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By

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1969

Price 4s 6d. net

C.P. No. 1027*

August, 1967

Calculation of Compressible Turbulent Boundary Layer on a Flat Plate

- by -

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SUMMARY

Calculations are presented of the compressible turbulent boundary layer on a flat plate. They have been made by a new, general, accurate and economical procedure. The physical inputs chiefly comprise: (i) a form of the mixing-length hypothesis, and (ii) the assumption of a uniform effective Prandtl number. The predictions are compared with available experimental data and empirical correlations; the agreement is satisfactory. It is pointed out that the same method may be expected to give good predictions even in more complex situations.

*Replaces A.R.C.29 564

1. Introduction

1.1 Purpose of the present paper

A new calculation procedure for turbulent boundary layers has been put forward in Refs. 1 and 2. This procedure is mathematically accurate, economical and widely applicable. Such a convenient mathematical tool prepares the way for research into physical hypotheses. The present paper provides an illustration of such research.

The problem considered here is that of a compressible turbulent boundary layer on a smooth isothermal flat plate in air. Attention will be given to the effects of Mach number, and of wall-to-mainstream temperature ratio, on the frictional drag and heat-transfer coefficient at the wall.

The available prediction methods for flat-plate drag and heat transfer have been summarised in Refs. 3 and 4. These papers compare predictions of the methods with each other and with experimental data; and they provide empirical correlations which fit available experimental data with reasonable accuracy.

Despite their simplicity and accuracy over a restricted range of conditions, such empirical correlations can hardly form the basis of a general theory. For they can be easily extended beyond their range of validity only when further experimental data become available. It is hard to modify, for example, the correlations of Refs. 3 and 4 to account for the effects of pressure gradient or of non-uniform wall temperature.

It is desirable therefore to construct a general theoretical framework, and then to explore the implications of a simple but plausible hypothesis and compare the results with experimental data or empirical correlations. The present paper is a step in this direction; here our purpose is to test the theory of Refs. 1 and 2 for the case of the compressible turbulent boundary layer on a flat plate.

1.2 Scope and outline of the present contribution

The present paper will be based on the calculation procedure developed in Refs. 1 and 2. Some important features of the procedure will be outlined in Section 2. The method involves solution of partial differential equations by a finite-difference technique, and incorporates two novel features. Firstly, the grid is so chosen that it adjusts its width so as to fit the thickness of the boundary layer. Secondly, once-for-all Couette-flow integrations are used near the wall, where the longitudinal convection is negligible.

The effective viscosity is calculated from a form of Prandtl's[5] mixing-length hypothesis, and the effective Prandtl number is regarded as uniform *cross the layer. In Section 3, results are presented of the computations for the drag coefficient and Stanton number of a flat plate; these results are compared with experimental data. The conclusions are given in Section 4. Taken together with the results of Ref. 9, they imply that the implications of the mixing-length hypothesis agree well with experiment over a wide range of conditions.

2. Desoription of the Calculation Yethcd

Since the theory which we shall use has been described in Ref. 1, and in more detail in Ref. 2, we here present only the important points of the theory.

2.1 Partial differential equations

For the compressible boundary layer, we shall solve the partial differential equations which govern the streamwise velocity u and the stagnation enthalpy $\tilde{h}$. The independent co-ordinates will be x and ω, where x is the distance along the plate and ω is a non-dimensional stream function; ω is defined so that it equals zero at the wall (subscript S) and unity at the outer edge of the boundary layer (subscript G). Thus in $x \sim \omega$ co-ordinates we have:

Conservation of momentum:

$$\frac{\partial u}{\partial x} + \frac{\dot{m}''_S + \omega(\dot{m}''_G - \dot{m}''_S)}{(\psi_G - \psi_S)} \frac{\partial u}{\partial \omega} = \frac{\partial}{\partial \omega}\left\{\frac{\rho u \mu_{eff}}{(\psi_G - \psi_S)^2} \frac{\partial u}{\partial \omega}\right\} - \frac{1}{\rho u}\frac{dp}{dx}; \quad \ldots(2.1.1)$$

Conservation of stagnation enthalpy:

$$\frac{\partial \tilde{h}}{\partial x} + \frac{\dot{m}''_S + \omega(\dot{m}''_G - \dot{m}''_S)}{(\psi_G - \psi_S)} \frac{\partial \tilde{h}}{\partial \omega} = \frac{\partial}{\partial \omega}\left\{\frac{\rho u \mu_{eff}}{(\psi_G - \psi_s)^2 \sigma_{eff}} \frac{\partial \tilde{h}}{\partial \omega}\right\} +$$

$$+ \frac{\partial}{\partial \omega}\left\{\frac{\rho u \mu_{eff}}{(\psi_G - \psi_S)^2}\left(1 - \frac{1}{\sigma_{eff}}\right)\frac{\partial(u^2/2)}{\partial \omega}\right\} \quad \ldots(2.1.2)$$

All the symbols are systematically defined in Nomenclature. It should suffice here to note that μ_{eff} stands for the effective viscosity, that $\dot{m}''_S$ is the mass-transfer rate through the wall, and that $-\dot{m}''_G$ represents the rate of entrainment into the boundary layer.

2.2. Physical hypotheses

<u>The effective viscosity.</u> We shall use a form of Prandtl's[5] mixing-length hypothesis for evaluating the effective viscosity. Thus,

$$\mu_{eff} = \rho \ell^2 \left| \partial u / \partial y \right|, \quad \ldots(2.2.1)$$

where ℓ is the mixing length. Further, we shall postulate the following variation of ℓ across the layer:

$$\left.\begin{array}{l} cl < y \; 6 \; \lambda y_\ell / K \; : \; \ell = Ky \\ \lambda y_\ell / K < Y \; : \; \ell = \lambda y_\ell \end{array}\right), \quad \ldots(2.2.2)$$

where/

where λ and K are constants, y is the distance from the wall, and y_ℓ is the distance (from the wall) of a point at which the velocity equals 0.99 times the free-stream velocity. A similar variation of the mixing-length was first proposed by Hudimoto[6], and its suitability has been confirmed by the experimental data collected by Escudier[7]. Also, Maise and McDonald.8 have shown, from experimental data for compressible boundary layers that, up to the Mach number of 5, the effect of compressibility on mixing length is negligible. Further, Spalding[9], by use of eq. (2.2.1) and (2.2.2) has obtained predictions which are in good agreement with a wide variety of experimental data.

The effective Prandtl number. We shall assume that the effective Prandtl number σ_{eff} is uniform across the boundary layer. The available experimental data collected by Kestin and Richardson[10] roughly conform to this behaviour.

Values of the constants. The values of the constants will be chosen so as to procure good agreement with experimental data in some simple cases. We shall take K as 0.435, λ as 0.09, and the effective Prandtl number σ_{eff} as 0.9 throughout the present work. It is important to note that the same set of hypotheses and the same values of constants were used in Ref. 9; that reference and the present paper, taken together, demonstrate that satisfactory agreement with experiment can be obtained, over a wide range of conditions, by use of the above-mentioned set of hypotheses.

2.3 The region near a wall

The hypotheses given above are applicable to only the filly-turbulent part of the boundary layer, where the laminar contribution is negligible. Near the wall, however, both the turbulent and laminar viscosities play comparable roles. As mentioned earlier, the smallness of the longitudinal convection in the vicinity of a wall enables us to use the once-for-all Couette-flow integrations for this region. The special hypotheses, giving the effective viscosity for the wall-near region, manifest themselves through these integrations. We shall once again omit details and use Ref. 2 where the Couette-flow integrations and the useful relationships extracted from them, have been described and explained. Here the reader should note that we use van Driest's[11] hypothesis for the variation of the effective viscosity near the walk; the resulting "universal law of the wall" is used as an asymptote for the profile in the fully-turbulent part of the layer.

2.4 Entrainment rate

Two mass-transfer rates, $\dot{m}''_S$ and $\dot{m}''_G$, appear in the partial differential equations (2.1.1) and (2.1.2). Of these $\dot{m}''_S$ will be taken as zero, because we shall deal with only the Impermeable-wall case; the other quantity, $\dot{m}''_G$, is the negative of the entrainment rate through the outer edge of the boundary layer. If we apply eq. (2.1.1) at the outer edge (i.e., at $\omega = 1$) and use the mixing-length hypothesis, it can be shown, after some algebraic manipulation, that:

$$\dot{m}''_G = - 2 \rho_G \ell_G^2 \left| \partial^2 u/\partial y^2 \right| G. \qquad \ldots(2.4.1)$$

We shall use this equation (or rather a finite-difference form of it) in our calculation procedure. As a consequence of the definition of the stream function ψ, we can obtain the relation:

$$\frac{d\left(\psi_G - \psi_S\right) /}{dx}$$

$$\frac{d}{dx}(\psi_G - \psi_S) = \dot{m}''_S - \dot{m}''_G, \qquad \ldots(2.4.2)$$

which will be used to calculate $(\psi_G - \psi_S)$ as the integration proceeds.

2.5 Solution by finite-difference method

Ref. 1 describes the finite-difference procedure which we shall use for the solution of the conservation equations of Section 2.1. Ref. 2 provides further details and also a computer programme based on this solution procedure. Here it is necessary to describe the method only in general terms.

The main novelties of the solution procedure are the choice of the ω co-ordinate in conjunction with the entrainment law, and the use of the Couette-flow relationships near the wall. The entrainment law ensures that the width of the grid always equals the thickness of the layer in which the dependent variables vary significantly; this makes the computation efficient. Further saving of computational effort comes from the use, near the wall, of the results of earlier integrations for the one-dimensional layer there.

The finite-difference procedure is of implicit type, and the difference equations have been made linear so that no iteration is necessary. The difference equations allow solution by a simple recurrence-formula technique.

The number of grid lines across the layer was six and the size of the forward step was adjusted so that the quantity of fluid entrained during the step equalled 10% of the amount of fluid already flowing in the layer. Repetition of some of the computations with smaller steps in both x and ω directions, showed that the above-mentioned grid size gave sufficient accuracy.

It may be of interest to the reader to know that, with this grid size, 1000 integrations can be performed in one minute of computing time on the IBM 7090 computer. This computing time is considerably less than that required for the procedures which have been reported elsewhere in the literature.

2.6 Specification of the fluid properties

For the computations to be presented in this paper, the density has been taken as inversely proportional to the absolute temperature, and the viscosity variation as given by

$$\frac{\mu}{\mu_G} = \left(\frac{T}{T_G}\right)^{0.76}, \qquad \ldots(2.6.1)$$

where the subscript G denotes the free-stream quantities. The laminar Prandtl number is 0.7. The specific heats are regarded as constant; their ratio is 1.4. The stagnation enthalpy $\tilde{h}$ is related to the specific enthalpy h via:

$$\tilde{h} = h + u^2/2. \qquad \ldots(2.6.2)$$

3./

3. Results of the Computations

3.1 The flat-plate drag

Uniform-property flow We now present the results of our computations, starting with the simplest case: the flat-plate boundary layer with uniform fluid properties. Fig. 1 shows the comparison of our prediction (the full line) with the experimentally-based correlation of Spalding and Chi, Ref. 3, (shown by the dots) for the drag of the flat plate. Here R_x is the length Reynolds number. The agreement is good. throughout; indeed the values of k and λ (mentioned earlier) have been chosen so as to procure good agreement precisely for this case.

Adiabatic plate: effect of Mach number. The equation for the stagnation enthalpy $\tilde{h}$ (2.1.2) can be solved as soon as a thermal boundary condition at the wall has been specified; by prescribing zero heat flux through the mall, we obtain results for the adiabatic-wall case. Fig. 2 shows how the drag varies with Mach number; here the ratio of the actual drag to the drag under uniform-property condition has been plotted, for the same value of the Reynolds number R_x. The fill curve represents our prediction and the dots display some experimental data collected by Schlichting[12]. The agreement of the theory with experiment is very satisfactory.

Effect of Mach number and temperature ratio. When finite heat transfer takes place through the wall, the temperature and density fields are affected; consequently, the drag values change. We present some computations for the isothermal-wall case with various wall-to-mainstream temperature ratios. In Fig. 3 the drag ratio is plotted against Mach number for different values of TS $/T_G$. The full lines show our predictions; the broken curves represent the Spalding-Chi correlation, which is based upon a large number of experimental data. The agreement may be regarded as satisfactory.

3.2 The flat-plate Stanton number

Uniform-property case. Once again we begin with the case in which the fluid properties remain almost uniform. In Fig. 4 is shown the comparison of cur Stanton-number prediction with the experimental data of Reynolds, Kays and Kline[13] (which are in agreement with the Chi-Spalding[4] recommendation that the Reynolds-analogy factor equals 1.16). We have chosen 0.9 as the value of the effective Prandtl number with reference to these data; obviously therefore, the agreement is quite good.

Effect of temperature ratio. Even at low Mach numbers, non-uniformities of density can be introduced by large temperature differences across the layer. In Fig. 5, we show the influence of wall-to-mamstream temperature ratio on the Stanton number. It oanbe seen that a wall colder than the free stream is less effective in increasing the Stanton number than is a wall hotter than the free stream in decreasing it. We now compare, in Fig. 6, our predictions with the experimental data of Chi and Spalding[4], obtained at low Mach number for various values of T_G/T_S. The predictions agree well with the experiments1 results.

Effect of Mach number and temperature ratio. The ratio of the actual Stanton number to the one under uniform-property conditions for the same R_x has been plotted, in Fig. 7, against the Mach number far different temperature ratios. The broken curves show the Chi-Spalding correlations; these are drawn only where the correlations are based upon experimental data. The agreement, once again, is satisfactory.

In all these computations the Reynolds-analogy factor was very nearly equal to 1.16 - a value, recommended by Chi and Spalding[4]. The value of the reoovery factor was around 0.93, whereas experimental data suggest a value of

about/

about 0.9. In this connection, it should be remembered that the value of the recovery factor mainly depends upon the value of the effective Prandtl number; it can be shown that the recovery factor should be larger than the effective Prandtl number (see Ref. 14). Probably, if the effective Prandtl number of the turbulent region were dropped to about 0.86, the recovery factor would be in good agreement with experiment, and the heat-transfer prediction would be scarcely affected.

4. Conclusions

(1) Successful predictions of the drag and the Stanton number have been obtained for a compressible turbulent boundary layer on a flat plate. The predictions agree well with available experimental data and empirical correlations.

(2) What is more important than the particular results presented here is that they have been obtained by use of a generally-applicable calculation method, based on a single effective viscosity hypothesis. Thus the present work serves as a demonstration of the capabilities of the solution procedure of Ref. 1, and of the realism of the mixing-length hypothesis.

(3) The same equations and the same set of hypotheses and constants, have been used in Ref. 9 where the predictions have been shown to agree well with experimental data for uniform-property flows in the presence of various pressure gradients and non-uniform wall temperature. The present paper deals with non-uniform-property case with zero pressure gradient. It can be reasonably expected that the same method and hypotheses will give good agreement with experiment when the pressure, the fluid properties and the wall temperature are all non-uniform.

5. Acknowledgements

The author wishes to thank Professor Spalding of Imperial College, London for helpful advice and encouragement. Thanks are also due to I.C.I.(India) Pvt. Ltd., for the tenure of a scholarship during the performance of the work reported here.

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Nomenclature/

Nomenclature

c_f	drag coefficient $\left(\equiv 2\tau_S/(\rho_G u_G^2)\right)$
$c_{f,o}$	drag ocefficient under uniform-property condition
h	specific enthalpy
$\tilde{h}$	stagnation enthalpy
K	a mixing-length constant
ℓ	the mixing length
$\dot{m}''$	mass-transfer rate across a boundary
p	pressure
R_x	length Reynolds number $\left(\equiv \rho_G u_G x/\mu_G\right)$
St	the Stanton number
St_o	the Stanton number under uniform-property condition
T	absolute temperature
u	velocity in the x direction
x	distance along the plate
Y	distance from and normal to the wall
y_ℓ	a characteristic thickness of the layer; distance from the wall of a point where $u = 0.99\ u_G$.
λ	a mixing-length constant
μ	laminar viscosity of the fluid

μ_{eff}✓

μ_{eff}	effective viscosity
ρ	density of the fluid
σ_{eff}	the effective Prandtl number
τ	local shear stress
ψ	a stream function ($d\psi \equiv \rho u \, dy$)
ω	dimensionless stream function $\left(\equiv (\psi - \psi_S)/(\psi_G - \psi_G)\right)$

Subscripts

G	free stream
S	wall

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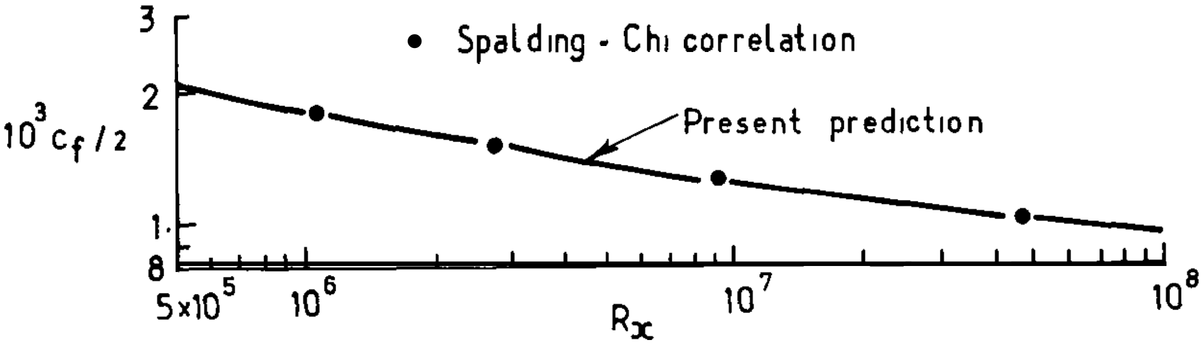

FIG.1 FLAT-PLATE DRAG WITH UNIFORM PROPERTIES.
COMPARISON WITH SPALDING - CHI CORRELATION

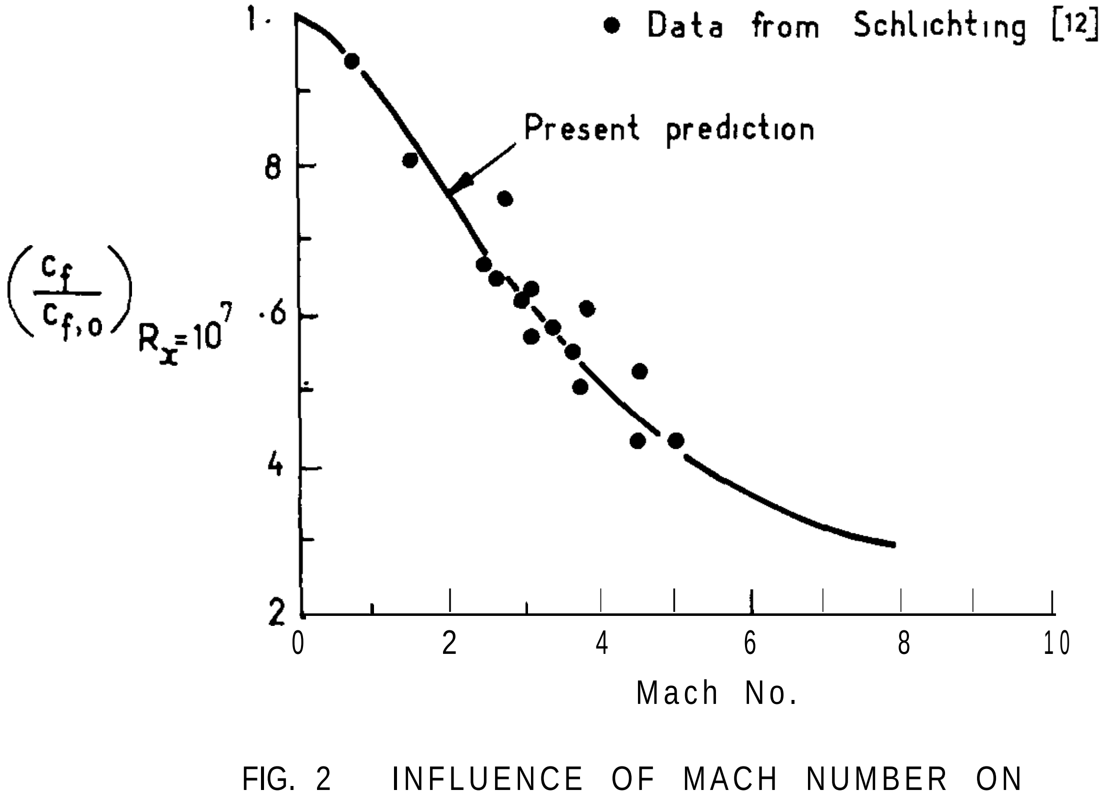

FIG. 2 INFLUENCE OF MACH NUMBER ON
ADIABATIC FLAT- PLATE DRAG
COMPARISON WITH EXPERIMENTAL DATA

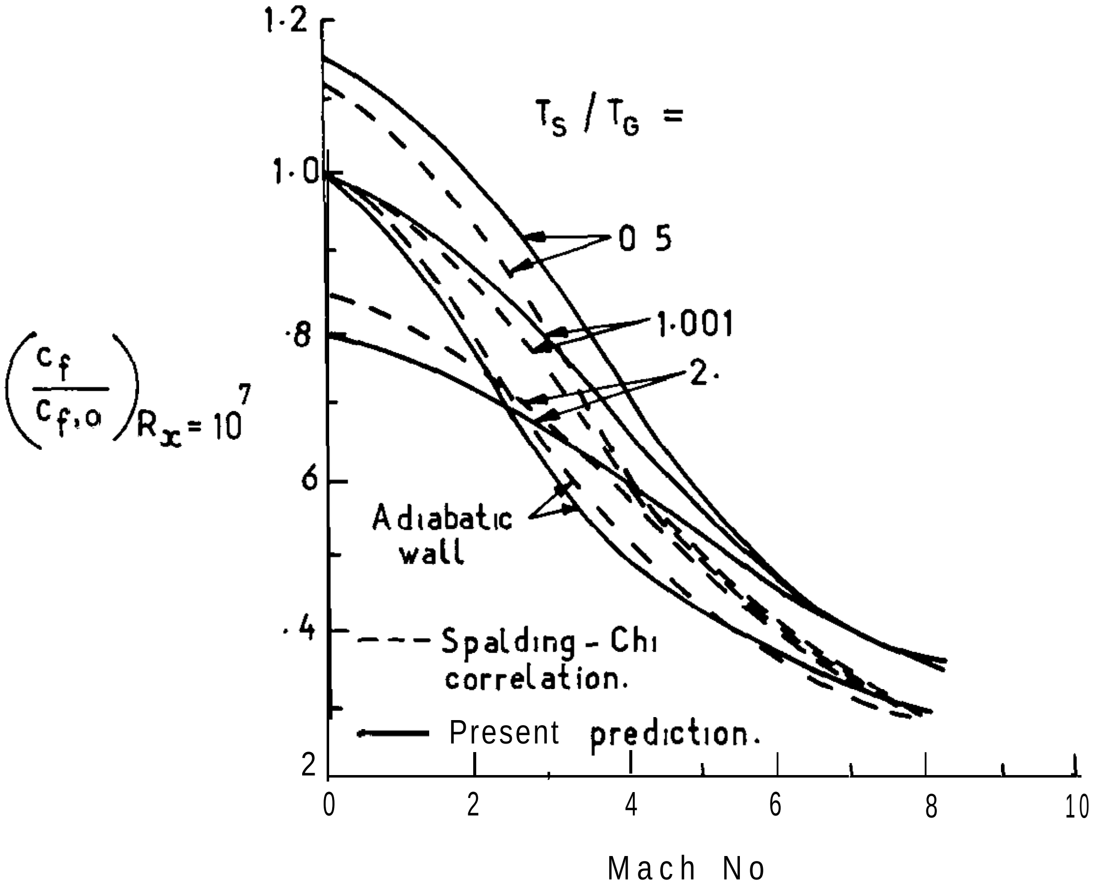

FIG. 3 THE DRAG OF A COMPRESSIBLE BOUNDARY LAYER ON FLAT PLATE. COMPARISON WITH SPALDING - CHI CORRELATION.

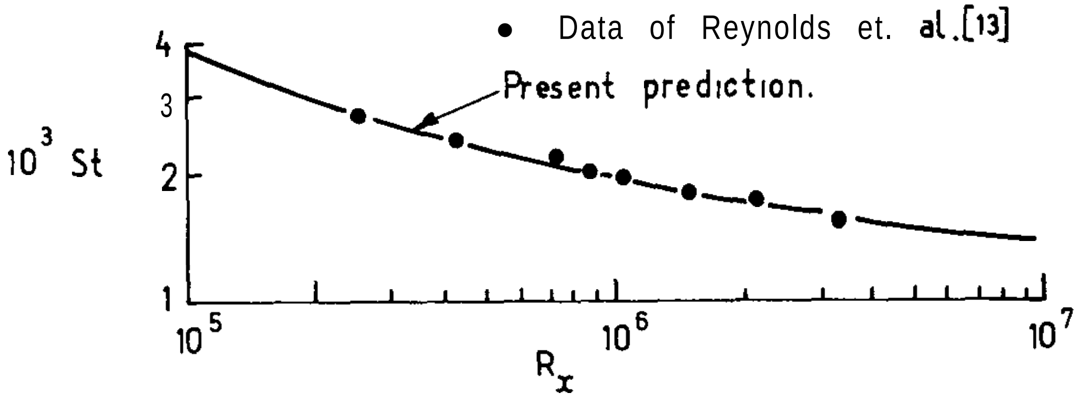

FIG 4 COMPARISON WITH UNIFORM-PROPERTY FLAT-PLATE DATA FOR HEAT TRANSFER

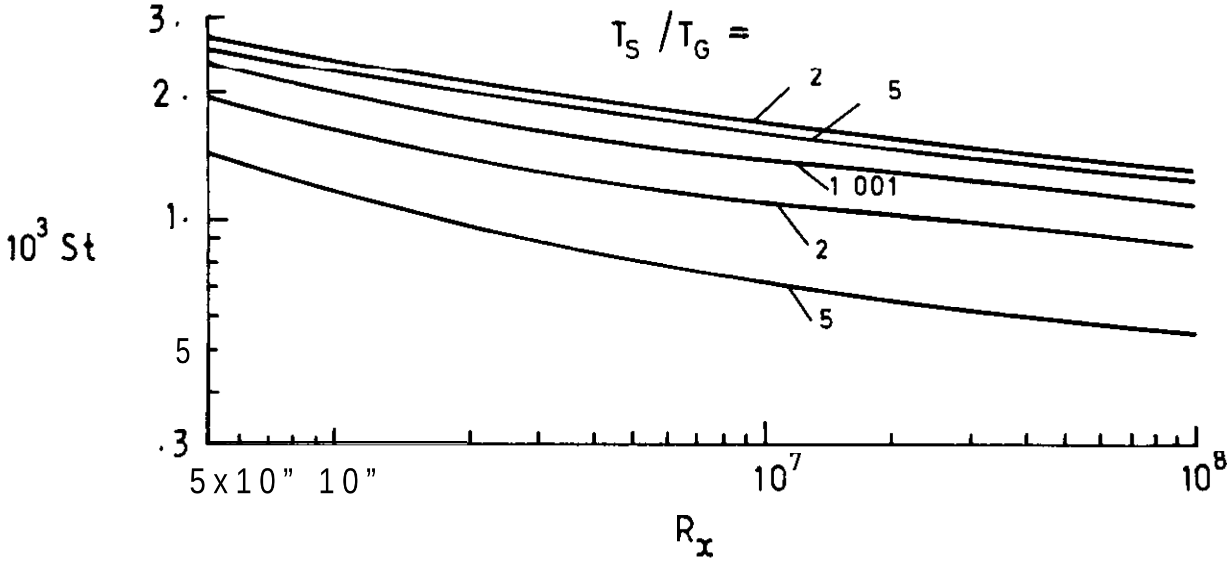

FIG 5 FLAT-PLATE STANTON NUMBERS AT LOW MACH NUMBER AND FOR VARIOUS TEMPERATURE RATIOS.

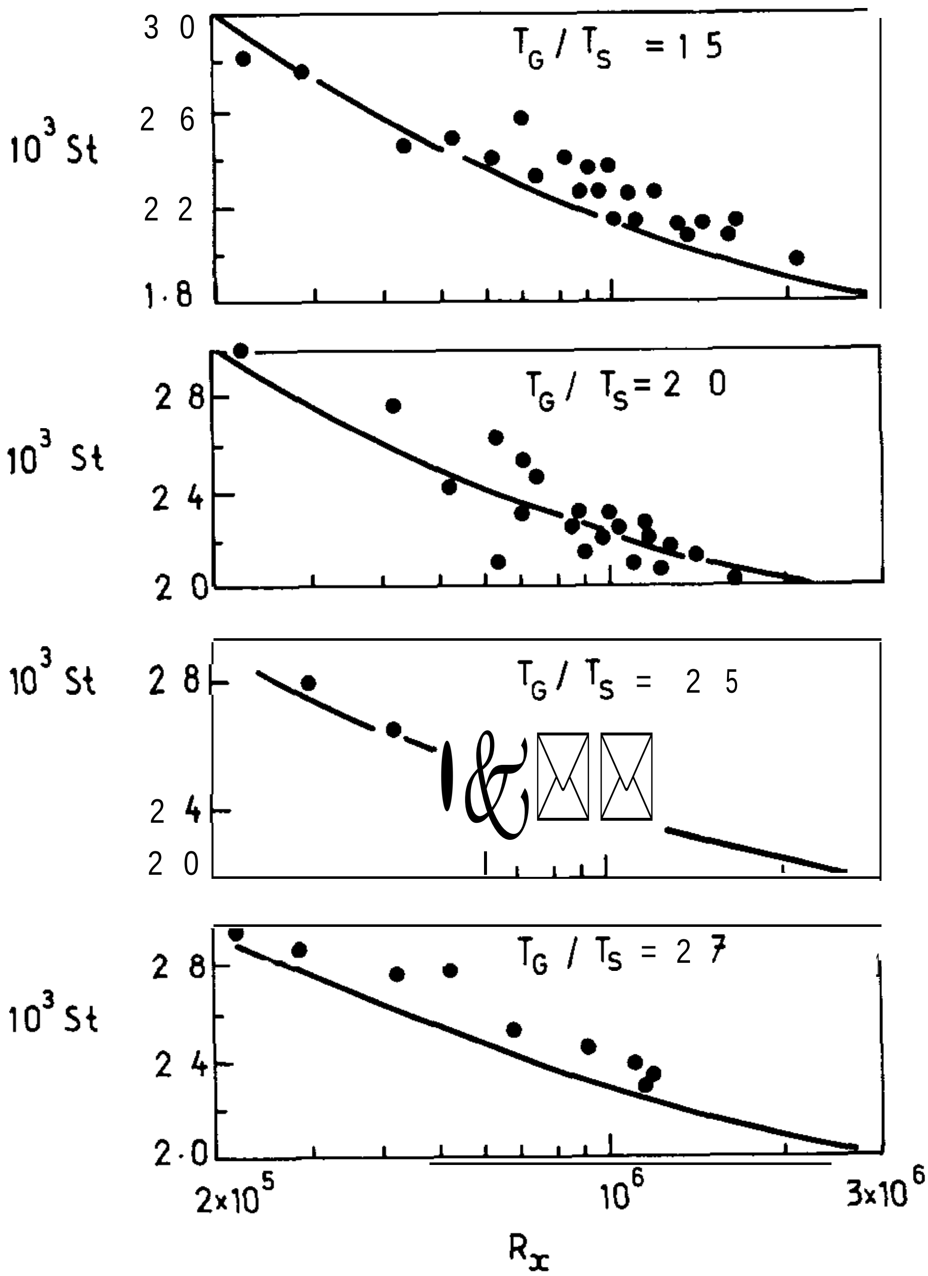

FIG 6 COMPARISON WITH FLAT-PLATE STANTON - NUMBER DATA FOR LARGE TEMPERATURE RATIOS

DOTS REPRESENT DATA OF CHI AND SPALDING [4], THE FULL LINES REPRESENT PRESENT PREDICTIONS

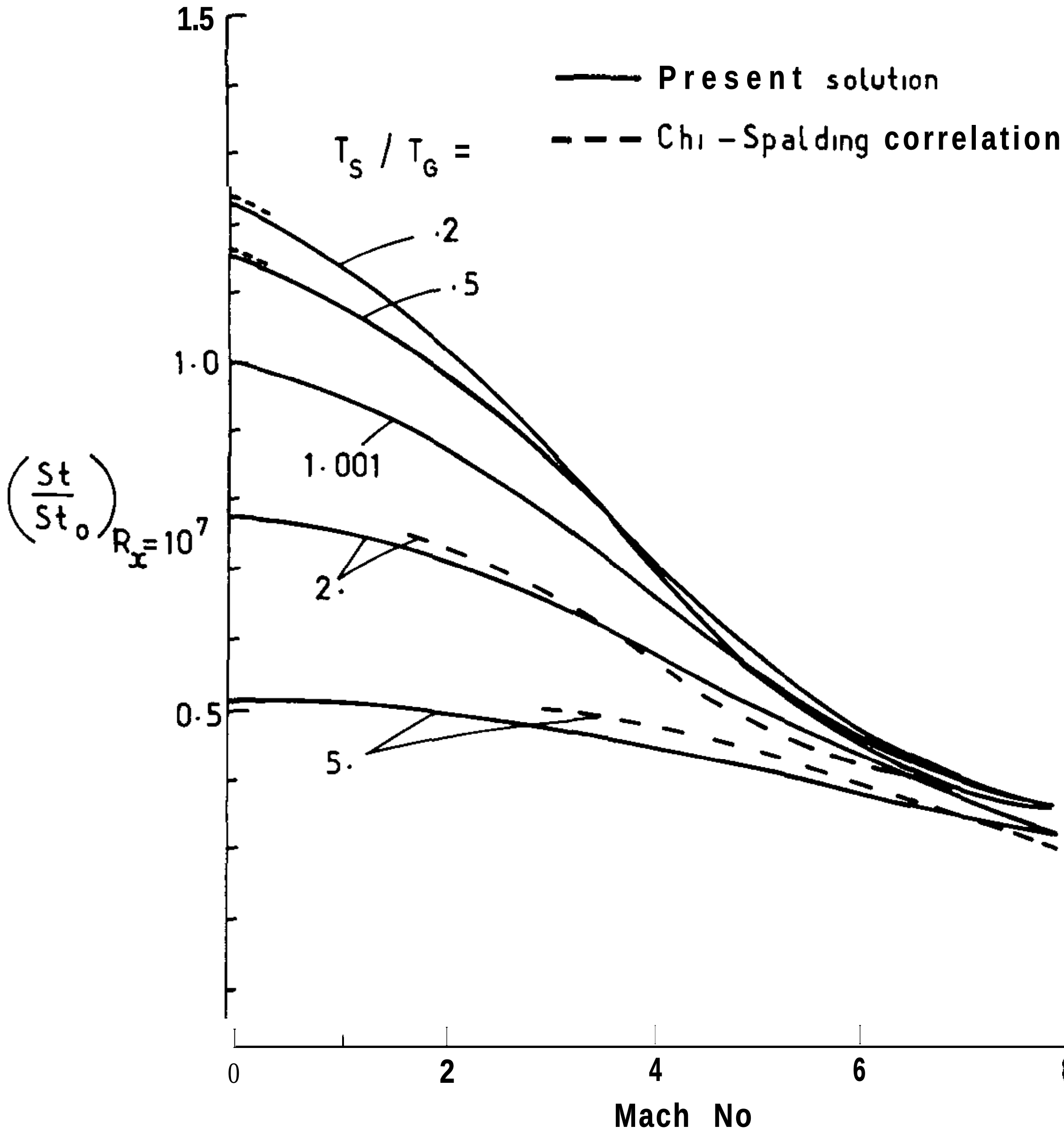

FIG 7 EFFECT OF MACH NUMBER AND TEMPERATURE RATIO ON FLAT-PLATE STANTON NUMBER.

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