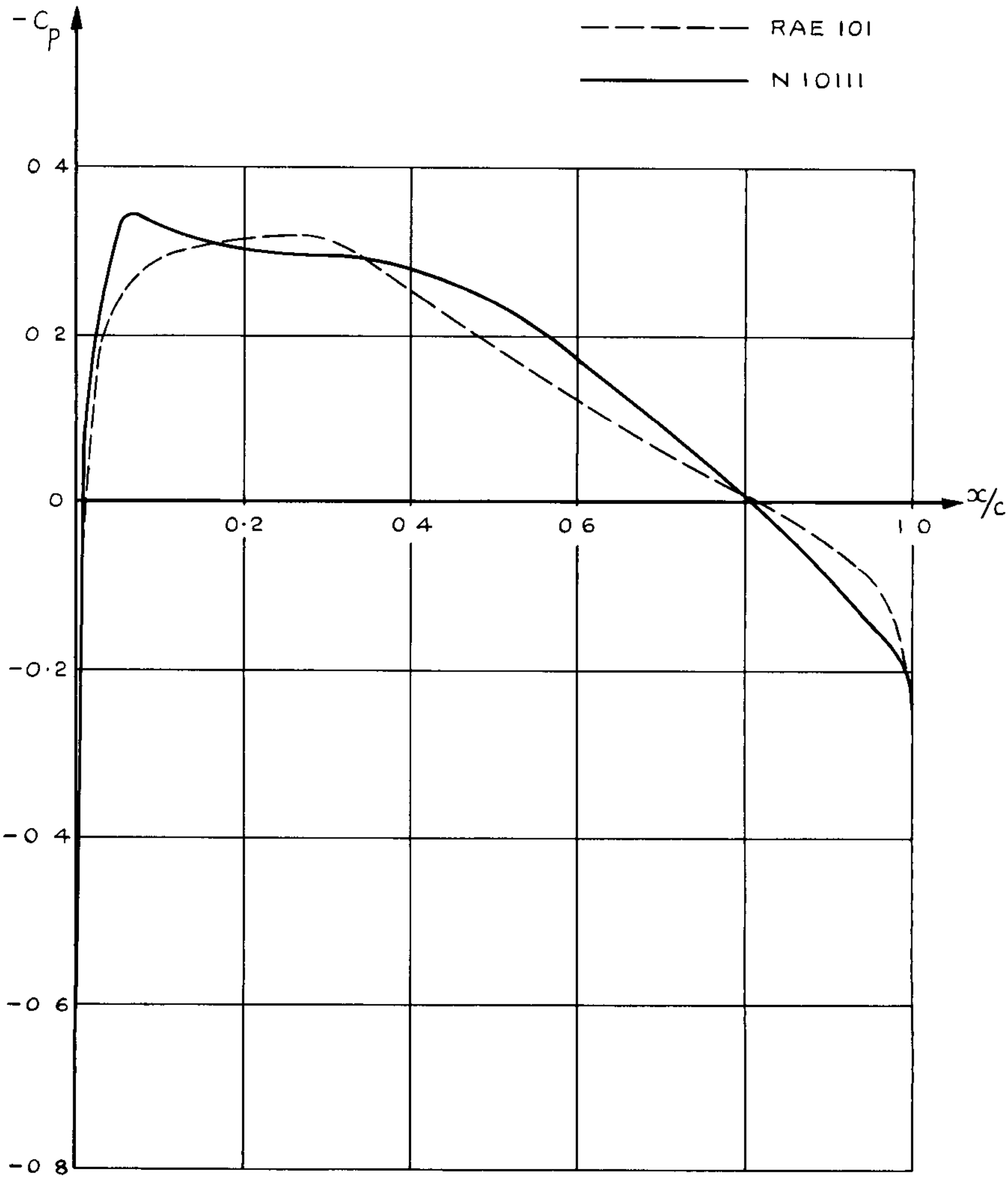

Fig. 7 Incompressible pressure distributions at zero incidence

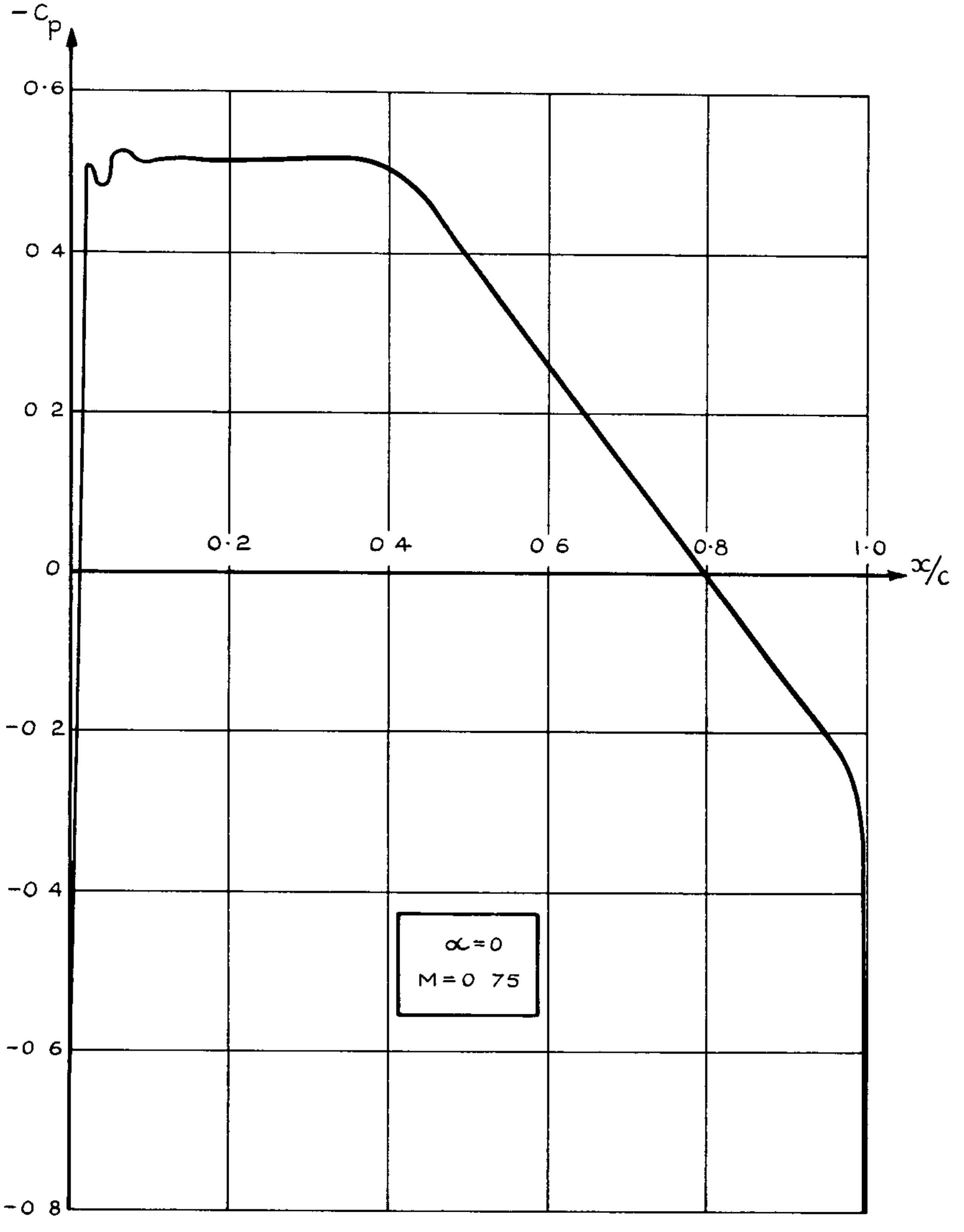

Fig. 8 Pressure distribution for intermediate section iterate N1016

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