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Actuator Disc Theories Applied to the Design of Axial Compressors

By

A. D. Carmichael and J. H. Horlock,

Cambridge University Engineering Laboratory

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Actuator Disc Theories Applied to the Design of Axial Compressors

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Summary

A relatively simple method is given for finding the flow conditions in the proximity of blade rows when the variation in whirl or tangential velocity is known. The radial variations in axial velocity that would be attained in radial equilibrium are first estimated using the radial equilibrium equations and then modified to allow for the interference effect of the adjacent blade rows on the axial velocity distribution between the blade rows.

The second part of the paper gives some solutions to the radial equilibrium equations for a useful general variation in whirl velocity. This variation gives constant radial work rotors and includes the free vortex and constant reaction designs as particular solutions.

The combination of these results gives a design method which should be useful in the design of axial compressors.

Notation

C_u	Tangential or whirl velocity
C_x	Axial velocity
U	Peripheral velocity
ΔW	Rotor Work
λ	$\dfrac{C_{u_1} + C_{u_2}}{2U} \doteq 1$ - 'Reaction'
x	Distance from actuator disc (always positive)
l	Blade height
r	Non-dimensional radius referred to the mid-radius
a	$\lambda_m U_m$
b	$\dfrac{\Delta W}{2U_m}$
P	Total pressure
p	Static pressure
ρ	Density

A	Blade length to axial width, aspect ratio

Subscripts

o	Conditions far ahead of the entry guide vanes
1	Conditions far downstream of entry guide vanes and far upstream of the rotor rows
2	Conditions far downstream of the rotor rows and far upstream of the stator rows
oo	Conditions ahead of entry guide vanes
o1	Conditions between entry guide vanes and the rotor row
o2	Conditions between the rotor rows and the stator rows
m	Conditions at mid-radius
E, R and S	Refer to values of x measured from entry guide vane, rotor and stator actuator discs respectively.

Introduction

It has been shown in Refs. 1 and 2 that radial equilibrium is not set up close to blade rows and that a more accurate estimate of the conditions can be obtained by using an Actuator Disc Theory. It appears from the literature on the subject (Refs. 3, 4 and 5) that the numerical solutions in particular cases using the Actuator Disc Theory are time consuming. However, for design purposes the problem may be simplified so that little more is required once the radial equilibrium solutions have been obtained.

Part I

Actuator Disc Theory

In general, radial equilibrium is not set up close to a blade row in an axial turbo-machine and Ref. 3 shows that at an actuator disc replacing the blade row (i.e., where the discontinuity in whirl or tangential velocity is said to occur), the axial velocity is approximately the arithmetic mean of the upstream and downstream values at infinity and that radial equilibrium conditions are approached exponentially on each side of the disc.

The axial velocity at a distance x from an isolated actuator disc in incompressible flow, is,

$$C_x = C_{x_1} + \left(\frac{C_{x_2} - C_{x_1}}{2}\right)e^{-kx/l} \quad \text{upstream of the disc}$$

and

$$C_x = C_{x_2} - \left(\frac{C_{x_2} - C_{x_1}}{2}\right)e^{-kx/l} \quad \text{downstream of the disc}$$

where/

where:-

C_{x_1}, C_{x_2} = axial velocities at infinity upstream and downstream of the disc (i.e., the "radial equilibrium" solutions corresponding to the whirl velocity distribution)

l = blade height

k = constant, a function of the hub/tip ratio $(k \doteqdot \pi)$.

The conditions near several actuator discs are obtained by superposition of the effects of the radial flow field from each disc. Thus for the simplest case of a single stage compressor of three rows, the axial velocity distributions are:-

<u>Between Entry Guide Vanes (E.G.V.) and the Rotor Row</u>

$$C_{x_{01}} = C_{x_1} + \left(\frac{C_{x_2} - C_{x_1}}{2}\right)\left(e^{-\pi x_R/l} - e^{-\pi x_S/l}\right) - \left(\frac{C_{x_1} - C_{x_0}}{2}\right)e^{-\pi x_E/l} \quad \ldots (1a)$$

<u>Between the Rotor Row and the Stator Row</u>

$$C_{x_{02}} = C_{x_2} - \left(\frac{C_{x_2} - C_{x_1}}{2}\right)\left(e^{-\pi x_R/l} + e^{-\pi x_S/l}\right) - \left(\frac{C_{x_1} - C_{x_0}}{2}\right)e^{-\pi x_E/l} \quad \ldots (2a)$$

For the more complicated case of a compressor of more than one stage it can be shown that:-

<u>Between Stator Rows (or E.G.V.s.) and Rotor Rows</u>

$$C_{x_{SR}} = C_{x_1} + \left(\frac{C_{x_2} - C_{x_1}}{2}\right)\left[\left(\underset{\text{upstream}}{\Sigma e^{-\pi x_S/l} - \Sigma e^{-\pi x_R/l}}\right) + \left(\underset{\text{downstream}}{\Sigma e^{-\pi x_R/l} - \Sigma e^{-\pi x_S/l}}\right)\right] - \left(\frac{C_{x_1} - C_{x_0}}{2}\right)e^{-\pi x_E/l} \quad \ldots (1b)$$

<u>Between Rotor Rows and Stator Rows</u>

$$C_{x_{RS}} = C_{x_2} - \left(\frac{C_{x_2} - C_{x_1}}{2}\right)\left[\left(\underset{\text{upstream}}{\Sigma e^{-\pi x_R/l} - \Sigma e^{-\pi x_S/l}}\right) + \left(\underset{\text{downstream}}{\Sigma e^{-\pi x_S/l} - \Sigma e^{-\pi x_R/l}}\right)\right] - \left(\frac{C_{x_1} - C_{x_0}}{2}\right)e^{-\pi x_E/l} \quad \ldots (2b)$$

These expressions for the multi-stage compressor can be simplified if it is assumed that the aspect ratio and blade height are the same for all stages. This assumption is reasonable since the series converges rapidly.

Then/

Then:-

$$C_{x_{SR}} = C_{x_1} + \left(\frac{C_{x_2} - C_{x_1}}{2}\right)\left(\beta_u e^{-\pi x_S/l} + \beta_d e^{-\pi x_R/l}\right) - \left(\frac{C_{x_1} - C_{x_0}}{2}\right) e^{-\pi x_E/l} \quad \ldots (1c)$$

and

$$C_{x_{RS}} = C_{x_2} - \left(\frac{C_{x_2} - C_{x_1}}{2}\right)\left(\beta_u e^{-\pi x_R/l} + \beta_d e^{-\pi x_S/l}\right) - \left(\frac{C_{x_1} - C_{x_0}}{2}\right) e^{-\pi x_E/l} \quad \ldots (2c)$$

where β_u is a function of the aspect ratio and the number of blade rows (except E.G.V. row) upstream of the point considered and β_d in the same function of the number of downstream blade rows. β is plotted in Fig. 1 where

$$\beta = 1 + \sum_{r=2}^{r=n} (-e^{\pi/A})^{r-1}$$

for n blade rows, upstream or downstream. $\lim_{n \to \infty} \beta = \dfrac{1}{(1 + e^{-\pi/A})}$

In the middle stage of a multi-stage compressor the E.G.V. interference is negligible and $\beta_u = \beta_d = \beta_\infty$. Then:-

$$C_{x_{SR}} = C_{x_1} + \left(\frac{C_{x_2} - C_{x_1}}{2}\right) \beta_\infty \left(e^{-\pi x_S/l} + e^{-\pi x_R/l}\right) \quad \ldots (1d)$$

and

$$C_{x_{RS}} = C_{x_2} - \left(\frac{C_{x_2} - C_{x_1}}{2}\right) \beta_\infty \left(e^{-\pi x_R/l} + e^{-\pi x_S/l}\right) \quad \ldots (2d)$$

where β_∞ corresponds to an infinite number of blade rows.

In the above expressions it has been assumed that C_{x_1} and C_{x_2} are the same for all blade rows, which might be inaccurate. In cases where the conditions are varying through a compressor expression (1b) and (2b) with the function (C - C)/2 included in the rows upstream or downstream) so that it is quite reasonable to assume that $\beta = 1$, i.e., that two adjacent actuator discs need only be considered. β falls rapidly at aspect ratios exceeding this value and all the rows should be allowed for.

Position of the Actuator Disc

Experimental data at present available suggest that the actuator discs should be placed at the centre of pressure of the blade row, i.e., at approximately 1/3 chord from the leading edge at design conditions. It should be noted that in the above equations x, the distance from the point considered to the actuator disc, is always positive.

Part II

The equations in Part I of this note apply to the solution of the flow in a turbo machine for any general variation in whirl velocity and the corresponding "radial equilibrium" solutions for axial velocity distributions. In Part II a particular variation of whirl velocity is considered and the "radial equilibrium" solution is obtained. Such solutions may then be used in Equations (1) and (2) to give the axial velocities between the rows, and the air angle distribution.

Whirl Velocity Variation

If it is assumed that the rotor work is to be constant radially,

$$\Delta W = U\,(C_{u_2} - C_{u_1}) = \text{Constant} \qquad \ldots(3)$$

i.e.,

$$C_{u_2} = C_{u_1} + \frac{\Delta W}{U} = C_{u_1} + \frac{1}{r}\left(\frac{\Delta W}{U_m}\right) \qquad \ldots(4)$$

The reaction of the stage is also of interest and a symbol

$$\lambda = \frac{C_{u_1} + C_{u_2}}{2U} = \frac{C_{u_1} + C_{u_2}}{2rU_m} \doteqdot 1 - \text{Reaction} \qquad \ldots(5)$$

may be defined where λ would be exactly equal to (1 - Reaction) if the axial velocity across the blade row at any radius were unchanged. Then eliminating from (4) and (5) the following expressions are obtained:-

$$C_{u_1} = r\lambda U_m - \frac{\Delta W}{2rU_m} \qquad \ldots(6)$$

and

$$C_{u_2} = r\lambda U_m + \frac{\Delta W}{2rU_m} \qquad \ldots(7)$$

where/

where λ is a function of radius.

If

$$r\lambda = \lambda_m r^n$$

then

$$\frac{\lambda}{\lambda_m} = r^{n-1} \qquad \ldots(8)$$

and

$$r\lambda U_m = (\lambda_m U_m) r^n = a r^n \quad \text{(say)} \qquad \ldots(9)$$

also

$$\frac{1}{r}\left(\frac{\Delta W}{2U_m}\right) = \frac{b}{r} \quad \text{(say)} \qquad \ldots(10)$$

giving the following expressions for whirl velocity:-

$$C_{u_1} = a r^n - \frac{b}{r} \qquad \ldots(11)$$

$$C_{u_2} = a r^n + \frac{b}{r} \qquad \ldots(12)$$

where $a = \lambda_m U_m$ and $b = \dfrac{\Delta W}{2U_m}$.

The variation in λ is governed by the index n. From equation (8) λ is constant when $n = 1$ giving approximately 'constant reaction' twist, and 'free vortex' twist occurs when $n = -1$. Another interesting blade twist is obtained when $n = 0$ and has been termed 'exponential' blading in Ref. 7, and results in approximately constant inlet angle to the stator row.

Radial Equilibrium Solutions for Use in Equations (1) and (2)

The radial equilibrium equation -

$$\frac{1}{\rho}\frac{dp}{dr} = \frac{C_u^2}{r}$$

can/

can be written as

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{C_u^2}{r} + C_u\frac{dC_u}{dr} + C_x\frac{dC_x}{dr}$$

which gives the following differential equation when the total pressure is constant across the annulus:-

$$\frac{C_u^2}{r} + C_u\frac{dC_u}{dr} + C_x\frac{dC_x}{dr} = 0$$

The solution when $C_u = ar^n \pm b/r$ is

$$\left(\frac{C_x}{C_{x_m}}\right)^2 = 1 - 2\left(\frac{a}{C_{x_m}}\right)^2(n+1)\left[\frac{1}{2n}(r^{2n}-1) \pm \left(\frac{1}{n-1}\right)\left(\frac{b}{a}\right)(r^{n-1}-1)\right] \quad \ldots(13)$$

$$= 1 - \left(\frac{a}{C_{x_m}}\right)^2\left[\psi \pm \left(\frac{b}{a}\right)\phi\right].$$

Table I gives details of the 'exponential' and 'constant reaction' blade twists. An example is given in the Appendix where the radial equilibrium solution for a 'constant reaction' design has been used in conjunction with the actuator disc equations to obtain the axial velocities and air angles for a particular stage design. The results of these calculations are given in Figs. 2 and 3. The functions ψ and ϕ, which are useful in design, are plotted in Fig. 4.

Comments

The largest differences between the design air angles obtained from radial equilibrium theory and actuator disc theories occur at the outside radius and may amount to about a degree in the case of the 'exponential' twist, but are larger for 'constant reaction' twist. In the example given in Appendix I the change in axial velocity across the rotor is appreciable, giving rise to large interference effects, and the axial velocity distributions given by the actuator disc equations differ considerably from the radial equilibrium values except ahead of the first stage rotors (Fig. 1). There may be a considerable gradient of work with radius for stages of closely spaced blade rows designed using the radial equilibrium theory.

From the equations for C_{x_1} and C_{x_2} given in Table 1, the difference between the axial velocities is greatest when (approximately)

$\left|ab\left(\frac{1-r}{r}\right)\right|$ is greatest (for the 'exponential' designs)

and

$\left|ab \log_e r\right|$ is greatest (for the 'constant reaction' designs)

i.e., when $\lambda_m \Delta W$ is large and $r = \frac{\text{radius}}{\text{mean radius}}$ is maximum or minimum.

Thus/

Thus the interference effects are likely to be most important in a stage of low hub/tip ratio and high stage-temperature rise (neglecting reaction). With conventional blade loadings however, a low hub/tip ratio stage has a low temperature rise and a high hub/tip ratio stage has a high temperature rise so that interference effects are largely independent of hub/tip ratio.

The expression for whirl velocity and axial velocity for an incompressible flow turbine are similar to those for a compressor. Using compressor notation, i.e., that whirl velocity is positive in the direction of rotation, the whirl and axial velocity variations for radial equilibrium theory before a turbine rotor correspond to those after the rotor for a compressor and vice versa, i.e., for a turbine $C_{u_1} = ar^n + b/r$ and $C_{u_2} = ar^n - b/r$.

The actuator disc equations are unaltered so that the interference effects for a turbine are generally much larger than for a compressor because of the larger work per stage and for most practical cases would be a maximum for a 'constant reaction' impulse turbine.

References

Nos.	Author(s)	Title, etc.
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5	S. L. Bragg and W. R. Hawthorne	Some exact solutions of the flow through annular cascade actuator discs. Journal of Aeronautical Sciences, Vol. 17, No. 4, April, 1950.
6	J. F. Louis and J. H. Horlock	A graphical method of predicting the off-design performance of a compressor stage. Current Paper 320. April, 1956.
7	A. D. Carmichael and G. M. Lewis	Radial equilibrium in axial compressors. (Unpublished.).

Table 1 - Radial Equilibrium Equations for Special Cases (See Fig. 3)

n	0	+ 1
Type	Exponential	Constant Reaction
C_{u_1}	$a - \frac{b}{r}$	$ar - \frac{b}{r}$
C_{u_2}	$a + \frac{b}{r}$	$ar + \frac{b}{r}$
$\frac{\lambda}{\lambda_m}$	$\frac{1}{r}$	1.0
$\left(\frac{C_{x_1}}{C_{x_{m_1}}}\right)^2$	$1 - 2\left(\frac{a}{C_{x_{m_1}}}\right)^2 \left[\log_e r + \left(\frac{b}{a}\right)\left(\frac{1 - r}{r}\right)\right]$	$1 - 2\left(\frac{a}{C_{x_{m_1}}}\right)^2 \left[(r^2 - 1) - 2\left(\frac{b}{a}\right)\log_e r\right]$
$\left(\frac{C_{x_2}}{C_{x_{m_2}}}\right)^2$	$1 - 2\left(\frac{a}{C_{x_{m_2}}}\right)^2 \left[\log_e r - \left(\frac{b}{a}\right)\left(\frac{1 - r}{r}\right)\right]$	$1 - 2\left(\frac{a}{C_{x_{m_2}}}\right)^2 \left[(r^2 - 1) + 2\left(\frac{b}{a}\right)\log_e r\right]$

where $a = \lambda_m U_m$

and $b = \frac{\Delta W}{2U_m}$

Appendix I/

Appendix I

Example

To find the axial velocity distribution and air angles for a compressor stage of closely spaced blades having the following design details.

$\Delta T = 13°C$ | Mid-radius reaction = 60%

Tip speed = 1000 ft/sec | $\lambda = 0.4$

Blade aspect ratio $l/c = 3.5$ | Average axial velocity = 525 ft/sec

Hub/tip ratio = 0.4 | Blade twist = Constant reaction

Assuming hub/tip is constant through the stage and also the average axial velocity is that at mid radius.

Radial Equilibrium Solution

From Table 1

$$\left(\frac{C_{x_1}}{C_{x_{m_1}}}\right)^2 = 1 - 2\left(\frac{a}{C_{x_{m_1}}}\right)^2\left[(r^2 - 1) - 2\left(\frac{b}{a}\right)\log_e r\right] \qquad \ldots(a)$$

$$\left(\frac{C_{x_2}}{C_{x_{m_2}}}\right)^2 = 1 - 2\left(\frac{a}{C_{x_{m_2}}}\right)^2\left[(r^2 - 1) + 2\left(\frac{b}{a}\right)\log_e r\right] \qquad \ldots(b)$$

Where

$$a = \lambda_m U_m = 280$$

and

$$b = \frac{\Delta W}{2U_m} = \frac{gJC_p\Delta T}{2U_m} = \frac{10,820 \times 13}{2 \times 700} = 100$$

then

$$2\left(\frac{a}{C_{x_{m_1}}}\right)^2 = 2\left(\frac{280}{525}\right)^2 = 0.57 \quad \text{and} \quad 2\left(\frac{b}{a}\right) = \frac{2 \times 100}{280} = 0.715$$

Radius/

	Radius	root	mid	tip
	r	0.57	1.0	1.43
(c)	$(r^2 - 1)$	-0.675	-	1.04
	$\log_e r$	-0.56	-	0.358
(d)	$2\ (b/a) \log_e r$	-0.40	-	0.256
	Difference (c)-(d)	-0.275	-	0.784
	$\left(\frac{C_{x_1}}{C_{x_{m_1}}}\right)$ from (a)	1.075	1.0	0.745
	Sum (C) + (d)	1.075	-	1.296
	$\left(\frac{C_{x_2}}{C_{x_{m_2}}}\right)$ from (b)	1.270	1.0	0.51
	C_{x_1}	564	525	392
	C_{x_2}	666	525	268
	$\frac{C_{x_1} - C_{x_0}}{2}$	19.5	0	-66.5
	$\frac{C_{x_2} - C_{x_1}}{2}$	51.0	0	-62.0

Actuator Disc Effects

Using equations (1c), (2c), (1d) and (2d) from Appendix I as the aspect ratio is greater than 2.0.

The actuator discs are assumed to be situated at 1/3 chord from leading edge. Then neglecting the effect of stagger:-

$$\frac{x}{l} \text{ from leading edge } = \frac{x}{c} \times \frac{c}{l} = \frac{0.33}{3.5} = 0.095 \quad \therefore e^{-0.095\pi} = 0.74$$

$$\frac{x}{l} \text{ from trailing edge } = \frac{0.66}{3.5} = 0.19 \quad \therefore e^{-0.19\pi} = 0.55$$

First Stage of Multistage Compressor

Between E.G.V.s and Rotors (Equation (1c))

(A)	C_{x_1}	564	525	392

$\beta_u = 0;\ \beta_d = 0.71$ from Fig. 4

(B)/

(B)	$0.71 \left(\frac{C_{x_2} - C_{x_1}}{2} \right) e^{-\pi x_R/l}$	27	-	-33
(C)	$\left(\frac{C_{x_1} - C_{x_0}}{2} \right) e^{-\pi x_E/l}$	11	-	-34
	$C_{x_{01}}$ = (A) + (B) - (C)	580	525	393

Between Rotors and Stators (Equation (2c))

(D)	C_{x_2}	666	525	268

$\beta_u = 1.0$; $\beta_d = 0.71$; then $\beta_o e^{-\pi x_R/l} + \beta_d^{\pi x_S/l} = 0.550 + 0.525$

(E)	$1.075 \left(\frac{C_{x_2} - C_{x_1}}{2} \right)$	55	-	-67
(F)	$\left(\frac{C_{x_1} - C_{x_0}}{2} \right) e^{-\pi x_E/l}$	6	-	-20
	$C_{x_{02}}$ = (D) - (E) - (F)	605	525	355

Middle Stages of Multistage Compressor

Between Stators and Rotors (Equation (1d))

(G)	C_{x_1}	564	525	392

$\beta = 0.71$ therefore $0.71 \left(e^{-\pi x_R/l} + e^{-\pi x_E/l}\right) = 0.71 \times 1.29$
$= 0.92$

(H)	$0.92 \left(\frac{C_{x_2} - C_{x_1}}{2} \right)$	47	-	-57
	$C_{x_{SR}}$ = (G) + (H)	611	525	335

Between Rotors and Stators (Equation (2d))

(I)	C_{x_2}	666	525	268
(J)	$0.92 \left(\dfrac{C_{x_2} - C_{x_1}}{2} \right)$	47	-	-57
	$C_{x_{RS}}$ = (I) - (J)	619	525	325
	$C_{x_{01}}/C_{x_{m1}}$	1.105	1.0	0.75
	$C_{x_{02}}/C_{x_{m2}}$	1.15	1.0	0.676
	$C_{x_{SR}}/C_{x_m}$	1.165	1.0	0.64
	$C_{x_{RS}}/C_{x_m}$	1.180	1.0	0.62

Air Angles

First Stage

ar	160	280	400
b/r	175	100	70
C_{u_1} = ar - b/r	-15	180	330
C_{u_2} = ar + b/r	335	380	470
$C_{x_{01}}$	580	525	393
$U/C_{x_{01}}$	0.690	1.336	2.540
$\tan \alpha_0 = C_{u_1}/C_{x_{01}}$	-0.025	0.343	0.840
$\tan \alpha_1 = U/C_{x_{01}} - \tan \alpha_0$	0.715	0.993	1.700
$C_{x_{02}}$	605	525	355
$U/C_{x_{02}}$	0.661	1.336	2.820
$\tan \alpha_3 = C_{u_2}/C_{x_{02}}$	0.554	0.724	1.325

$\tan \alpha_2$/

$\tan \alpha_2 = U/C_{x_{02}} - \tan \alpha_3$	0.107	0.612	1.499
α_0	-1.4	19.0	40.0
α_1	35.6	44.8	59.6
α_2	6.1	31.5	56.3
α_3	29.0	35.9	53.0

The air angles are shown graphically in Fig. 2. It should be noted that α_2 is greater than α, at the outside radius if a radial equilibrium solutions is assumed.

FIG. 1.

Interference factor β
(Appendix 1)

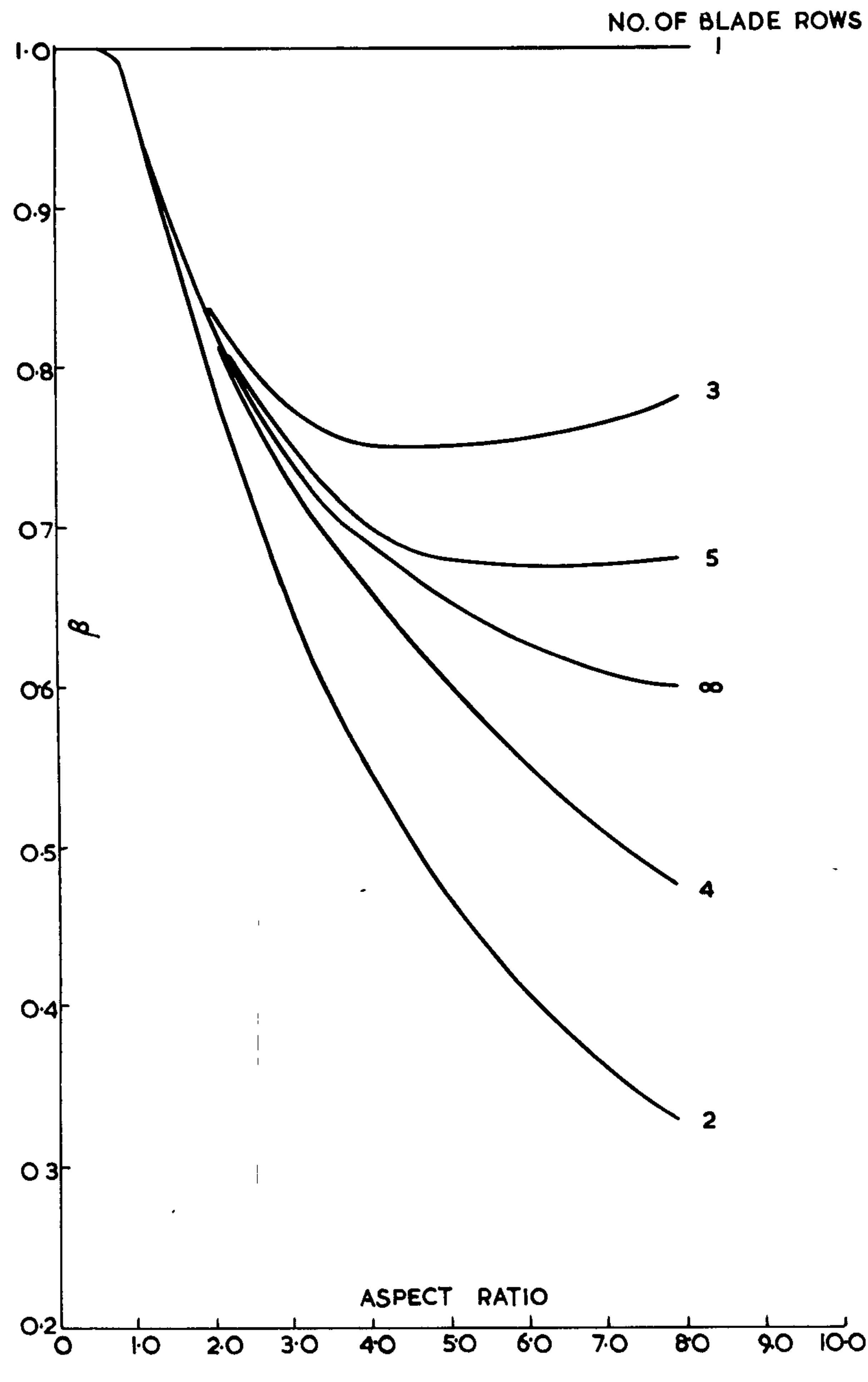


Fig. 2. Axial Velocity Distributions for the Example.

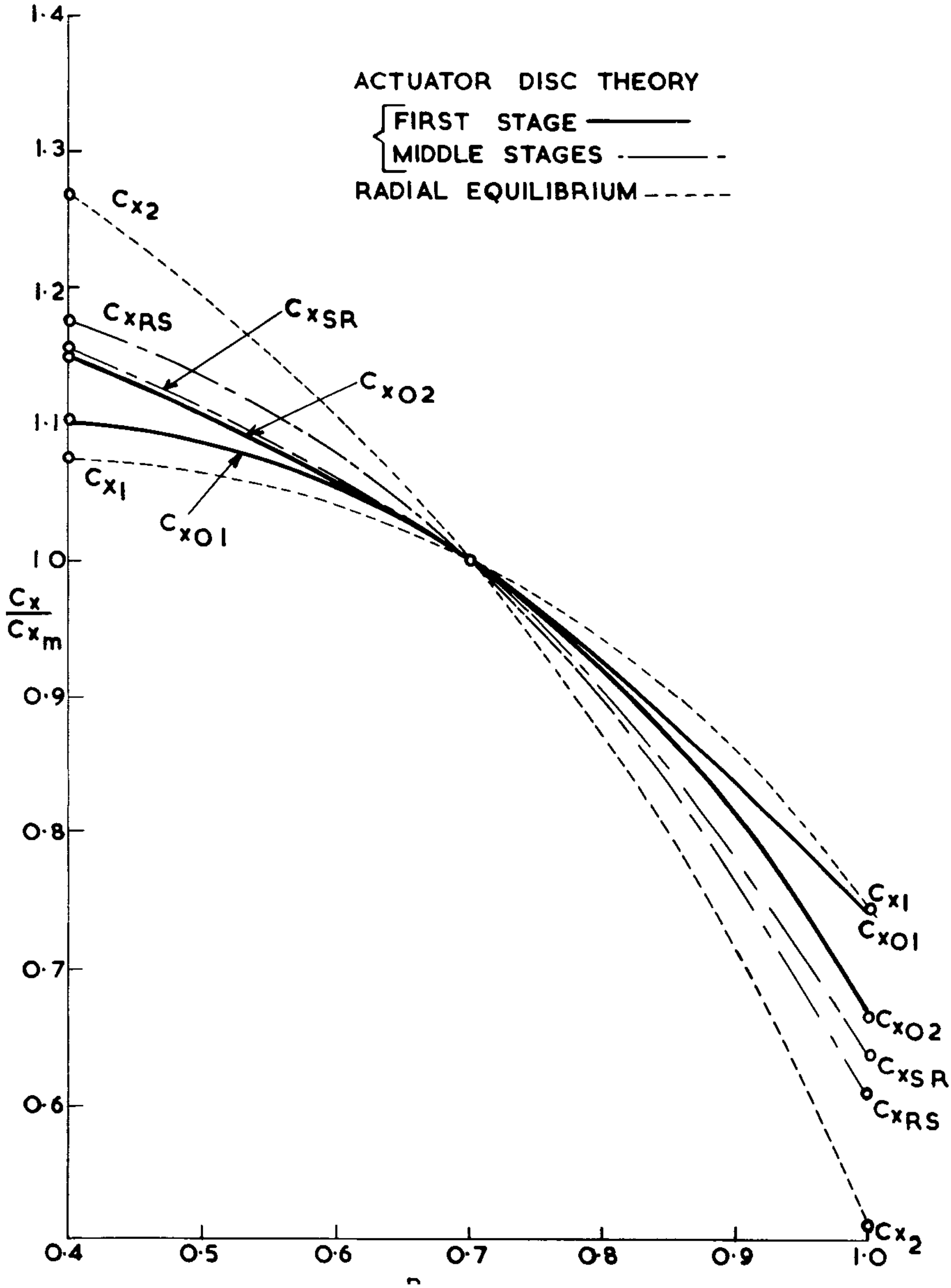


Fig.3. Air Angles for the Example

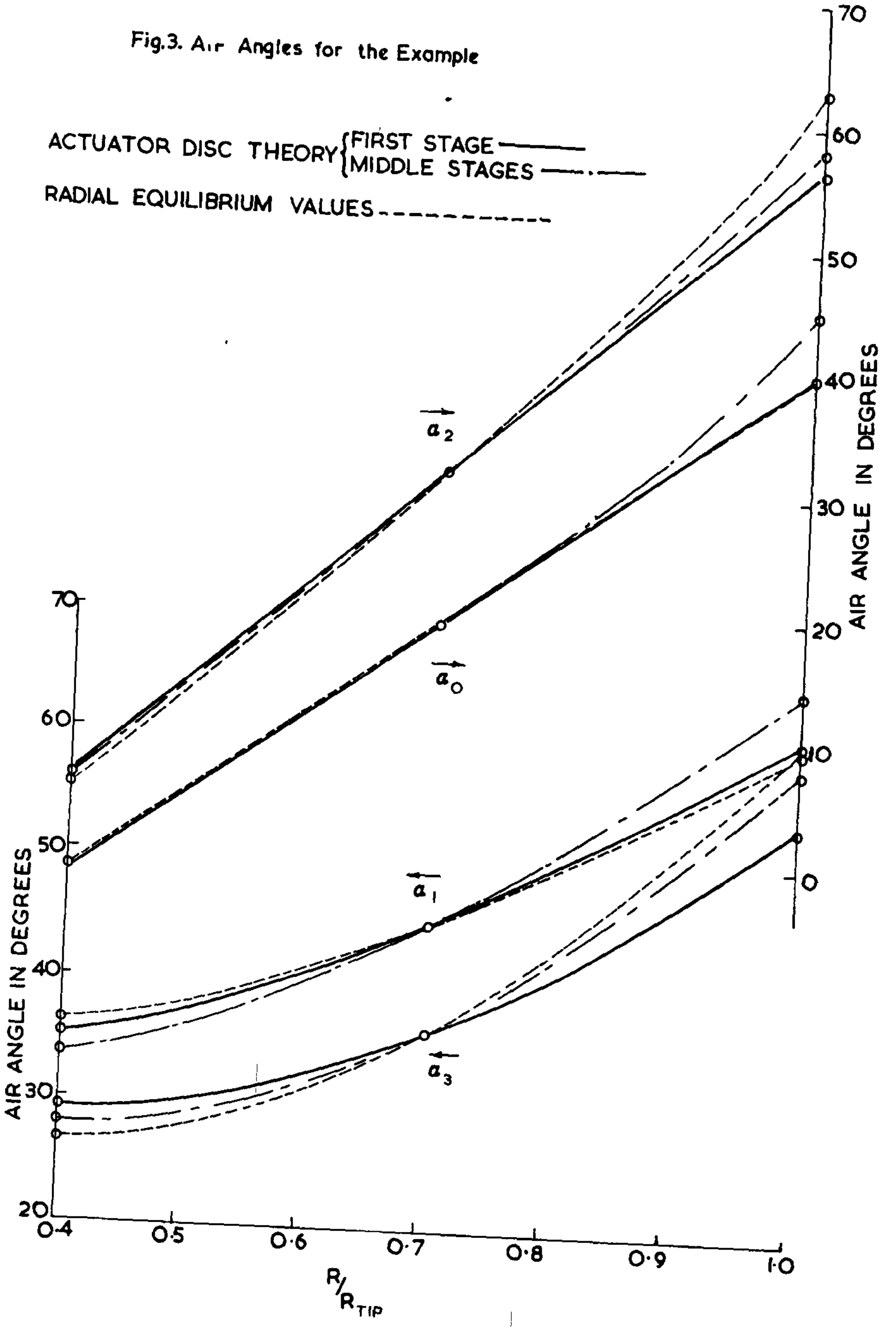
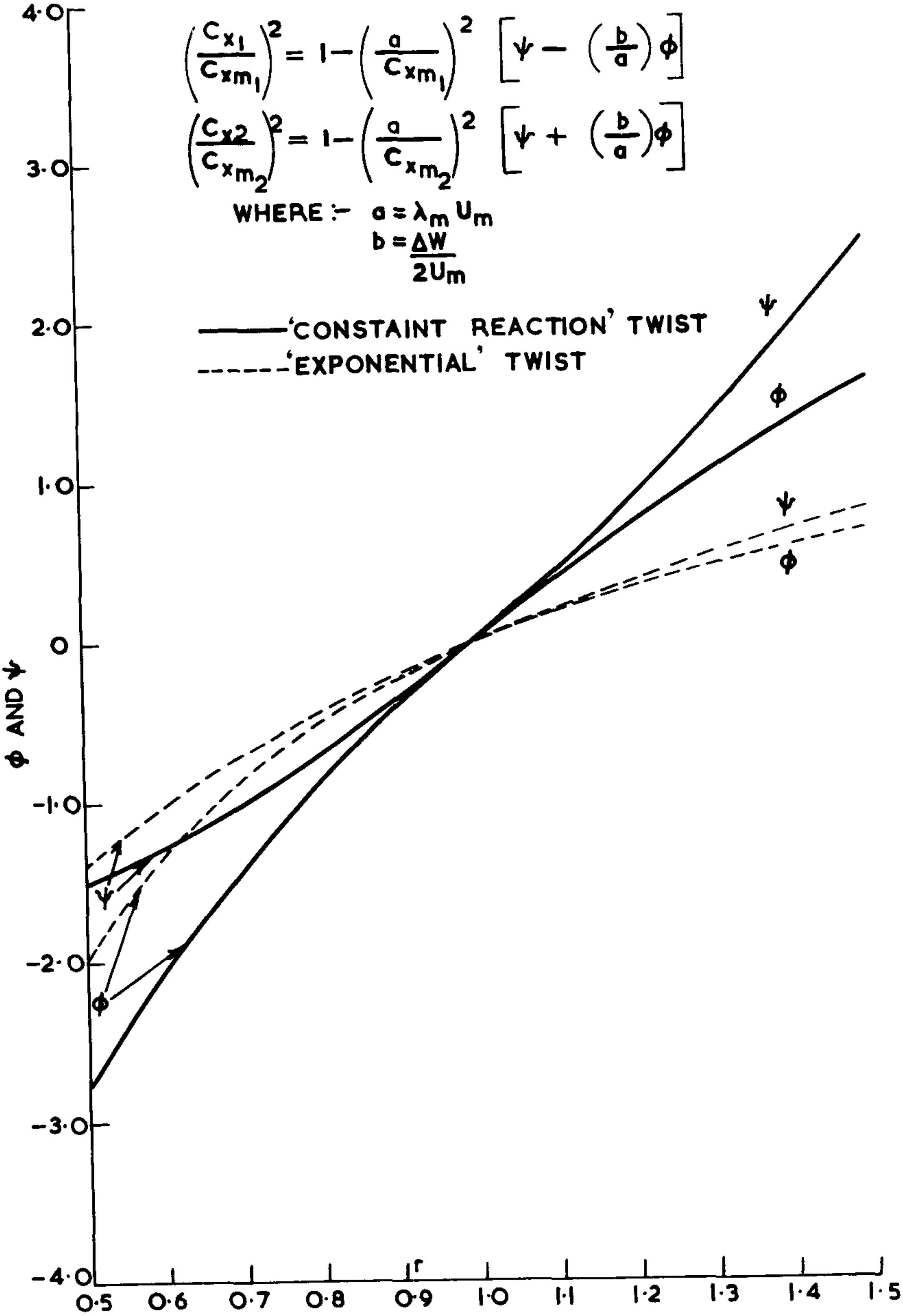


Fig. 4. Radial Equilibrium Functions



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