A Review of Theoretical Work Relevant to the Problem of Heat Transfer Effects on Laminar Separation

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SUMMARY

The results of various theories concerning the effects of heat transfer on laminar separation are briefly discussed. All the theories agree in predicting that wall temperature has an important influence on separation conditions.

Introduction

The effects of heat transfer on laminar boundary-layer separation have been discussed in several theoretical papers. Most of them postulate arbitrarily fixed pressure distributions which are unrelated to the rate of growth of the boundary layer. In this respect the theories are unrealistic. However one would expect there to be a qualitative agreement between experiment and the theoretical findings concerning the effects of wall temperature on separation conditions. Hence, when the experiments reported in Ref.1 were planned, a survey was made of the relevant theoretical work, and this brief note is the outcome.

1. The case with linear adverse velocity gradient

Exact integration of the boundary layer equations for cases with external velocity distributions of the form

\[ U_i = U_o - \beta x = U_o (1 - \xi) \]

where \( x \) is the distance from the leading edge, would probably give results for the effects of heat transfer and Mach number on \( \xi_b \), the value of \( \xi \) at separation, as shown in Fig.1. Here curves are drawn of \( \xi_b \) as a function of Mach number, for various uniform wall temperatures \( T_w \). \( T_o \) is the free-stream temperature and \( T_w = (1 + 0.2 M_o) T_o \) is the wall temperature for zero heat transfer, all temperatures being measured on the absolute scale. The figure shows the general shape of the curves with a very rough indication of the numerical values. Secondary effects, such as those due to the difference between the Prandtl number \( \sigma \) and 1, are ignored.

Notes
(a) Fig.1 shows a bigger variation of \( \xi_b \) with wall temperature at high Mach numbers than is predicted by Ref.2, according to which \( \xi_b \) depends primarily on the ratio \( T_w/T_o \), and is almost independent of Mach number. In support of this theoretical result, it is argued in Ref.2 that the density of the fluid near the wall is the primary relevant variable. However there are other important factors.

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Thus with a cooled wall at a high Mach number, the temperature in the middle of the boundary layer is high, and the consequent difference between the viscosity at the wall and that in the middle of the boundary layer has an important effect on the profile shape upstream.

(b) Refs. 3 and 4 agree in predicting that at moderate Mach numbers there is a considerable increase of \( \xi_s \) with \( M_0 \) for fixed ratios of \( T_w/T_0 \). The result of Ref. 3 is obtained by numerical integration and is exact apart from any computational errors.

(c) The asymptotic limit of \( \xi_s \) at high Mach numbers for finite wall temperatures is estimated from Ref. 5. The whole field of Fig. 1 could be covered by the method of Ref. 5, though the computation would be somewhat tedious. However it can be simplified for very large Mach numbers. Relations are presented in Ref. 5 between various shape parameters of the boundary layer profiles as functions of, effectively, \( T_w/T_2 \). One of the shape parameters is related to the skin friction and another is related to the pressure gradient. For infinite Mach numbers \( T_w/T_2 \rightarrow 0 \) for all finite values of \( T_w \), and for \( T_w/T_2 = 0 \) the relation between the skin friction parameter and the pressure gradient parameter is not single-valued. The limit \( \xi_s = 0.34 \) in Fig. 1 was determined on the assumption that separation occurs when the pressure gradient parameter reaches its maximum. At this point, according to the solution, the skin friction \( c_f \) would be small and decreasing with infinite \( d\xi/dx \). However the solution would break down before \( c_f \) actually became zero, presumably because of the approximate method of integration employed and the nature of the assumed velocity profiles. Thus the numerical value of the upper limit of \( \xi_s \) in Fig. 1 may be rather inaccurate.

(d) The method of Ref. 5 can also be used to show that \( \xi_s \rightarrow 0 \) as \( M_0 \rightarrow \infty \) if the wall temperature is maintained at a sufficiently large fraction of the zero heat transfer value \( T_z \). This limiting fraction is somewhere between 1 and 0.6, and for smaller values of \( T_w/T_2 \), \( \xi_s \) does not tend to zero as \( M_0 \rightarrow \infty \).

(e) For wall temperatures in the region of the zero heat transfer value \( T_z \), \( \xi_s \) is roughly proportional to \( T_w/T_z \) at \( M_0 = 0 \). (c.f. Ref. 4) However at higher Mach numbers it is clear from Fig. 1 that the variation with temperature is bigger. Thus \( \xi_s \) probably varies roughly as \( T_w/T_z \) at \( M_0 = 3 \) for \( T_w \) in the region of \( T_z \).

2. The case with velocity profiles of constant shape and zero skin friction everywhere

It is shown in Ref. 6 that with suitable pressure distributions and uniform wall temperatures similar solutions can be obtained for the boundary-layer velocity profiles. These solutions are related to those for incompressible flow with zero heat transfer and external velocity distributions of the form \( U = \chi x \). For the solutions of Ref. 6, it is of interest to consider how the non-dimensional pressure gradient

\[
\frac{dP}{dx} \propto \frac{T_w}{U} \quad \text{varies with Mach number and wall temperature } T_w \text{ for the boundary layer that is everywhere on the point of separation. When the Mach number is small } \frac{dP}{dx} \text{ varies roughly as } T_w^{1/2} \text{ for } T_w \text{ in the region of the zero heat transfer value } T_z. \text{ At large Mach numbers, it follows from the analysis of Ref. 6 that}
\]

\[
\xi_s \propto \frac{T_w}{U}
\]

*As defined in Ref. 5.
\[
- 3 -
\]
\[
x \frac{dp}{dx} = - \frac{2yn}{y-1} \frac{x}{M_o^{2y-1}} \int_0^{M_o \frac{3y-1}{y-1}} dx
\]

where \( y \) is the ratio of the specific heats, and \( M_o \), the local Mach number at the edge of the boundary layer, satisfies the equation

\[
\frac{d^2 M_o}{dx^2} + \left( \frac{2y - 1}{y-1} \right) \frac{dM_o}{dx} = 0.
\]

The constant \( m \), which is related to the exponent in the incompressible \( u \propto x^m \) solutions, takes different values at different wall temperatures to give separation everywhere. From (2), \( M_o \propto x^{-K} \) where

\[
K = -\frac{2m(Y-1)}{(Y-1) + (3Y-1)m},
\]

provided that \( 0 < m < -\left( \frac{Y-1}{3Y-1} \right) \). This proviso is satisfied for values of \( T_w \) greater than about 0.4 \( T_\infty \), and under these conditions at large Mach numbers \( M_o, \frac{dp}{dx} = -2yn \left( \frac{Y-1}{Y-1} + (3Y-1)m \right) \) from (1). Hence from the figures given in Ref.6 for the separation values of \( m \) in terms of \( T_w/T_\infty \), \( \frac{dp}{dx} \) varies at large Mach numbers roughly as \( T_w^{-m} \), where \( 1.5 < n < 2 \), for \( T_w \) in the region of \( T_\infty \).

Thus the variation of \( \frac{dp}{dx} \) with temperature increases as the Mach number increases, in much the same way that the variation of \( \xi_\alpha \) with temperature in Fig.1 increases.

3. Cases with an abrupt pressure gradient provoking separation without any pressure increase, according to a Pohlhausen-type analysis

In incompressible flow, methods of the well known Pohlhausen type, based on the momentum integral equation and a single parameter family of assumed velocity profiles, indicate that an abrupt adverse gradient, such as is sketched in Fig.2, causes immediate separation with no pressure increase if the gradient exceeds a certain critical finite value. This result is probably untrue and is due to the false assumption that the velocity profile at separation is always of the same shape. However separation would probably occur with only a small increase of pressure under the conditions which are critical according to the theory.

A similar situation is predicted by Ref.4 for compressible flow with heat transfer. The case of a stagnation point flow with a favourable pressure gradient followed by an abrupt adverse gradient is considered. It is found that the adverse gradient needed to provoke immediate separation is somewhat decreased by increasing wall temperature \( T_w \). However the effect is not large because contrary effects operate in the regions of favourable and unfavourable gradients. For the case as sketched in Fig.2 where the pressure is constant for a certain distance from the leading edge, and there is then a constant adverse gradient, it may be shown that according to the theory of Ref.4, immediate separation occurs if

\[
\frac{dC_p}{dx} > \frac{6hT}{(1+0.2m^2)(1-\frac{T_w}{T_\infty})}
\]

where/
where \( C_p \) is the pressure coefficient and \( \mathcal{M} \) is the free-stream Mach number. Thus approximately 
\[
x \frac{dC_p}{dx} \propto \left( \frac{T_w}{T_z} \right)^{-0.73}
\]
for \( T_w \) in the region of \( T_z \). For zero heat transfer, when \( T_w = T_z \), \( x \frac{dC_p}{dx} \) according to (3) takes the values 0.291, 0.239, 0.155 and 0.102 at Mach numbers of 1.5, 2, 3 and 4 respectively. These are fairly close to the experimental data for the maximum value of \( x \frac{dC_p}{dx} \) reached at the upstream end of the laminar foot for separation from a flat wall in supersonic flow. (c.f. Ref.7.) The variation with \( T_w/T_z \) indicated by (3) is less than that predicted for \( \left( x \frac{dC_p}{dx} \right)_{\text{max}} \) in Ref.7, but nevertheless it is greater than as observed experimentally.

### References

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The variation of $\frac{\gamma}{T_s} = \frac{(U_0 - U)}{U_0}$ at separation with Mach number and wall temperature for the linear adverse velocity gradient case.

Type of pressure distribution which can lead to separation with no pressure increase, according to methods of the Pohlhausen type.