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A Note on Some Integrals in Aerodynamics

By

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A note on some integrals in aerodynamics

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SUMMARY

Some double integrals in aerodynamics are difficult to evaluate because of the singularities in the range of integration. The Dirac delta function has been found useful in evaluating such integrals. Some examples of its use are given.

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1 Introduction

In aerodynamics some double integrals occur, where, it has been thought, the order of integration cannot be inverted. The reason usually given for this is the presence of singularities of the integrand in the range of integration. It is shown here that many of the single integrals which occur in the evaluation of these double integrals are usually evaluated incorrectly and a term involving the Dirac delta function has been omitted. When this additional term is included the order of integration can be inverted. Some examples are given.

2 Preliminary Results

2.1 The Dirac delta function $\delta(x)$ is the 'derivative' of the Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 . \end{cases}$$

It is zero for $x \neq 0$ and at $x = 0$ it is infinite in such a way that

$$\int_a^b f(x)\, \delta(x)\, dx = f(o)$$

whenever the origin is an internal point of (a,b). We shall need to use a delta function with the property

$$\int_0^\infty f(x)\, \delta(x)\, dx = f(o) .$$

We may consider this to be the limiting case of the equation

$$\int_0^\infty f(x)\, \delta(x-\varepsilon)\, dx = f(\varepsilon)$$

as $\varepsilon \to 0$.

Since some writers use a delta function with the property

$$\int_0^\infty f(x)\, \delta(x)\, dx = \tfrac{1}{2} f(o)$$

we shall give the delta function used in this paper the suffix R to show that it has the property

$$\int_0^\infty f(x)\, \delta_R(x)\, dx = f(o) .$$

A rigorous definition of a delta function is given by Temple[1]. It depends on the theory of weak convergence. A sequence of functions $f_n(x)$ is said to be weakly convergent if the sequence of functions

$$F_n(\phi) = \int_a^b f_n(x)\ \phi(x)\ dx$$

converges for each function $\phi(x)$ which is continuous and has continuous derivatives of all orders in the interval (a,b). The sequence $F_n(\phi)$ may tend to a limit $F(\phi)$ for all such $\phi(x)$ when no limit function $f(x)$ exists or when even though $f(x)$ exists the integral

$$\int_a^b f(x)\ \phi(x)\ dx$$

does not. The class of sequences $\{f_n(x)\}$ which give the same limit $F(\phi)$ are then said to define a weak function $f(x)$. The Dirac delta function is a weak function.

The theory is similar to Cantor's definition of a real number. A real number is defined to be the class of sequences of rational numbers with a given property. For example the real number $\sqrt{2}$ is defined to be the class of sequences $\{x_n\}$ of rational numbers x_n with the property $x_n^2 \to 2$ as $n \to \infty$.

2.2 The trigonometric functions $\{\sin n\theta\}$ form a complete orthogonal set in the range $(0,\pi)$. It is easily seen that the Fourier sine series of $\delta(\theta-\phi)$ where both θ and ϕ are in the range $(0,\pi)$ is given by the equation

$$\frac{\pi}{2}\,\delta(\theta-\phi) = \sum_1^\infty \sin n\theta \sin n\phi \ . \qquad (1)$$

Similarly the Fourier cosine series of $\delta(\theta-\phi)$ is given by

$$\frac{\pi}{2}\,\delta(\theta-\phi) = \tfrac{1}{2} + \sum_1^\infty \cos n\theta \cos n\phi \ . \qquad (2)$$

As $\phi \to 0$, when θ is in the range $(0,\pi)$, this becomes

$$\frac{\pi}{2}\,\delta_R(\theta) = \tfrac{1}{2} + \sum_1^\infty \cos n\theta \qquad (3)$$

where

$$\int_o^\pi f(\theta)\ \delta_R(\theta)\ d\theta = f(o) \ .$$

In equations (1), (2), (3) the left hand side is the weak limit of the sums formed from the first n terms of the series on the right.

2.3 The integral

$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\phi}\, d\theta = \pi \frac{\sin n\phi}{\sin\phi}$$

is assumed to be known.

We see by using this result that the Fourier cosine series of the function

$$\frac{1}{\cos\theta - \cos\phi}$$

in the range $(0,\pi)$ is

$$\frac{1}{\cos\theta - \cos\phi} = 2\sum_1^\infty \frac{\sin n\phi}{\sin\phi} \cos n\theta \, . \tag{4}$$

The series is divergent, but its first Cesaro sum, its (C,1) sum is $\frac{1}{\cos\theta - \cos\phi}$. We shall also need the Fourier sine series of $\cot\frac{1}{2}\theta$ in $(0,\pi)$, it is

$$\cot\tfrac{1}{2}\theta = 2\sum_1^\infty \sin n\theta \, . \tag{5}$$

Here again the series is divergent but its first Cesaro sum is $\cot\frac{1}{2}\theta$. Both results are true in the sense of weak convergence.

2.4 If

$$F(p) = \int_0^\infty e^{-pz} f(z)\, dz$$

is the Laplace transform of $f(z)$ and if $G(p)$ is the Laplace transform of $g(z)$ then the Laplace transform of the convolution integral

$$\int_0^z f(t)\, g(z-t)\, dt$$

is $F(p)\,G(p)$.

3 Examples

3.1 The integral equation

$$g(x) = \frac{1}{2\pi} \int_{-1}^{+1} \frac{f(\xi)}{x - \xi} d\xi$$

occurs in two dimensional subsonic aerofoil theory and in slender body theory. The solution needed in slender body theory is

$$f(x) = -\frac{2}{\pi} \frac{1}{\sqrt{1-x^2}} \int_{-1}^{+1} \frac{g(\xi)\sqrt{1-\xi^2}}{x-\xi} d\xi .$$

Because of the singularities in the integrals it is not easy to show that this is in fact a solution of the integral equation. Lomax, Heaslett and Fuller[2] verify this result by using their theory of residuals. (A residual is defined to be the change in the value of a double integral when the order of integration is reversed.) We shall show that the result can be verified more simply by introducing the delta function. The two methods are basically the same.

We have

$$\frac{1}{2\pi} \int_{-1}^{+1} \frac{f(\xi)}{x-\xi} d\xi = -\frac{1}{\pi^2} \int_{-1}^{+1} \frac{d\xi}{(x-\xi)\sqrt{1-\xi^2}} \int_{-1}^{+1} \frac{g(\eta)\sqrt{1-\eta^2}}{\xi-\eta} d\eta$$

$$= -\frac{1}{\pi^2} \int_{-1}^{+1} g(\eta)\sqrt{1-\eta^2}\, d\eta \int_{-1}^{+1} \frac{d\xi}{\sqrt{1-\xi^2}\,(x-\xi)(\xi-\eta)} . \quad (6)$$

If we evaluate the inner integral by the usual methods we get

$$\int_{-1}^{+1} \frac{d\xi}{\sqrt{1-\xi^2}\,(x-\xi)(\xi-\eta)} = \frac{1}{\eta - x} \int_{-1}^{+1} \frac{1}{\sqrt{1-\xi^2}} \left[\frac{1}{x-\xi} - \frac{1}{\eta-\xi}\right] d\xi$$

$$= 0 \qquad\qquad x \neq \eta .$$

When $x = \eta$ the two first order singularities at $\xi = x$ and $\xi = \eta$ coalesce to form one second order singularity. If the right hand side of equation (6) is to reduce to $g(x)$ then this inner integral must behave like a delta function at $x = \eta$. We shall show that this is so by using the series (4) and (1).

If we put $\xi = -\cos\theta$, $x = -\cos\phi$, $\eta = -\cos\psi$ the inner integral becomes

$$-\int_0^{\pi} \frac{d\theta}{(\cos\theta - \cos\phi)(\cos\theta - \cos\psi)} .$$

Using the series (1) this becomes

$$- \int_0^\pi \left\{ 2 \sum_1^\infty \frac{\sin n\phi}{\sin\phi} \cos n\theta \right\} \left\{ 2 \sum_1^\infty \frac{\sin m\psi}{\sin\psi} \cos m\theta \right\} d\theta = -2\pi \frac{\sum_1^\infty \sin n\phi \sin n\psi}{\sin\theta \sin\psi}$$

$$= -\pi^2 \frac{\delta(\phi-\psi)}{\sin\phi \sin\psi} .$$

Therefore

$$\frac{1}{2\pi} \int_{-1}^{+1} \frac{f(\xi)}{x-\xi} d\xi = \int_0^\pi g(\psi) \sin^2\psi \frac{\delta(\phi-\psi)}{\sin\phi \sin\psi} d\psi$$

$$= g(\phi) = g(x) .$$

3.2 There would at first sight seem to be no difficulty in evaluating integrals of the form

$$\int_0^\pi F(\phi)\, d\phi \int_0^\pi \cot \tfrac{1}{2}\theta \frac{\sin\theta}{\cos\theta - \cos\phi} d\theta$$

but it will be shown that unless the inner integral is evaluated correctly by using the delta function a wrong result will be obtained. This is because $\cot \frac{1}{2}\theta$ has a singularity at $\theta = 0$ which combines with the singularity in

$$\frac{1}{\cos\theta - \cos\phi}$$

when $\phi = 0$ to form a singularity of higher order.

The inner integral is

$$I(\phi) = \int_0^\pi \cot \tfrac{1}{2}\theta \frac{\sin\theta}{\cos\theta - \cos\phi} d\theta$$

$$= \int_0^\pi \left\{ 2 \sum_1^\infty \sin n\theta \right\} \left\{ 2 \sum_1^\infty \sin m\theta \cos m\phi \right\} d\theta$$

$$= 2\pi \sum_1^\infty \cos m\phi$$

$$= 2\pi \left\{ -\frac{1}{2} + \frac{\pi}{2} \delta_R(\phi) \right\} \qquad 0 \leq \phi \leq \pi$$

$$= -\pi + \pi^2 \delta_R(\phi) .$$

The value of this integral is usually given as $-\pi$. The correct value of the double integral is then

$$-\pi \int_0^\pi F(\phi)\, d\phi + \pi^2 F(o)$$

and not

$$-\pi \int_0^\pi F(\phi)\, d\phi \, .$$

3.3 The Abel integral equation

$$w(y) = \int_0^y \frac{g(x)}{(y-x)^{\frac{1}{2}}}\, dx \tag{7}$$

occurs frequently in supersonic aerodynamics. Its solution is

$$g(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{w(t)}{(x-t)^{\frac{1}{2}}}\, dt = -\frac{1}{2\pi}\int_0^x \frac{w(t)}{(x-t)^{3/2}}\, dt \, . \tag{8}$$

Lomax et al.[2] verify this result by using the theory of residuals. We shall here verify the result by using the delta function.

We have

$$\int_0^y \frac{g(x)}{(y-x)^{\frac{1}{2}}}\, dx = -\frac{1}{2\pi}\int_0^y \frac{dx}{(y-x)^{\frac{1}{2}}}\int_0^x \frac{w(t)}{(x-t)^{3/2}}\, dt$$

$$= -\frac{1}{2\pi}\int_0^y w(t)\, dt \int_t^y \frac{dx}{(y-x)^{\frac{1}{2}}(x-t)^{3/2}} \, .$$

The principal value of the inner integral

$$\int_t^y \frac{dx}{(y-x)^{\frac{1}{2}}(x-t)^{3/2}} \tag{9}$$

is zero when $y \neq t$. It can be seen that when $y = t$ the integral must behave like a delta function. We shall show below that the value of the integral is $-2\pi\, \delta_R(y-t)$.

With this result we have

$$\int_0^y \frac{g(x)}{\sqrt{y-x}}\,dx = -\frac{1}{2\pi}\int_0^y w(t)\,\{-2\pi\,\delta_R(y-t)\}\,dt$$

$$= w(y)\ .$$

3.31 The integral (8) is equal to

$$I(z) = \int_0^z \frac{du}{(z-u)^{\frac{1}{2}}\,u^{3/2}}$$

where $z = y-t$. To evaluate this integral we consider the integral

$$I(z,\alpha) = \int_0^z \frac{du}{(z-u)^{\frac{1}{2}}\,u^{3/2-\alpha}}$$

where α is a complex parameter. The integral exists for $R(\alpha) > \frac{1}{2}$.

We shall show by analytic continuation that the value of the function, defined for $R(\alpha) > \frac{1}{2}$, by the integral

$$\int_0^a I(z,\alpha)\,f(z)\,dz$$

is $-2\pi\,f(o)$ at $\alpha = 0$, and so

$$I(z) = -2\pi\,\delta_R(z)\ .$$

The integral $I(z,\alpha)$ is equal to

$$z^{\alpha-1}\int_0^1 (1-t)^{-\frac{1}{2}}\,t^{\alpha-3/2}\,dt = z^{\alpha-1}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}\ .$$

Therefore

$$\int_0^a I(z,\alpha)\,f(z)\,dz = \frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}\int_0^a z^{\alpha-1}\,f(z)\,dz$$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)}\left\{\left[\frac{1}{\alpha}\,z^{\alpha}\,f(z)\right]_0^a - \frac{1}{\alpha}\int_0^a z^{\alpha}\,f'(z)\,dz\right\}$$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha+1)}\left\{a^{\alpha}\,f(a) - \int_0^a z^{\alpha}\,f'(z)\,dz\right\}.$$

For $\alpha = 0$ we get, by analytic continuation,

$$\int_0^a I(z,o)\, f(z)\, dz = \Gamma(\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}) \left\{ f(a) - \int_0^a f'(z)\, dz \right\}$$

$$= -\,2\pi\, f(o)$$

and so $I(z) = -\,2\pi\,\delta_R(z)$, i.e.

$$\int_t^y \frac{dx}{(y-x)^{\frac{1}{2}}(x-t)^{3/2}} = -\,2\pi\,\delta_R(y-t)\;.$$

3.32 We shall now obtain the result of 3.3 by a slightly different method.

Since $I(z,\alpha)$ is a convolution integral its Laplace transform is

$$\int_0^\infty e^{-pz}\, I(z,\alpha)\, dz = \frac{\Gamma(\frac{1}{2})}{p^{\frac{1}{2}}}\,\frac{\Gamma(\alpha-\frac{1}{2})}{p^{\alpha-\frac{1}{2}}}$$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(\alpha-\frac{1}{2})}{p^{\alpha}}\;.$$

Although the integral does not exist in the usual sense for $\alpha = 0$ it can by analytic continuation be given the value

$$\Gamma(\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}) = -\,2\pi\;.$$

We can now show that

$$\int_0^y I(y-t)\, w(t)\, dt = -\,2\pi\, w(x)\;.$$

The Laplace transform of the left hand side of this equation is

$$\int_0^\infty e^{-pu}\, I(u)\, du \int_0^\infty e^{-pv}\, w(v)\, dv = -\,2\pi \int_0^\infty e^{-pv}\, w(v)\, dv$$

and so the value of the original integral is $-\,2\pi\, w(x)$.

3.4 Lomax et al. discuss the change of order of integration of the double integral

$$\int_0^x \frac{d\eta}{\eta^{\frac{1}{2}}} \int_\eta^x \frac{d\xi}{(\xi-\eta)^{3/2}}\;.$$

The value of the inner integral is

$$-\frac{2}{(x-\eta)^{\frac{1}{2}}} .$$

It can be evaluated either by the method of finite parts or by the Laplace transform method given in paragraph 3.32. The value of the double integral is then

$$-2\int_0^x \frac{d\eta}{\eta^{\frac{1}{2}}\,(x-\eta)^{\frac{1}{2}}} = -2\pi .$$

If we invert the order of integration we get

$$\int_0^x d\xi \int_0^\xi \frac{d\eta}{\eta^{\frac{1}{2}}\,(\xi-\eta)^{3/2}} .$$

We have shown in 3.3 that the value of the inner integral is $-2\pi\,\delta_R(\xi)$ and so the value of the double integral is again -2π.

The order of integration of the double integral can be inverted if the integral

$$\int_0^\xi \frac{d\eta}{\eta^{\frac{1}{2}}\,(\xi-\eta)^{3/2}}$$

is given its true value $-2\pi\,\delta_R(\xi)$ and not zero its finite part value.

4 Conclusions

It is shown in this paper that there is no need to construct special methods to deal with the inversion of improper integrals in aerodynamics but that if the Dirac delta function is used the problems can be dealt with by the ordinary methods of analysis.

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