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# A Generalisation of the Nyquist Stability Criterion with particular reference to Phasing error

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# A generalisation of the Nyquist stability criterion with particular reference to phasing error

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SUMMARY

The effect of phasing error on the stability of a two-dimensional linear servomechanism is considered and it is shown that the system will be stable if the phase margin at the cut-off frequency exceeds the phasing error.

The more general case of a number of identical servos with cross-coupling is investigated and a generalisation of the Nyquist criterion for stability is formulated.

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## 1 Introduction

The general theory of coupled linear servomechanisms is extremely complicated[1]. In the case where the servos controlling each element or coordinate are identical, however, rather simple criteria for stability can be given. An important practical case is that of phasing error in two dimensions and this will be considered first. A statement of the criterion in this case is given towards the end of section 2.

Consider, for example, a radar set whose axis is required to follow a moving object. There are two elements to be controlled: namely, the left-right and up-down positions of the radar axis. If the error signals are resolved along the correct axes and the error components are used to control the appropriate motion of the radar axis, then the system can be regarded as two separate one-dimensional servo-mechanisms. If, however, owing to imperfections in the resolver mechanism the error signals are resolved along axes displaced from the correct axes, the system can no longer be regarded as two distinct one-dimensional servomechanisms, since errors in the position of each element occur as input components to both the control mechanisms.

The above system is an example of the so-called phasing error problem in a two-dimensional system. The effect of this phasing error on the stability of the system will be shown below. A more general system consisting of a number of interacting variables will then be considered, and a generalisation of Nyquist's stability criterion will be derived.

## 2 Effect of phasing error on stability

Let $y_1$, $y_2$ be the coordinates of a target point in a plane and $x_1$, $x_2$ be the coordinates of the follow-up point. It will be supposed that the follow-up point is moved by two identical linear servos each acting parallel to one of the axes of coordinates. Normally, the error vector is resolved along the coordinate axes and the error components $y_1-x_1, y_2-x_2$ used to drive the appropriate servos. If there is a phasing error $\alpha$, however, the error vector is resolved along a pair of axes obtained from the coordinate axes by rotating them an angle $\alpha$, with the result that the motion of the system is represented by the equations

$$\begin{aligned} Z(D).x_1 &= Y(D)[(y_1 - x_1) \cos \alpha + (y_2 - x_2) \sin \alpha] \\ Z(D).x_2 &= Y(D)[(y_2 - x_2) \cos \alpha - (y_1 - x_1) \sin \alpha] , \end{aligned} \tag{1}$$

where $Y(D)$, $Z(D)$ are polynomials in $D = d/dt$.

This is a pair of linear simultaneous differential equations in $x_1$ and $x_2$. On solving for $x_1$ and $x_2$, one obtains

$$\begin{aligned} (Z^2 + 2ZY \cos \alpha + Y^2)x_1 &= (ZY \cos \alpha + Y^2)y_1 + (ZY \sin \alpha)y_2 \\ (Z^2 + 2ZY \cos \alpha + Y^2)x_2 &= (ZY \cos \alpha + Y^2)y_2 - (ZY \sin \alpha)y_1 \end{aligned} \tag{2}$$

The stability of the system is therefore determined, in the usual way, by the polynomial

$$G(s) = Z(s) + 2Z(s).Y(s) \cos \alpha + Y^2(s) .$$

For the system to be stable it is necessary and sufficient that the real parts of all the roots of G(s) should be negative.

In order to transform this to a useful practical criterion two steps are performed. The first is to factorise G(s) into two factors which gives an intermediate criterion and the second may be compared with the derivation of the Nyquist criterion.

G(s) is factorised in terms of Y(s)/Z(s), giving

$$G(s) = [Z(s) + e^{i\alpha} Y(s)][Z(s) + e^{-i\alpha} Y(s)].$$

Writing Y(s)/Z(s) = F(s), the transfer function of the system, the stability criterion takes on the following intermediate form. The system will be stable if and only if the roots of the two equations $F(s) + e^{\pm i\alpha} = 0$ all lie in the left half of the complex s-plane.

Thus, as s traverses the imaginary axis and the infinite semicircle in the right-hand s-plane, $F(s) + e^{\pm i\alpha}$ must enclose no zeros for stability, i.e. F(s) must not enclose the points $-e^{\pm i\alpha}$. But this statement is precisely the Nyquist criterion[3] with the conjugate points $-e^{\pm i\alpha}$ replacing the point $-1 + i0$.

Now the points $-e^{\pm i\alpha}$ lie on the unit circle about the origin, each subtending, with the real axis, an angle $\alpha$ at the origin. It is thus seen that the system will be stable if and only if $\alpha < \phi_c$, where $\phi_c$ is the angle subtended by the point at which $F(j\omega)$ cuts the unit circle. But $\phi_c$ is simply the phase margin of the system in the absence of phasing error, so that the stability criterion takes the simple form:

THE SYSTEM WILL BE STABLE IF AND ONLY IF THE PHASING ERROR IS LESS THAN THE PHASE MARGIN IN THE ABSENCE OF PHASING ERROR.

The degree of stability is readily assessed, since the phase margin is simply $\phi_c - \alpha$.

As an example consider the case where

$$F(s) = k(1 + sT)/s^2$$

As s traverses the imaginary axis (with an indent at the origin) and an infinite circle in the right half s-plane, F(s) traverses the parabola $F(j\omega) = \frac{k}{\omega^2}(-1 - j\omega T)$ and part of an infinite circle to the right (Fig.1). The phase margin $\phi_c$ (in the absence of phasing error) is given by the equation $\sin^2 \phi_c = kT^2 \cos \phi_c$ so that, with a phasing error $\alpha$, the condition for stability is that $kT^2 > \sin^2 \alpha / \cos \alpha$.

## 3 Generalisation to a class of multi-element servomechanisms

The two-dimensional system represented by equations (1) has the following two properties.

(1) The transfer function $F(s) = Y(s)/Z(s)$ is the same for the servos controlling each coordinate.

(2) The input to each servo is a linear function of the error components.

A system of n elements possessing the above properties is called a uniform n-dimensional linear servomechanism. It should be noted that the gains of the separate servos need not be identical, since the appropriate factors may be included in the coefficients of the error components.

The remainder of this paper is concerned with the derivation of a criterion for the stability of such a system.

## 4 The stability polynomial

Let $y_i$ $(i = 1,2,\ldots,n)$ be the inputs to, $x_i$ $(i = 1,2,\ldots,n)$ be the outputs from and $F(s) = Y(s)/Z(s)$ be the transfer function of a uniform n-dimensional linear servomechanism. Then the motion of the system is represented by the n equations

$$Z(D).x_i = Y(D) \left[ \sum_{j=1}^{n} a_{ij} \, (y_j - x_j) \right], \quad (i = 1,2,\ldots,n) \qquad (2)$$

where the $a_{ij}$ are constants (real or complex).

These equations may be looked upon as a set of n linear simultaneous equations for the $x_i$ in terms of the $y_i$ and the solution may be written in the form

$$P_i \, (D).x_i = \sum_{j=1}^{n} Q_{ij} \, (D) \, y_j \, , \qquad (3)$$

where the $P_i$ (D) and $Q_{ij}$ (D) are polynomials in D.

In order to determine these polynomials it is convenient to write equations (2) in vector notation.

Let $\overline{x}$ be the column vector whose components are $x_i (i = 1,\ldots,n)$, $\overline{y}$ be the column vector whose components are $y_i$ $(i = 1,\ldots,n)$ and let A be the matrix whose i,j-th element is $a_{ij}$. Then equations (2) can be replaced by the single equation

$$Z(D) \, \overline{x} = Y(D).A(\overline{y} - \overline{x}) \qquad (4)$$

It is seen that the servo is completely described by its transfer function and by the matrix A which will be called the coupling matrix.

Rearranging equation (4)

$$[Z(D).I + Y(D).A] \, \overline{x} = Y(D).A \, \overline{y} \, , \qquad (5)$$

where I is the n-th order unit matrix.

The solution of this equation, in component form, is well-known to be

$$|M|\, x_i = Y(D) \sum_{j=1}^{n} \sum_{k=1}^{n} M_{ji}\, a_{jk}\, y_k \,, \qquad (6)$$

where $M = [m_{ij}]$ is the matrix $[Z(D).I + Y(D).A]$ and $M_{ji}$ is the co-factor of $m_{ji}$ in $|M|$. For a proof, see, for example, Ref.4 p.443.

In writing equation (6) it is tacitly assumed that $|M| \neq 0$; the case $|M| = 0$ is trivial.

It follows that the system will be stable, that is to say, each and all the separate elements are under stable control, if and only if the roots of the polynomial

$$G(s) = |Z(s)\; I + Y(s)\; A| \qquad (7)$$

lie in the left half of the complex s-plane.

## 5 The generalised Nyquist criterion

As in the phasing error case, G(s) can be factorised and the stability criterion related to the behaviour of the transfer locus of the system.

From (7),

$$G(s) = Y^n(s) \left| \frac{Z(s)}{Y(s)} \cdot I + A \right|$$

$$= Y^n(s) \prod_{i=1}^{n} \left( - \frac{Z(s)}{Y(s)} - \lambda_i \right)$$

$$= (-1)^n \prod_{i=1}^{n} (Z(s) + Y(s).\lambda_i)$$

where the $\lambda_i$ are the eigenvalues of A, i.e. the roots of the polynomial $|\lambda I - A| = 0$.

The stability condition is thus that the roots of $F(s) + \lambda_i^{-1} = 0$ lie in the left half of the s-plane, for every $\lambda_i$ .

The final step is to transform from a criterion in the s-plane to one in the F-plane. To do this use is made of the principle of the argument in analysis.

Let $H(s)$ be a regular function of s within and on a closed contour C save for P poles within the contour, so that if s describes C in the positive sense, $H(s)$ describes a closed contour $\Gamma$ in the H-plane. Then the theorem states that if the point s encircles Z zeros and P poles (taking into account any multiplicity of zeros and poles), the point $H(s)$ in the H-plane encircles the origin $N = Z-P$ times in a positive sense.

This theorem will now be applied to the function $H(s) = F(s) + \lambda_i^{-1}$ .

Since the map of a contour in the s-plane on to the F-plane can be obtained from the corresponding map on the H-plane by a shift of the origin (without axis rotation) to the point $H = + \lambda_i^{-1}$ it follows that the contour C in the s-plane, described in a positive sense, will map into a contour $\Gamma$ in the F-plane which will encircle the point $-\lambda_i^{-1}$ in a positive sense $N = Z-P$ times.

The contour C is taken to consist of the imaginary axis from $-j\infty$ to $+j\infty$ closed by a large semi-circle in the right half plane with the proviso that poles of $F(s)$, i.e. zeros of $Z(s)$, lying on the imaginary axis are detoured by small semi-circles so as to exclude them from the contour.

If the system is stable this contour must have no zeros of $H(s)$ on or within it, i.e. $Z = 0$, so that the contour $\Gamma$ in the F-plane, which is called the transfer locus of the servo, must encircle the point $-\lambda_i^{-1}$ exactly $-P$ times in a positive sense, where P is the number of poles of $H(s)$, i.e. the number of zeros of $Z(s)$, lying in the right half plane.

The generalisation of the Nyquist criterion can now be stated.

<u>Theorem</u>

If $F(s) = Y(s)/Z(s)$ is the transfer function and A the coupling matrix of a uniform n-dimensional servomechanism, the servo will be stable if and only if the transfer locus encircles each of the points $-\lambda_i^{-1}$ $(i = 1,2,\ldots,n)$ exactly $-P$ times in a positive sense, where P is the number of zeros of $Z(s)$ lying in the right half of the s-plane and the $\lambda_i$ are the eigenvalues of the matrix A.

If the coupling matrix A is composed of real elements then the eigenvalues of A are either real or occur in conjugate pairs and if the transfer function $F(s)$ is a real function of s then the transfer locus is symmetrical about the real axis.

Thus, in practice, as in the application of the ordinary Nyquist criterion, it is only necessary to consider the eigenvalues and transfer locus lying on or below the real axis.

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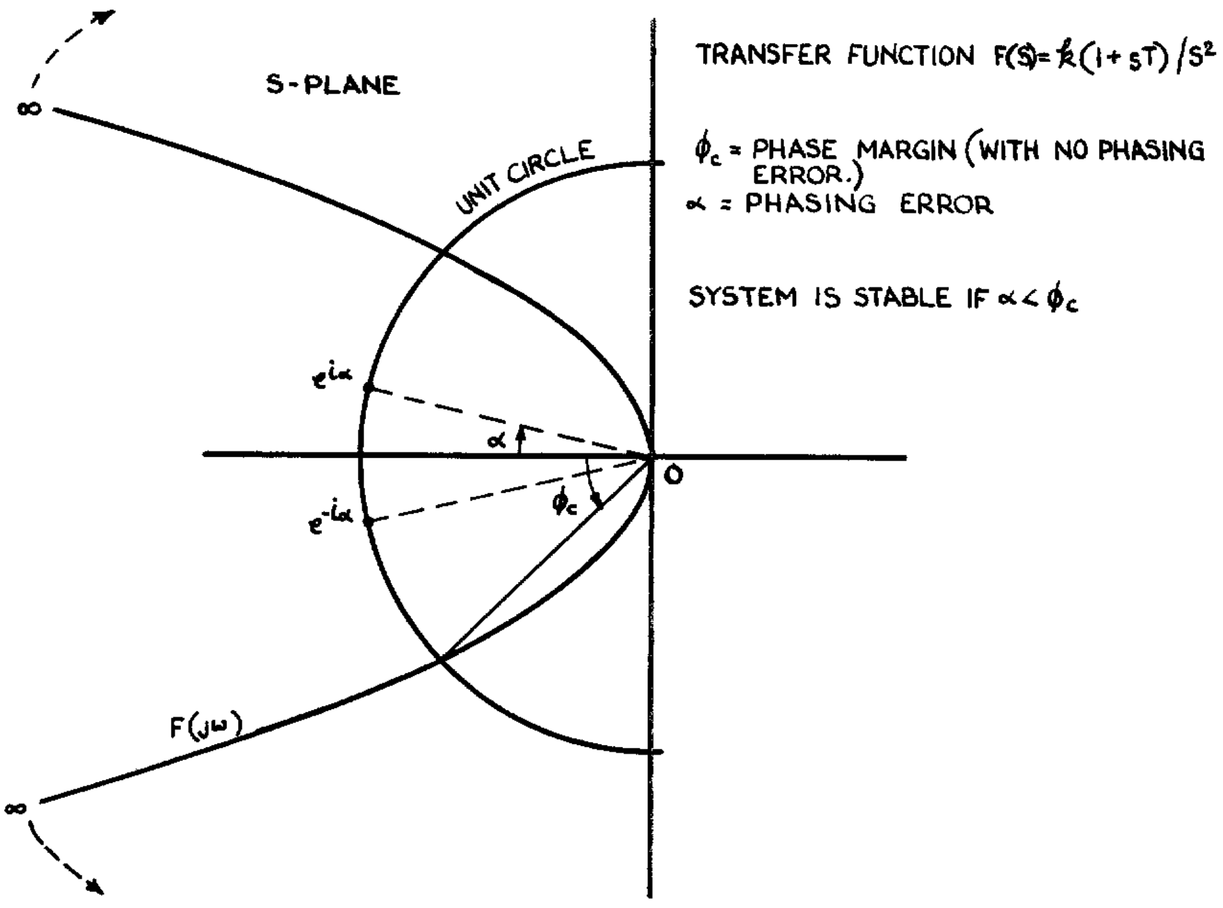


FIG. 1. THE NYQUIST CRITERION IN THE PRESENCE OF PHASING ERROR.

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