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Preliminary Analysis for a Jet Flap System in Two-Dimensional Inviscid Flow

By

E. C. Maskell and S. B. Gates

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Corrigendum

Delete para. 3.4 and substitute:

3.4 External flow due to vorticity at jet boundaries

The external flow associated with a polar element of the jet is that due to vortex elements $d\Gamma'_1$, $d\Gamma'_2$ at the jet boundaries, where

$$d\Gamma'_1 = u_1 \left(R - \frac{\delta}{2}\right) d\theta$$

$$d\Gamma'_2 = -u_2 \left(R + \frac{\delta}{2}\right) d\theta .$$

The complex velocity due to the vortices $d\Gamma'_1$, $d\Gamma'_2$ at $z = \pm\, \delta/2$ is given by

$$\frac{dw}{dz} = -\frac{i}{2\pi}\left(\frac{d\Gamma'_1}{z - \frac{\delta}{2}} + \frac{d\Gamma'_2}{z + \frac{\delta}{2}}\right)$$

$$= -\frac{i}{2\pi}\left\{\frac{d\Gamma'_1 + d\Gamma'_2}{z} + \frac{(d\Gamma'_1 - d\Gamma'_2)\delta}{2z^2}\right\}.$$

The first term in the bracket represents the k distribution on the centre line of the jet; the second represents a doublet of strength $(d\Gamma'_1 - d\Gamma'_2)\,\frac{\delta}{2}$. But

$$d\Gamma'_1 + d\Gamma'_2 = U\delta\left(\frac{\rho V^2}{\rho_o U^2} - 1\right) d\theta$$

$$\frac{d\Gamma'_1 - d\Gamma'_2}{z} = U\left(1 - \frac{\delta^2}{4R^2}\,\frac{\rho V^2}{\rho_o U^2}\right) R d\theta .$$

Hence

$$\frac{dw}{dz} = -\frac{iU\delta}{2\pi}\left\{\left(\frac{\rho V^2}{\rho_o U^2} - 1\right)\frac{1}{z} + \left(1 - \frac{\delta^2}{4R^2}\,\frac{\rho V^2}{\rho_o U^2}\right)\frac{R}{z^2}\right\} d\theta \qquad (21)$$

from which it follows that the doublet term can be neglected if δ is sufficiently small.

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Preliminary Analysis for a Jet Flap System in two-dimensional inviscid flow

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SUMMARY

Work so far done on jet flap analysis in two dimensional flow has been somewhat obscured by uncertainties as to the physical assumptions made by various workers. In these notes an attempt is made to clarify the background of the subject in two ways:-

(1) by establishing some properties of a thin jet in a uniform field of flow

(2) by stating some momentum theorems connected with a jet issuing from a body in a uniform field of flow.

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1 Introduction

Attempts made so far to work out a theory of the jet flap in two dimensional inviscid incompressible flow have produced much disagreement, mainly perhaps through failure to appreciate clearly the assumptions made by various workers. It may be useful therefore to try to establish some common ground of agreement by assembling the elements of the problem and stating a few theorems which are likely to be used in its solution. The analysis assumes an incompressible main flow, but goes a step further than usual in treating the jet flow as compressible, the incompressible jet thus appearing as a special case.

2 Analysis of a thin jet in a uniform field of flow

Geometrically the jet is defined (Fig.1) by the radius of curvature R of its centre line at distance s along its curve, and by its thickness δ perpendicular to the centre line. We define the jet as thin when $(\delta/R)^2$ can everywhere be neglected.

The flow is everywhere irrotational except at the boundaries AB, CD of the jet, which are vortex lines across which the pressure is continuous but the velocity and the density are both discontinuous.

The main flow has stagnation pressure H_o and stagnation density ρ_o, which is the density of the incompressible flow.

The jet flow has stagnation pressure H and stagnation density ρ_J. If the jet flow is treated as incompressible the densities are constant at ρ_J in the jet and ρ_o outside.

In all practical cases H is much greater than H_o and the difference $K = H - H_o$ is clearly a basic parameter. We shall treat the jet flow as isentropic, so that the density is a function of the pressure only.

The velocity is V, the pressure is p and the density is ρ at the centre line, and so R, δ, V, p, ρ are functions of s. It will be useful to express the properties of the jet in terms of these quantities and the stagnation pressures.

It should be emphasised that the jet has only three independent physical quantities that are constant:- the mass flow m, the stagnation pressure, and the stagnation density*. In general, for instance, the momentum flow μ varies along the jet.

We can now analyse the conditions in a small element of the jet cut off by two adjacent radii (Fig.2).

At the upper boundary of the jet the pressure is p_1 and the velocity and density are v_1, ρ_1 inside the jet and u_1, ρ_o outside it. Suffix 2 denotes corresponding quantities at the lower boundary.

The usual equations for irrotational flow in the jet are

$$\left.\begin{aligned} \frac{\partial v}{\partial r} + \frac{v}{r} &= 0 \\ \frac{\partial p}{\partial r} &= \frac{\rho v^2}{r} \end{aligned}\right\} \qquad (1)$$

where v, p, r, ρ are respectively velocity, pressure, radius of curvature and density.

---

* Stagnation density has been used for convenience instead of the more usual stagnation temperature.

It follows that across any section of the jet we have

$$r\,v \;=\; \text{constant} \;=\; R\,V \qquad (2)$$

whence

$$\left.\begin{aligned} v_1 &= \left(1 + \frac{\delta}{2R}\right) V \\ v_2 &= \left(1 - \frac{\delta}{2R}\right) V \end{aligned}\right\} \qquad (3)$$

and

so that the variation of v across the jet is small, and pressure differences are related by the Bernoulli equation in its differential form

$$\frac{dp}{\rho} + v\,dv \;=\; 0$$

Hence, we get

$$\frac{p_2 - p_1}{\rho} \;=\; V(v_1 - v_2) \;=\; \frac{V^2\delta}{R} \qquad (4)$$

and

$$\left.\begin{aligned} \frac{p_1 + p_2}{2} &= p \\ \frac{\rho_1 + \rho_2}{2} &= \rho \end{aligned}\right\} \qquad (5)$$

Now let U be the velocity in the main flow when the pressure is p. Then Bernoulli's equation gives for the main flow

$$p_1 + \tfrac{1}{2}\rho_o u_1^2 = p_2 + \tfrac{1}{2}\rho_o u_2^2 = p + \tfrac{1}{2}\rho_o U^2 = H_o \qquad (6)$$

The following relations now follow from (6), using (4) and (5):-

$$\tfrac{1}{2}\rho_o\,(u_1^2 - u_2^2) \;=\; p_2 - p_1 \;=\; \frac{\delta}{R}\,\rho\,V^2$$

$$\tfrac{1}{2}\rho_o\,\frac{u_1^2 + u_2^2}{2} + \frac{p_1 + p_2}{2} \;=\; H_o$$

or

$$\tfrac{1}{2}\rho_o\,\frac{u_1^2 + u_2^2}{2} \;=\; H_o - p \;=\; \tfrac{1}{2}\rho_o U^2$$

Thus we have

$$\left.\begin{aligned} \frac{u_1^2 - u_2^2}{2} &= \frac{\rho}{\rho_o}\frac{\delta}{R}V^2 \\ \text{and} \qquad \frac{u_1^2 + u_2^2}{2} &= \bar{U}^2 \end{aligned}\right\} \tag{7}$$

Hence

$$\left.\begin{aligned} u_1^2 &= \bar{U}^2 + \frac{\rho}{\rho_o}\frac{\delta}{R}V^2 \\ u_2^2 &= \bar{U}^2 - \frac{\rho}{\rho_o}\frac{\delta}{R}V^2 \end{aligned}\right\} \tag{8}$$

and if U is the mean of the main stream velocities on the boundaries of the jet we have in general

$$U = \frac{u_1 + u_2}{2} = \frac{\bar{U}}{2}\left\{\left(1 + \frac{\rho}{\rho_o}\frac{\delta}{R}\frac{V^2}{\bar{U}^2}\right)^{\frac{1}{2}} + \left(1 - \frac{\rho}{\rho_o}\frac{\delta}{R}\frac{V^2}{\bar{U}^2}\right)^{\frac{1}{2}}\right\} \tag{9}$$

If $\frac{\rho}{\rho_o}\frac{\delta}{R}V^2$ is small compared with $\bar{U}^2$ these reduce to

$$\left.\begin{aligned} u_1 &= \bar{U}\left(1 + \frac{\delta}{2R}\frac{\rho V^2}{\rho_o \bar{U}^2}\right) \\ u_2 &= \bar{U}\left(1 - \frac{\delta}{2R}\frac{\rho V^2}{\rho_o \bar{U}^2}\right) \end{aligned}\right\} \tag{10}$$

and so

$$\left.\begin{aligned} U &= \bar{U} \\ u_1 - u_2 &= \frac{\delta}{R}\frac{\rho V^2}{\rho_o \bar{U}} \end{aligned}\right\} \tag{11}$$

To complete what is wanted for determining the properties of the jet, convenient equations for isentropic one dimensional flow along the centre line are

$$\left.\begin{aligned} V^2 &= \frac{2\gamma}{\gamma - 1}\frac{H}{\rho_J}\left\{1 - \left(\frac{p}{H}\right)^{1-\frac{1}{\gamma}}\right\} \\ \frac{\rho}{\rho_J} &= \left(\frac{p}{H}\right)^{1/\gamma} \end{aligned}\right\} \tag{12}$$

If the jet flow can be treated as incompressible, so that the velocity of sound is infinite and $\gamma \to \infty$, these become

$$V^2 = \frac{2}{\rho_J} (H - p), \text{ Bernoulli's equation}$$

and $$\rho = \rho_J ;$$

3 Properties of a thin jet

3.1 The mass flow m is given by

$$m = \rho \int_{R-\frac{\delta}{2}}^{R+\frac{\delta}{2}} v \, dr = \rho RV \int_{R-\frac{\delta}{2}}^{R+\frac{\delta}{2}} \frac{dr}{r} \qquad \text{from (2)}$$

$$= \rho V \delta \qquad (13)$$

This must be constant along the jet.

3.2 The momentum flow μ is given by

$$\mu = \rho \int_{R-\frac{\delta}{2}}^{R+\frac{\delta}{2}} v^2 \, dr = \rho R^2 V^2 \int_{R-\frac{\delta}{2}}^{R+\frac{\delta}{2}} \frac{dr}{r^2} \qquad \text{from (2)}$$

$$= \rho V^2 \delta = mV \qquad (14)$$

Thus μ varies as V along the jet.

3.3 Circulation

$d\Gamma$, the circulation clockwise associated with the polar element of Fig.2, is the sum of $d\Gamma_1$ the circulation round its upper arc and $d\Gamma_2$ the circulation round its lower arc, where

$$d\Gamma_1 = (u_1 - v_1) \left(R - \frac{\delta}{2}\right) d\theta$$

$$d\Gamma_2 = - (u_2 - v_2) \left(R + \frac{\delta}{2}\right) d\theta$$

But since there is no circulation in the jet

$$v_1 \left(R - \frac{\delta}{2}\right) = v_2 \left(R + \frac{\delta}{2}\right)$$

Hence $$\frac{d\Gamma}{d\theta} = (u_1 - u_2) R - (u_1 + u_2) \frac{\delta}{2}$$

Thus if k is the circulation per unit length of the centre line

$$k = \frac{d\Gamma}{ds} = (u_1 - u_2) - U \frac{\delta}{R}$$

But from (7) $$(u_1 - u_2)U = \frac{\rho}{\rho_o} \frac{\delta}{R} V^2$$

It follows that

$$k = U \frac{\delta}{R} \left(\frac{\rho V^2}{\rho_o U^2} - 1 \right) \tag{15}$$

$$= \frac{\mu}{\rho_o UR} - \frac{U\delta}{R} \qquad \text{using (14)} \tag{16}$$

If $\frac{\rho}{\rho_o} \frac{\delta}{R} V^2$ is small compared with $\bar{U}^2$, U can be replaced by $\bar{U}$ in (15) and (16). In this case we can relate k to K, the difference in the stagnation pressures, as follows.

Inside the jet we have

$$p + \tfrac{1}{2} \rho V^2 L = H = H_o + K \tag{17}$$

where $$L = \frac{2}{\gamma M^2} \left\{ \left(1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma - 1}} - 1 \right\}$$

and M is the local Mach number.

Also by definition

$$p + \tfrac{1}{2} \rho_o \bar{U}^2 = H_o \tag{18}$$

Hence from (17) and (18)

$$\frac{\rho V^2}{\rho_o \bar{U}^2} - 1 = \frac{K}{L(H_o - p)} - \left(1 - \frac{1}{L} \right)$$

and so from (15), putting U = U we have finally, using (18),

$$k = \frac{\delta}{R}\sqrt{\frac{2}{\rho_o(H_o - p)}}\left\{\frac{K}{L} - (H_o - p)(1 - \frac{1}{L})\right\} \tag{19}$$

For incompressible jet flow, when M = 0 and L = 1, this reduces to

$$k = \frac{K\delta}{R}\sqrt{\frac{2}{\rho_o(H_o - p)}} \tag{20}$$

3.4 External flow due to vorticity at jet boundaries

It might be thought that the external flow due to the vorticity associated with the jet could be calculated from a distribution k on the centre line of the jet. To show that this procedure is not good enough we have to return to the elements $d\Gamma_1$, $d\Gamma_2$.

Consider the flow due to vortices $d\Gamma_1$, $d\Gamma_2$ at $z = \pm\frac{\delta}{2}$.

The complex velocity is given by

$$\frac{dw}{dz} = -\frac{i}{2\pi}\left(\frac{d\Gamma_1}{z - \frac{\delta}{2}} + \frac{d\Gamma_2}{z + \frac{\delta}{2}}\right)$$

$$= -\frac{i}{2\pi}\left\{\frac{d\Gamma_1 + d\Gamma_2}{z} + \frac{(d\Gamma_1 - d\Gamma_2)\delta}{2\,z^2}\right\}$$

The first term in the bracket represents the k distribution on the centre line of the jet; the second represents a vortex doublet of strength $(d\Gamma_1 - d\Gamma_2)\,\frac{\delta}{2}$.

But $$d\Gamma_1 + d\Gamma_2 = U\delta\left(\frac{\rho V^2}{\rho_o U^2} - 1\right)d\theta$$

$$\frac{d\Gamma_1 - d\Gamma_2}{2} = -UR\left(\frac{V}{U} - 1\right)d\theta$$

Hence $$\frac{dw}{dz} = -\frac{iU\delta}{2\pi}\left\{\left(\frac{\rho V^2}{\rho_o U^2} - 1\right)\frac{1}{z} - \left(\frac{V}{U} - 1\right)\frac{R}{z^2}\right\} \tag{21}$$

from which it follows that the doublet term is not in general negligible at distances of the order of R.

4 Conditions at the ends of the jet

4.1 The source. Mathematically the jet may be supposed to issue from a source of energy in the main stream, with given pressure, velocity, width and angle to the main stream. Thus, using suffix e for this condition, p_e, V_e, δ_e and θ_e (but not R_e) are given at s = 0. In reality, as in the jet flap problem, the source is an orifice in a surface, and it is one of the difficulties of the subject to specify precisely the conditions at the orifice itself.

4.2 <u>Far downstream.</u> The jet must clearly end by being parallel to the main stream. Using suffix ∞ for this end we must have

$p = p_\infty$, the undisturbed main stream pressure

$R \to \infty$

$k = 0$

Thus from (12), (13), (14) we have

$$\left.\begin{aligned} \rho_\infty V_\infty \delta_\infty &= m \\ m V_\infty &= \mu_\infty \\ V_\infty^2 &= \frac{2\gamma}{\gamma - 1} \frac{H}{\rho_J} \left\{ 1 - \left(\frac{p_\infty}{H}\right)^{1-\frac{1}{\gamma}} \right\} \\ \frac{\rho_\infty}{\rho_J} &= \left(\frac{p_\infty}{H}\right)^{1/\gamma} \end{aligned}\right\} \qquad (22)$$

These equations determine the ultimate width, velocity, density and momentum flow of the jet. When the jet flow is incompressible we have

$$\rho_\infty = \rho_J$$

and

$$V_\infty^2 = \frac{2}{\rho_J}(H - p_\infty)$$

This last equation may also be written

$$\tfrac{1}{2} \rho_J V_\infty^2 - \tfrac{1}{2} \rho_0 U_\infty^2 = K$$

4.3 We can now examine the difference between the momentum flows at the ends of the jet.

It follows from (14) and (12) that

$$\mu_\infty - \mu_e = m (V_\infty - V_e)$$

and

$$V_\infty^2 - V_e^2 = \frac{2\gamma}{\gamma - 1} \frac{H^{1/\gamma}}{\rho_J} \left\{ p_e^{\frac{\gamma-1}{\gamma}} - p_\infty^{\frac{\gamma-1}{\gamma}} \right\}$$

whence we have

$$\frac{\mu_\infty - \mu_e}{m} = \frac{2\gamma}{\gamma - 1}\frac{H^{1/\gamma}}{\rho_J}\frac{\left(p_e^{\frac{\gamma-1}{\gamma}} - p_\infty^{\frac{\gamma-1}{\gamma}}\right)}{V_e + V_\infty} \tag{23}$$

which reduces to

$$\frac{\mu_\infty - \mu_e}{m} = \frac{2}{\rho_J}\frac{p_e - p_\infty}{V_e + V_\infty} \tag{24}$$

for incompressible jet flow.

These equations show that the ultimate momentum flow is greater or less than the issuing momentum flow according as the mean issuing pressure is greater or less than the undisturbed pressure of the main stream.

5 Jets of constant momentum flow

5.1 Equations (23), (24) show that if $p_e = p_\infty$ we can have a constant momentum jet in which V, ρ, p and δ are constant throughout its length, the pressures p_1, p_2 at its boundaries adjusting themselves to suit the curvature according to equation (4).

5.2 A more general deduction from equations (23) (24), and one which has a practical bearing on the jet flap problem, is that the variation of μ along the jet tends to zero, whatever its exit pressure, when V becomes very large. This suggests that the difficult mathematics of the jet flap problem may be simplified, while still retaining some validity, by proceeding from the thin jet of this analysis to the infinitely thin jet in which $\delta \to 0$ as $V \to \infty$ under certain conditions.

D. A. Spence for instance proposes the limiting case of the constant momentum jet, with incompressible flow, in which $V \to \infty$ as $\delta \to 0$ in such a way that the mass flow $\rho\delta V \to 0$ but the momentum flow $\rho\delta V^2 \to \bar{\mu}$, a finite constant. This system is governed by the simple equations:-

$$p_2 - p_1 = \frac{\bar{\mu}}{R} = \rho_o U k \tag{25}$$

It returns in fact to the 'momentum flux line' of the N.G.T.E. analysis.

5.3 It is not immediately obvious from equation (23) that the infinitely thin jet of finite momentum with compressible flow must necessarily have constant momentum. We can easily show, however, that this is true and that equation (25) remains valid even when the flow in the jet is assumed to be compressible.

The equations governing the jet flow are (12), (13) and (14). From (12) and (14) we have

$$\rho V^2 = \frac{2\gamma}{\gamma - 1} p \left\{ \left(\frac{H}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1 \right\} = \frac{\mu}{\delta} \tag{26}$$

and from (12), (13) and (14) we get

$$\frac{m^2}{\mu\delta} = \rho = \rho_J \left(\frac{p}{H}\right)^{1/\gamma} \tag{27}$$

We make $\delta \to 0$ in such a way that μ remains finite. Then, since p must remain finite, we have from (26)

$$\text{as} \quad \delta \to 0, \qquad \frac{2\gamma}{\gamma - 1}\, p \left(\frac{H}{p}\right)^{\frac{\gamma-1}{\gamma}} \to \frac{\mu}{\delta} \tag{28}$$

Now, from (27) and (28), we find that

$$\text{as} \quad \delta \to 0, \quad \mu^2 \to \frac{2\gamma}{\gamma - 1}\, \frac{H}{\rho_J}\, m^2 = \text{constant} = \bar{\mu}^2 \tag{29}$$

since H, ρ_J and m are all constant along the jet. Hence, the jet has constant momentum whatever the variation of static pressure may be.

From (28) we have

$$\text{as} \quad \delta \to 0, \quad \left(\frac{p}{H}\right)^{1/\gamma} \to \left(\frac{2\gamma}{\gamma - 1}\, \frac{p}{\mu}\right)^{\frac{1}{\gamma-1}} \delta^{\frac{1}{\gamma-1}}$$

so that, from (27),

$$\text{as} \quad \delta \to 0, \quad m^2 \to \rho_J\, \mu \left(\frac{2\gamma}{\gamma - 1}\, \frac{p}{\mu}\right)^{\frac{1}{\gamma-1}} \delta^{\frac{\gamma}{\gamma-1}}$$

Hence, if ρ_J remains finite,

$$m \to 0, \qquad \text{as} \quad \delta \to 0$$

and

$$V = \frac{\bar{\mu}}{m} \to \infty ,$$

and, from (27),

$$\rho \to 0, \text{ as } \quad \delta \to 0 .$$

We find, therefore, that if $\delta \to 0$ but μ remains finite, and ρ_J remains finite,

$$\mu \to \bar{\mu}, \text{ a constant along the jet}$$

$$m, \rho \to 0$$

and $$V, H \to \infty$$

and the infinitely thin jet is again governed by the equation (25), viz:

$$p_2 - p_1 = \frac{\bar{\mu}}{R} = \rho_0 U k$$

6 Momentum theorems for a jet flap system

In general, Fig.3, there is a duct through the body with entry A_oB_o and exit AB. A source of energy in the duct collects the air from PQ far upstream of the body and ejects it as the jet with boundaries AS, BR, which are parallel to the main stream far behind the body. The momentum flow is $\mu_{-\infty}$ for upstream, μ_i at entry, μ_e at exit and μ_∞ far downstream. The stagnation pressure is the same as that of the mainstream before entry: there is no vorticity along the dividing streamlines PB_1, QA_1. The stagnation pressure of the emerging jet is increased: there is vorticity on its boundaries AS, BR.

There are four stagnation points of the main flow, at A_1, B_1, A,B.

In source type flow, as in the N.G.T.E. experiment, Fig.4, the source of the jet air is outside the system; there is no forward entry. There are three stagnation points A_1, A, B.

It seems useful to write down momentum equations for the entry flow, the ducted body, the jet, and the whole system. The theorems for parts of the system (Paras.6.1, 6.2) apply to compressible but inviscid flow. In the theorems for the whole system (Para. 6.3) the jet flow may be viscous.

6.1 The ducted body

(1) Resultant force

Let F be the resultant force on the body and let the pressures p be taken into the two surfaces separated by the duct, and into the duct at its end.

Then the vector momentum equation for the flow through the duct is

$$\mu_e - \mu_i = \left(- \int_{AFA_o, BGB_o} + \int_{A_oB_o} - \int_{AB} \right) p ds \qquad (30)$$

and the vector equation for the resultant force F is

$$F = \left(\int_{ACA_oFA} + \int_{BDB_oGB} \right) p\,ds \tag{31}$$

$$= \left(\int_{ACA_o} + \int_{BDB_o} + \int_{AFA_o} + \int_{BGB_o} \right) p\,ds$$

$$= \left(\int_{ACA_o} + \int_{BDB_o} + \int_{A_oB_o} - \int_{AB} \right) p\,ds - (\mu_e - \mu_i) \quad \text{from (30)}$$

$$= \left(\int_{ACA_o} + \int_{BDB_o} \right) p\,ds + \left(J_e - \int_{AB} p\,ds \right) - \left(J_i - \int_{A_oB_o} p\,ds \right) \tag{32}$$

where J is the reaction to the momentum flow μ. In Fig.4 this becomes

$$F = \int_{ACDB} p\,ds + \left(J_e - \int_{AB} p\,ds \right) \tag{33}$$

Equation (33) is the basis of the N.G.T.E. experiment, where p and J_e are measured.

(2) Moment about origin O

To get the moment of the momentum flow μ about O we multiply each element of μ by its perpendicular distance from O and integrate along the line which the flow is crossing. Thus the moment of the momentum flow is $L\mu$ where L is a mean length. Also the moment of $p\,ds$ about O is $p\,\ell\,ds$, ℓ being the perpendicular from O to the line of action of p.

Hence the angular momentum equation corresponding to (30) is

$$L_e \mu_e - L_i \mu_i = \left(- \int_{AFA_o, BGB_o} + \int_{A_oB_o} - \int_{AB} \right) p\,\ell\,ds$$

and the moment equation corresponding to (31) is

$$N = \left(\int_{ACA_oFA} + \int_{BDB_oGB} \right) p\,\ell\,ds$$

The results corresponding to (32), (33) are

$$N = \left(\int_{ACA_o} + \int_{BDB_o} \right) p\, \ell ds + \left(L_e J_e - \int_{AB} p\, \ell ds \right) - \left(L_i J_i - \int_{A_o B_o} p\, \ell ds \right) \quad (34)$$

$$= \int_{ACDB} p\, \ell ds + \left(L_e J_e - \int_{AB} p\, \ell ds \right) \quad (35)$$

Equation (35) is the basis of the N.G.T.E. moment measurement.

6.2 Inflow and jet

Measuring p always into the tube of flow the vector momentum equations are

$$\int_{A_1 QPB_1 A_1} p\, ds = \mu_i - \mu_{-\infty} = J_{-\infty} - J_i \quad \text{for the inflow} \quad (36)$$

and

$$\int_{ASRBA} p\, ds = \mu_\infty - \mu_e = J_e - J_\infty \quad \text{for the jet} \quad (37)$$

It is useful to resolve parallel and perpendicular to the undisturbed flow. Thus if the jet issues at the mean angle τ and φ is the mean inclination of AB we have

$$J_e \sin\tau = \int_{AS,RB} p \cos\theta\, ds - p_e\, BA \cos\varphi \quad (38)$$

i.e. $$J_e \sin\tau + p_e\, BA \cos\varphi = L_2 = \int_{AS,RB} p \cos\theta\, ds$$

where L_2 = total lift reaction of the emerging jet
= lift reaction on the jet due to the external stream,

and

$$J_e \cos\tau - J_\infty = \int_{AS,RB} p \sin\theta + p_\infty\, SR - p_e\, BA \sin\varphi \quad (39)$$

i.e. $(J_e \cos\tau + p_e \, BA \sin\varphi) - (J_\infty + p_\infty \, SR) = D_2 = \int_{AS,RB} p \sin\theta$

where D_2 = drag reaction on the jet due to the external stream.

6.3 The whole system

In considering the whole system the restrictions imposed on the flow may usefully be relaxed. The assumption of a thin jet is retained, but the main flow is assumed to be compressible and the jet flow to be viscous.

The momentum theorem is applied over a control surface C, which is a large circuit enclosing the aerofoil and cutting the streamlines of the jet orthogonally (Fig.6).

R is a characteristic dimension of C (equal to the radius if C is circular),

W is that part of C lying within the jet,

and C' = C - W is that part of C lying wholly within the main flow.

On C' the Bernoulli function χ is constant if the main flow is homentropic, where

$$\chi = \int \frac{dp}{\rho} + \tfrac{1}{2}(u^2 + v^2)$$

Lift

The total lift force L applied to the aerofoil is the reaction to the y-components of all the external forces applied to the fluid within the large circuit C. In general this includes tangential forces on the body, as well as the reaction to the jet momentum.

The momentum equation for the fluid within C is, therefore,

$$- L = \int_C p \, dx + \int_C \rho v (v \, dx - u dy) \tag{40}$$

Writing

$$\left.\begin{aligned} p &= p_\infty + p' \\ u &= U_\infty + u' \\ v &= v' \end{aligned}\right\} \tag{41}$$

(40) becomes

$$- L = \int_C p' \, dx - \int_C \rho U_\infty v' \, dy + \int_C \rho \, v'(v' \, dx - u' \, dy) \tag{42}$$

At infinity downstream the jet streamlines are parallel to the free stream direction. Hence

$$\underset{R \to \infty}{\mathrm{Lt}} \left[\int_{W} p' dx - \int_{W} \rho U_{\infty} v' dy + \int_{W} \rho v'(v' dx - u' dy) \right] = 0$$

and (42) may be written

$$- L = \underset{R \to \infty}{\mathrm{Lt}} \int_{C'} \{(p' dx - \rho U_{\infty} v' dy) + \rho v'(v' dx - u' dy)\} \qquad (43)$$

Now on C' both u' and v' are $O(^1/R)$, and the Bernoulli function χ is constant, so that

$$p + \rho_{\infty} U_{\infty} u' = 0$$

terms of $O(^1/R^2)$ being neglected, and

$$\underset{R \to \infty}{\mathrm{Lt}} \int_{C'} \rho v'(v' dx - u' dy) = O\left(\frac{1}{R}\right)$$

Hence, (43) becomes

$$L = \underset{R \to \infty}{\mathrm{Lt}} \int_{C'} \rho_{\infty} U_{\infty} (u' dx + v' dy)$$

$$= \underset{R \to \infty}{\mathrm{Lt}} \int_{C} \rho_{\infty} U_{\infty} (u\, dx + v\, dy)$$

i.e.

$$L = \rho_{\infty} U_{\infty} \Gamma_{\infty} \qquad (44)$$

where Γ_{∞} is the clockwise circulation about any circuit C which encloses the aerofoil and cuts the jet orthogonally at infinity downstream.

Equation (44) is Taylor's well known lift-circulation theorem. The general proof given above is appreciably simpler than that previously given by Temple[1].

With an incompressible main stream ρ_{∞} does not differ from ρ_0, the stagnation density, and

$$L = \rho_0 U_{\infty} \Gamma_{\infty}$$

Thrust

The thrust T is the reaction to the x-components of the external forces applied to the fluid within C.

The momentum equation is

$$T = - \int_C p\, dy + \int_C \rho u (v\, dx - u\, dy) \tag{45}$$

Using (41), (45) becomes

$$T = - \int_C p' dy + \int_C \rho U_\infty (v\, dx - u\, dy)$$

$$+ \int_C \rho u' (v\, dx - u dy) \tag{46}$$

Now,

$$Q = \int_C \rho (v dx - u dy) = \text{total mass flow outwards over the circuit C}$$

$$= 0, \text{ for a ducted flow}$$

$$= m, \text{ for a source-type flow}$$

where m = mass flow emitted by the source. Hence (46) may be written

$$T = - \int_C p' dy + Q U_\infty + \int_C \rho u' (v dx - u dy) \tag{47}$$

Using (41), (47) becomes

$$T = - \int_C (p' + \rho U_\infty u') \, dy + Q U_\infty + \int_C \rho u' (v' dx - u' dy) \tag{48}$$

On C', as before, u', v' are $O(^1/R)$ and

$$p' + \rho_\infty U_\infty u' = 0$$

$$\int_{C'} \rho u' (v' dx - u' dy) = O(^1/R)$$

and $\rho \to \rho_\infty$ as $R \to \infty$

Hence, in the limit $R \to \infty$, (48) becomes

$$T = - \int_W (p' + \rho U_\infty u') \, dy + Q U_\infty - \int_W \rho u'^2 dy$$

where the integrals are taken in a clockwise direction.

Taking the integral across the jet in the sense y increasing, we get

$$T = \int_W [p' + \rho u'(U_\infty + u')] \, dy + Q \, U_\infty \qquad (49)$$

In the limit $R \to \infty$, p' is constant across the jet and

$$\int_W p' \, dy = - \rho_\infty U_\infty u'_W \delta_\infty$$

where δ_∞ = thickness of the jet at ∞

u'_W = value of u' at the outer edges of the jet

= $O(^1/R)$

Hence

$$\underset{R \to \infty}{\mathrm{Lt}} \int_W p' \, dy = 0$$

provided that δ_∞ remains small.

Then (49) becomes

$$T = \int_W \rho \, u'(U_\infty + u') \, dy + Q \, U_\infty$$

$$= \int_W \rho \, u(u - U_\infty) \, dy + Q \, U_\infty \qquad (50)$$

Now the jet momentum and mass flow crossing the section at infinity are

$$\mu_\infty = \int_W \rho \, u^2 \, dy$$

and

$$m_\infty = \int_W \rho \, u \, dy$$

Hence, (50) may be written

$$T = \mu_\infty - (m_\infty - Q)\,U_\infty \qquad (51)$$

where $Q = 0$ for a ducted flow,

and $Q = m$ for a source type flow.

For an inviscid jet there is no mixing so that

$$m = m_\infty$$

and we get $T = \mu_\infty - m_\infty U_\infty = \mu_\infty - mU_\infty$, for a ducted flow

and $T = \mu_\infty$, for a source-type flow.

7 Conclusion

It should perhaps be repeated that this analysis makes no attempt to solve the jet flap problem. The problem itself is of course to determine the jet so that its boundaries are streamlines of the outer flow, given the body and the conditions at the jet exit.

Acknowledgements

Acknowledgements are due to C.F. Griggs for assistance in the preparation of this paper.

List of Symbols

R	radius of curvature of the centre line of the jet
R	characteristic dimension of the circuit C in para.6.3
s	distance along the centre line of the jet
δ	thickness of the jet perpendicular to the centre line
H_o, ρ_o	stagnation pressure and density of the main stream
H, ρ_J	stagnation pressure and density of the jet
$K = H - H_o$	
V, p, ρ	velocity, pressure and density at the centre line of the jet
p_1, p_2	pressures at the upper and lower boundaries of the jet
v_1, ρ_1	velocity and density within the jet at its upper boundary
v_2, ρ_2	" " " " " " " " lower "
u_1	main stream velocity at the upper boundary of the jet
u_2	" " " " " lower " " " "
$\bar{U}$	main stream velocity corresponding to pressure p
U	mean velocity $\dfrac{u_1 + u_2}{2}$
M	local Mach number in the jet

List of Symbols (Cont'd.)

γ	ratio of specific heats inside the jet
m	mass flow in the jet
μ	momentum flow in the jet
$d\Gamma$	element of circulation about a bounding vortex sheet
k	circulation about the jet per unit length of the centre line
θ	inclination of the centre line to the undisturbed stream direction
z	complex co-ordinate x + iy
w	complex potential function
L =	$\frac{2}{\gamma M^2}\left\{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}-1\right\}$ in Para.3.3
L, ℓ	lengths defined in Para.6.1 (2)
L	total lift force
L_2	lift reaction of the emerging jet
D_2	drag reaction on the jet due to the external stream
J	reaction to the momentum flow μ
F′	resultant force
T	thrust force
N	moment about the origin
τ	mean angle of the jet at exit
φ	mean inclination of the exit (see Fig.5)
u,v	velocity components defined in Fig.6
χ	Bernoulli function $\int \frac{dp}{\rho} + \frac{1}{2}(u^2 + v^2)$
C	a large circuit enclosing the aerofoil and cutting the jet orthogonally see Para.6.3
W	that part of C lying within the jet
C′	that part of C lying wholly within the main flow
Γ_∞	the clockwise circulation about C when the jet is cut at infinity downstream
Q	total mass flow outwards across the circuit C

List of Symbols (cont'd.)

Suffix

$-\infty$	refers to conditions far upstream
∞	" " " " downstream
i	" " " at inlet
e	" " " " exit

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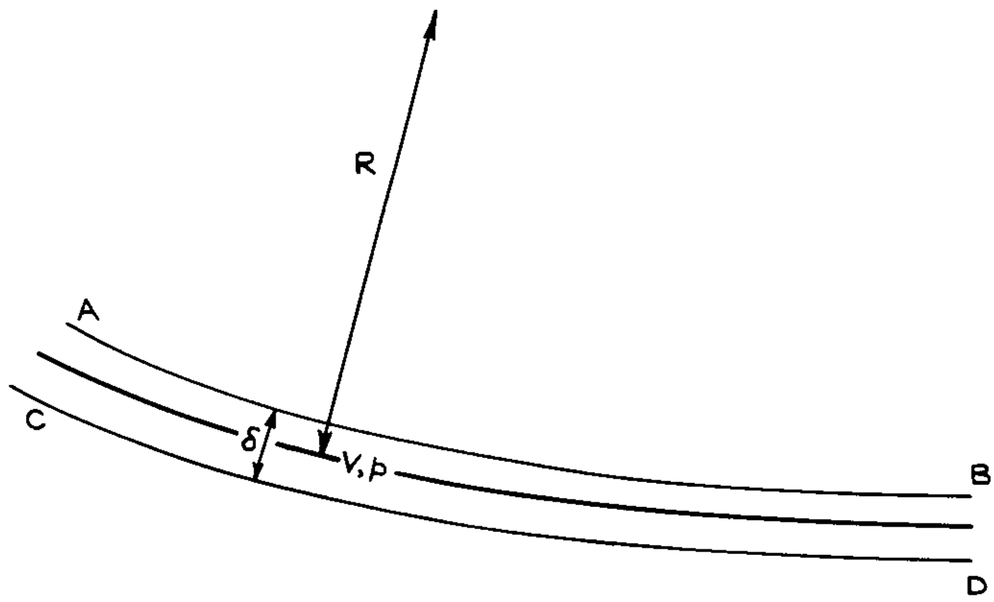

FIG. 1. SEE SECTION 2.

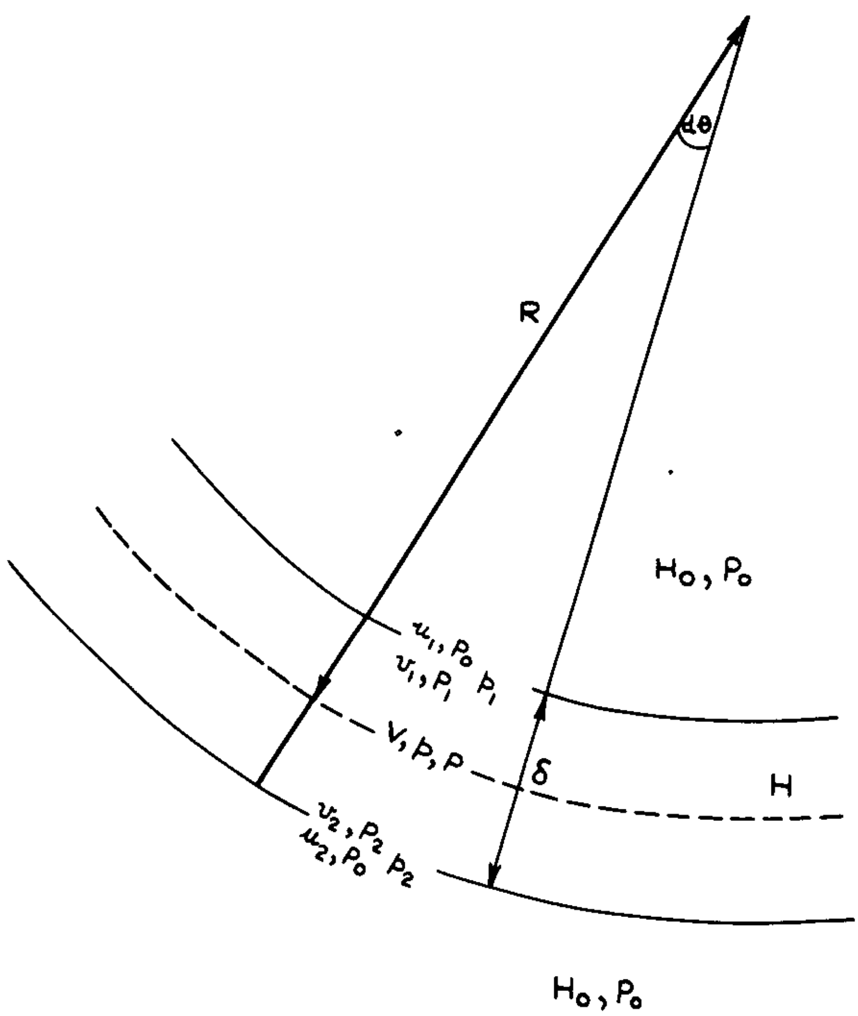

FIG. 2. SEE SECTION 2.

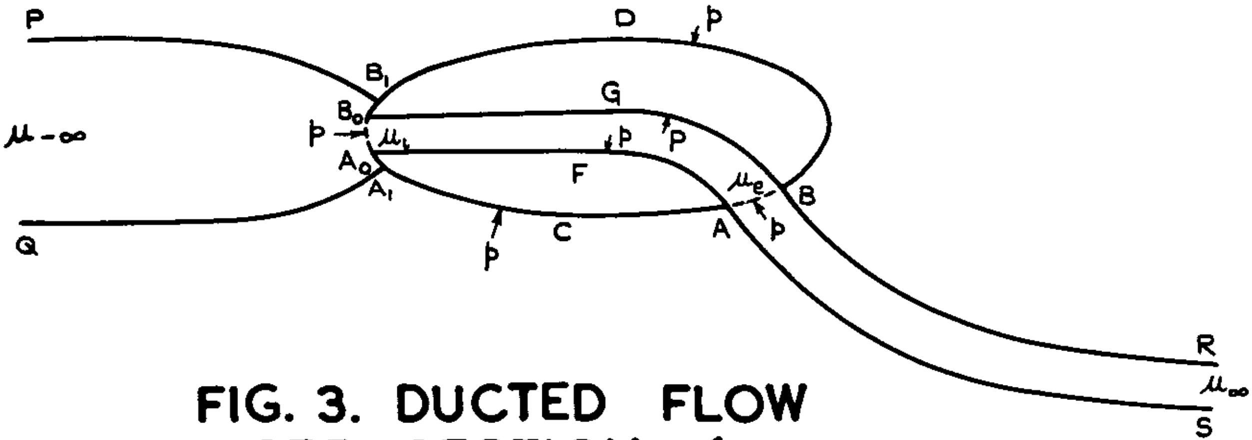

FIG. 3. DUCTED FLOW
SEE SECTION 6.

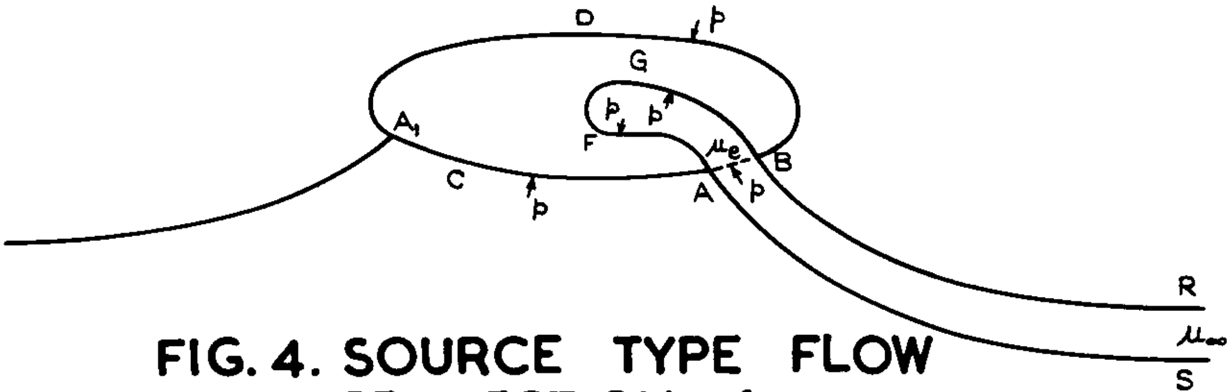

FIG. 4. SOURCE TYPE FLOW
SEE SECTION 6.

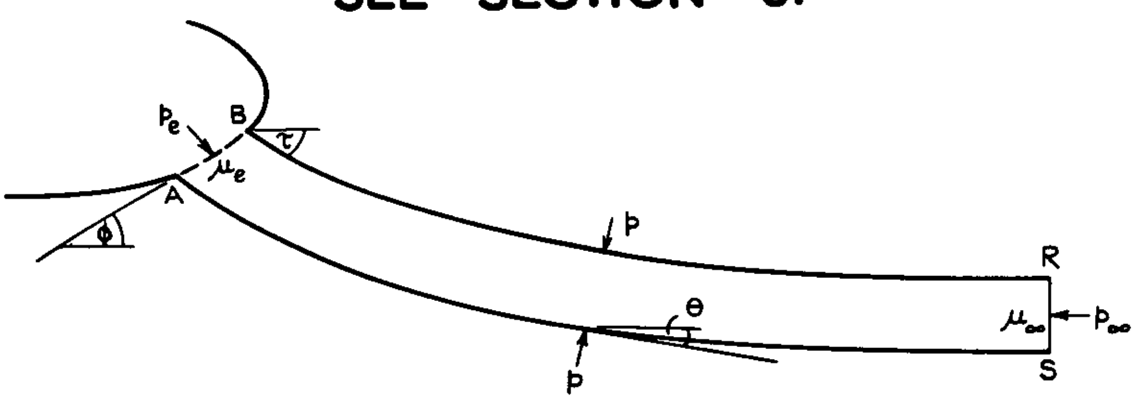

FIG. 5. SEE SECTION 6.2.

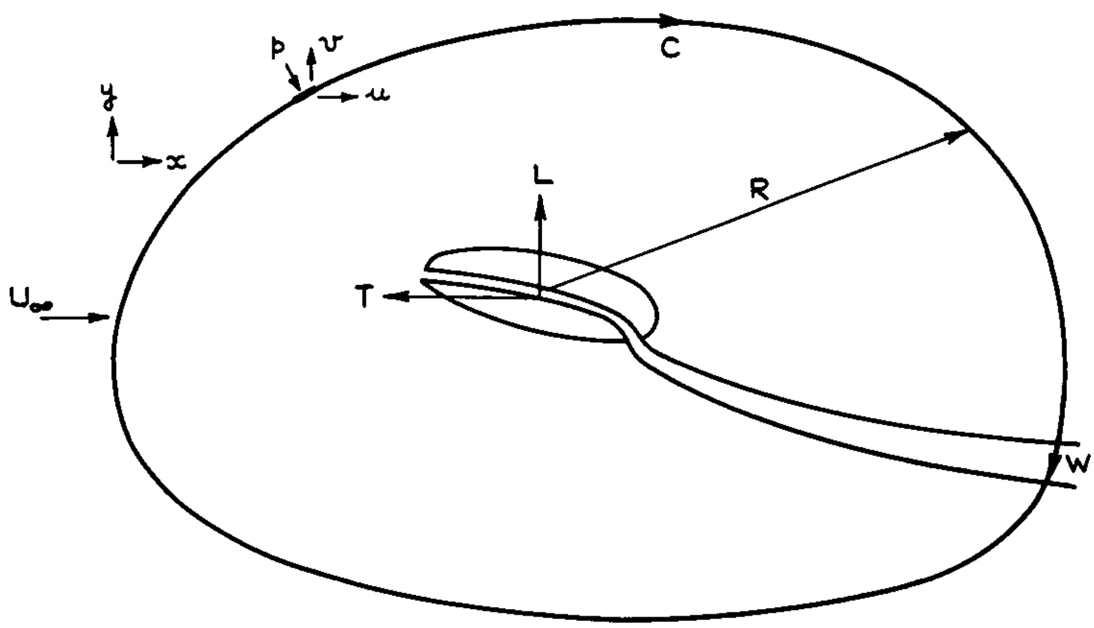

FIG. 6. SEE SECTION 6.3.

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