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The Buckling under Longitudinal Compression of a Simply Supported Panel that changes in Thickness across the Width

By

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The Buckling under Longitudinal Compression of a Simply Supported Panel that Changes in Thickness across the Width

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E. C. Capey, B.Sc.

SUMMARY

An exact solution is obtained for the buckling under longitudinal compression of a simply supported panel made up of three strips in which the central strip differs in thickness from the outer strips. The critical buckling stress is calculated numerically for a number of different ratios between thicknesses and widths of the central and outer strips. Some comparative results are given for the case when the longitudinal edges are clamped.

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1 Introduction

The integral construction of stringer sheet, whether by extrusion or by machining from the solid, makes possible a variation in skin thickness across the panel, with a consequent gain in efficiency[1].

This report presents an exact solution for the type of thickness variation across the width of the panel shown in Fig.1 when the edges are simply supported.

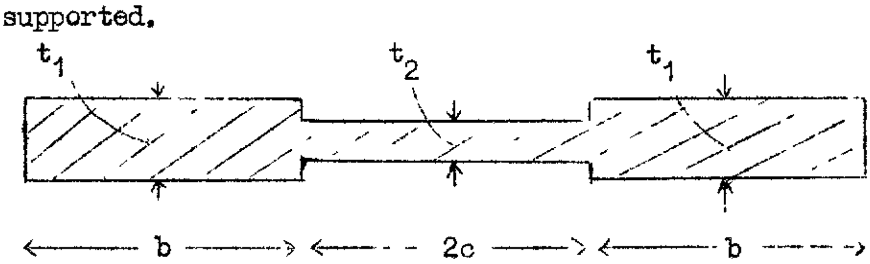

Fig.1

The cross-section of Fig.1 may be considered as an approximation to the more practical cross-section shown in Fig.2.

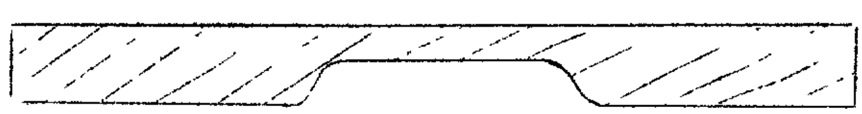

Fig.2

It is found that for a given width and cross-sectional area the highest buckling stress is about 40% greater than for a strip of constant thickness when

$$\frac{t_1}{t_2} \approx 0.36$$

and

$$\frac{c}{b} \approx 1 .$$

This configuration was examined for clamped edges and it was found that the buckling stress was 9% lower than for a clamped strip of constant thickness.

2 List of symbols

dimensions of panel:

- a = length
- b = width of outer strip
- c = half width of inner strip
- s = width of panel = $2b + 2c$

dimensions of panel	t	=	thickness of plate
	t_1	=	thickness of outer strip
	t_2	=	thickness of inner strip
	t_o	=	average thickness of panel = $(2bt_1 + 2ct_2)/s$
elastic properties	E	=	Young's modulus
	ν	=	Poisson's ratio (which is taken to be 0.3)
	G	=	Modulus of shear rigidity
axes	Ox, Oy	=	axes in plane of plate
	Oz	=	axis perpendicular to plate
	O_1x, O_1y_1, O_2x, O_2y_2	=	axes in plane of panel, as shown in Fig. 3
displacements	u, v, w	=	displacement of a point in the x, y and z directions respectively
	$f(y)$		defined by equation (9)
	w_1, w_2	=	displacement in the z-direction of points in the outer and inner strips respectively
stresses and forces	σ_x	=	compressive stress
	σ_{cr}	=	buckling stress
	$\bar{\sigma}$	=	buckling stress of a long plate with the same width, cross-sectional area and constant thickness
	τ_{xy}	=	shearing stress
	N_{xy}	=	shearing force per unit length
	η	=	$\sqrt{12\,(1-\nu^2)\,\sigma_{cr}/E}$
	M_y	=	moment per unit length
	Q_y	=	shearing stress per unit length
	M_{xy}	=	shearing moment per unit length
	m	=	number of half waves into which panel buckles
	K	=	$m\pi/a$ = $2\pi \times$ wave number of buckle
constants in equation (14)	α	=	$K\sqrt{(\eta/Kt + 1)}$
	β	=	$K\sqrt{(\eta/Kt - 1)}$

$$\left.\begin{aligned} \alpha_1 &= K\sqrt{(\eta/Kt_1 + 1)} \\ \beta_1 &= K\sqrt{(\eta/Kt_1 - 1)} \\ \alpha_2 &= K\sqrt{(\eta/Kt_2 + 1)} \\ \beta_2 &= K\sqrt{(\eta/Kt_2 - 1)} \end{aligned}\right\} \text{ constants in equations (15)}$$

$$\left.\begin{matrix} A, B, C, D \\ A_1, B_1, C_1, D_1 \\ A_2, B_2, C_2, D_2 \end{matrix}\right\} = \text{constants of integration in equations (14) and (15)}$$

$$H = \alpha_1 A_1$$

$$\text{ratios} \left\{\begin{aligned} \gamma &= t_2/t_1 \\ \zeta &= c/b \\ \mu &= \sigma_{cr}/\bar{\sigma} \\ \lambda &= \pi/Ks = \text{half wavelength of buckle/width of panel} \\ \varepsilon &= \eta/Kt_2 \end{aligned}\right.$$

Δ defined by equation (25)

Δ' defined by equation (42)

$$\begin{matrix}\text{functions} \\ \text{appearing} \\ \text{in } \Delta\end{matrix} \left\{\begin{aligned} a_1 &= 1 - \nu + \gamma\varepsilon \\ a_2 &= 1 - \nu - \gamma\varepsilon \\ a_3 &= (1 - \nu + \varepsilon)\,\gamma^3 \\ a_4 &= (1 - \nu - \varepsilon)\,\gamma^3 \\ b_1 &= \sqrt{(\gamma\varepsilon+1)}\,\coth\left\{\frac{\pi\sqrt{(\gamma\varepsilon+1)}}{2\lambda(1+\zeta)}\right\} \\ b_2 &= \sqrt{(\gamma\varepsilon-1)}\,\cot\left\{\frac{\pi\sqrt{(\gamma\varepsilon-1)}}{2\lambda(1+\zeta)}\right\} \\ b_3 &= -\sqrt{(\varepsilon+1)}\tanh\left\{\frac{\pi\zeta\sqrt{(\varepsilon+1)}}{2\lambda(1+\zeta)}\right\} \\ b_4 &= \sqrt{(\varepsilon-1)}\tan\left\{\frac{\pi\zeta\sqrt{(\varepsilon-1)}}{2\lambda(1+\zeta)}\right\} \end{aligned}\right.$$

3 Method of solution

Timoshenko has shown[2] that, when a flat plate of constant thickness t is subjected to a compressive stress in the x-direction, the equation of equilibrium is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{12(1-\nu^2)\,\sigma_x}{Et^2}\frac{\partial^2 w}{\partial x^2} . \qquad (1)$$

Equation (1) applies separately to each strip in the panel shown in Fig.3. The panel buckles symmetrically, so it is necessary to consider only one half of the panel. Using the notation of Fig.3, equation (1) is solved in Appendix I and the solution is

$$\left.\begin{aligned} w_1 &= \sin Kx\,(A_1 \sinh \alpha_1 y_1 + B_1 \sin \beta_1 y_1 + C_1 \cosh \alpha_1 y_1 + D_1 \cos \beta_1 y_1) \\ w_2 &= \sin Kx\,(A_2 \sinh \alpha_2 y_2 + B_2 \sin \beta_2 y_2 + C_2 \cosh \alpha_2 y_2 + D_2 \cos \beta_2 y_2) \end{aligned}\right\} \quad (2)$$

where K, A_1, B_1, C_1, D_1, A_2, B_2, C_2 and D_2 are constants of integration to be determined from the boundary conditions, and α_1, β_1, α_2 and β_2 are functions of t, σ_x and K .

Due to symmetry the constants A_2 and B_2 vanish. The other six boundary conditions are that the displacement w and the bending moment vanish at the edge $y_1 = 0$, and that at the junction between the two thicknesses there is continuity of displacement w , slope $\frac{\partial w}{\partial y}$, bending moment and shear.

It is shown in Appendix II that, for a given value of K, these conditions give a non-zero solution for the A's, B's, C's and D's (i.e. there is buckling), provided that

$$\Delta = \begin{vmatrix} 1 & a_1 & b_1 & a_2 b_1 \\ 1 & a_2 & b_2 & a_1 b_2 \\ 1 & a_3 & b_3 & a_4 b_3 \\ 1 & a_4 & b_4 & a_3 b_4 \end{vmatrix} = 0 \qquad (3)$$

The a's and b's are all functions of γ, ζ, ν, λ and μ, and consequently Δ is a function of these quantities.

The method of solution (taking ν to be 0.3), is to choose values of γ (i.e. t_2/t_1) and ζ (i.e. c/b), and a value of λ, then determine by trial and error the value of μ for which the determinant Δ vanishes. This process is repeated with a number of different values of λ , and μ is plotted against λ for that particular panel. The wavelength which gives the smallest μ, and consequently the smallest buckling stress for a given panel is the actual wavelength for a long panel.

The same mathematical procedure is applied in Appendix III to the case when the edges of the panel are clamped, and the buckling stress is calculated for a few values of γ and ζ.

4 Shear stiffness of panel

It is shown in Appendix IV that the shear stiffness, in the unbuckled state, of the panel in Fig.3 is

$$G t_o \left\{ \frac{(1+\zeta)^2 \gamma}{(\gamma+\zeta)(1+\gamma\zeta)} \right\} . \tag{4}$$

For a panel of constant thickness this expression becomes $G t_o$. The quantity in braces is necessarily $\leqslant 1$ for all values of γ and ζ. If $\gamma = 0.36$ and $\zeta = 1$, which are the conditions giving the highest buckling stress, then the shear stiffness is $0.78\ G t_o$.

5 Presentation of results

Figs.4 to 9 give the buckling stress and wavelength of buckle of a long panel in terms of γ and ζ. Figs.4, 5, 6 and 8 refer to the case when the sides of the panel are simply supported. Fig.7 gives some results for the clamped case, and the two are compared in Fig.9.

These results show that a simply supported panel of this type has the greatest buckling stress if $\gamma \approx 0.36$ and $\zeta \approx 1$. As the graph in Fig.3 goes down steeply on either side of $\gamma = 0.36$, the full advantage of having changes in thickness is not attained unless γ has a value close to 0.36. At this value there is a reduction in the shear stiffness of 22% and a reduction in the shear strength of 47%.

6 Conclusions

An exact solution is obtained for the buckling under longitudinal compression of a simply supported panel, made up of three strips, in which the central strip differs in thickness from the outer strips. It is shown that, for a given width and cross-sectional area, the buckling stress has a maximum value of 1.42 times that of a strip of constant thickness, when:

(a) The central strip is about twice as wide as each of the outer strips (i.e. $\zeta = 1$).

(b) The thickness of the central strip is about 0.36 times the thickness of the outer strips (i.e. $\gamma = 0.36$).

The buckling stress of a similar clamped panel has also been calculated. If the dimensions are determined by $\gamma = 0.36$ and $\zeta = 1$, the buckling stress is 0.91 times that of a corresponding clamped panel of constant thickness.

REFERENCES

No.	Author	Title, etc.
1	N.W. Parsons	The buckling under compression of a thin rectangular plate of variable thickness. ARC 17,231, November 1954.
2	S. Timoshenko	Theory of Elastic Stability, p.305.
3		p.330.
4		p.345.

APPENDIX I

General Solution of the Equation of Equilibrium for Plates under Compression

The equation of equilibrium of a flat plate under a compressive stress σ_x can be written as

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = -\frac{\eta^2}{t^2}\frac{\partial^2 w}{\partial x^2} \tag{5}$$

where

$$\eta^2 = 12\,(1-\nu^2)\,\sigma_{cr}/E\ , \tag{6}$$

which is independent of the thickness of the plate, and is therefore the same for all parts of the panel.

If the plate is simply supported at the ends $x = 0$ and $x = a$, then at these ends:

$$w = 0 \tag{7}$$

and

$$\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0\ . \tag{8}$$

Equations (7) and (8) are satisfied by

$$w = f(y)\sin(m\pi x/a)\ , \tag{9}$$

where m is an integer.

Substituting

$$K = m\pi/a \tag{10}$$

in equation (9), we obtain

$$w = \sin Kx\, f(y) \tag{11}$$

which, when substituted in equation (5), gives

$$f'''' - 2K^2 f'' + K^4 f = \frac{\eta^2 K^2}{t^2} f\ , \tag{12}$$

which may be put in the form

$$f'''' + (\beta^2 - \alpha^2)\, f'' - \alpha^2\beta^2 f = 0\ . \tag{13}$$

It can be shown that the general solution of equation (13) is:

$$f = A \sinh \alpha y + B \sin \beta y + C \cosh \alpha y + D \cos \beta y . \qquad (14)$$

The strip buckles symmetrically, so only one half of it need be considered. For the outer strip of this half of the panel:

$$\left.\begin{aligned} w_1 &= \sin Kx\,(A_1 \sinh \alpha_1 y_1 + B_1 \sin \beta_1 y_1 + C_1 \cosh \alpha_1 y_1 + D_1 \cos \beta_1 y_1), \\ &\text{and for the central strip} \\ w_2 &= \sin Kx\,(A_2 \sinh \alpha_2 y_2 + B_2 \sin \beta_2 y_2 + C_2 \cosh \alpha_2 y_2 + D_2 \cos \beta_2 y_2). \end{aligned}\right\} \quad (15)$$

Because of symmetry the constants A_2 and B_2 vanish.

The expressions for α_1, β_1, α_2 and β_2 can be simplified by substituting

$$\varepsilon = \eta / K t_2 \qquad (16)$$

to the form

$$\left.\begin{aligned} \alpha_1 &= K\sqrt{\gamma\varepsilon + 1} & \beta_1 &= K\sqrt{\gamma\varepsilon - 1} \\ \alpha_2 &= K\sqrt{\varepsilon + 1} & \beta_2 &= K\sqrt{\varepsilon - 1} \end{aligned}\right\} \quad (17)$$

APPENDIX II

Calculation of the Compressive Stress Required to Produce Buckling of a Simply Supported Panel

Boundary Conditions

At $y_1 = 0$ the panel is simply supported, so that

$$\left.\begin{aligned} w_1 &= 0 \\ \frac{\partial^2 w_1}{\partial y_1^2} + \nu \frac{\partial^2 w_1}{\partial x^2} &= 0 . \end{aligned}\right\} \qquad (18)$$

These boundary conditions are satisfied only if C_1 and D_1 in equations (15) vanish, leaving

$$\left.\begin{aligned} w_1 &= \sin Kx \, (A_1 \sinh \alpha_1 y_1 + B_1 \sin \beta_1 y_1) \\ w_2 &= \sin Kx \, (C_2 \cosh \alpha_2 y_2 + D_2 \cos \beta_2 y_2) \end{aligned}\right\} \qquad (19)$$

Junction Boundary Conditions

At the junction between the two thicknesses the following boundary conditions hold:

$$\left.\begin{aligned} (w_1)_b &= (w_2)_{-c} \\ \left(\frac{\partial w_1}{\partial y_1}\right)_b &= \left(\frac{\partial w_2}{\partial y_2}\right)_{-c} \\ M_y = \frac{-Et_1^3}{12\,(1-\nu^2)} \left(\frac{\partial^2 w_1}{\partial y_1^2} + \nu \frac{\partial^2 w_1}{\partial x^2}\right)_b &= \frac{-Et_2^3}{12\,(1-\nu^2)} \left(\frac{\partial^2 w_2}{\partial y_2^2} + \nu \frac{\partial^2 w_2}{\partial x^2}\right)_{-c} \\ Q_y - \frac{\partial M_{xy}}{\partial x} = \frac{-Et_1^3}{12(1-\nu^2)} \left(\frac{\partial^3 w_1}{\partial y_1^3} + \overline{2-\nu} \, \frac{\partial^3 w_1}{\partial x^2 \partial y_1}\right)_b \\ = \frac{-Et_2^3}{12(1-\nu^2)} \left(\frac{\partial^3 w_2}{\partial y_2^3} + \overline{2-\nu} \, \frac{\partial^3 w_2}{\partial x^2 \partial y_2}\right)_{-c} \end{aligned}\right\} \qquad (20)$$

When equations (19) are differentiated, and the differential coefficients are simplified using equations (17) and substituted into equations (20), we obtain:

$$\left.\begin{aligned}
&A_1 \sinh \alpha_1 b + B_1 \sin \beta_1 b - C_2 \cosh \alpha_2 c - D_2 \cos \beta_2 c = 0 \\
&\alpha_1 A_1 \cosh \alpha_1 b + \beta_1 B_1 \cos \beta_1 b + \alpha_2 C_2 \sinh \alpha_2 c - \beta_2 D_2 \sin \beta_2 c = 0 \\
&t_1^3 (\gamma\varepsilon+1-\nu) A_1 \sinh \alpha_1 b - t_1^3 (\gamma\varepsilon-1+\nu) B_1 \sin \beta_1 b - t_2^3 (\varepsilon+1-\nu) C_2 \cosh \alpha_2 c \\
&\qquad + t_2^3 (\varepsilon-1+\nu) D_2 \cos \beta_2 c = 0 \\
&t_1^3 (\gamma\varepsilon-1+\nu) A_1\alpha_1 \cosh \alpha_1 b_1 - t_1^3 (\gamma\varepsilon+1-\nu) B_1\beta_1 \cos \beta_1 b \\
&\qquad + t_2^3 (\varepsilon-1+\nu) C_2\alpha_2 \sinh \alpha_2 c + t_2^3 (\varepsilon+1-\nu) D_2\beta_2 \sin \beta_2 c = 0
\end{aligned}\right\} \quad (21)$$

Condition of Buckling

The condition of buckling is that the determinant of the coefficients of A_1, B_1, C_2 and D_2 vanishes, that is

$$\begin{vmatrix}
\sinh \alpha_1 b & \alpha_1 \cosh \alpha_1 b & t_1^3 (\gamma\varepsilon+1-\nu) \sinh \alpha_1 b & t_1^3 (\gamma\varepsilon-1+\nu)\, \alpha_1 \cosh \alpha_1 b \\
\sin \beta_1 b & \beta_1 \cos \beta_1 b & -t_1^3 (\gamma\varepsilon-1+\nu) \sin \beta_1 b & -t_1^3 (\gamma\varepsilon+1-\nu)\, \beta_1 \cos \beta_1 b \\
-\cosh \alpha_2 c & \alpha_2 \sinh \alpha_2 c & -t_2^3 (\varepsilon+1-\nu) \cosh \alpha_2 c & t_2^3 (\varepsilon-1+\nu)\, \alpha_2 \sinh \alpha_2 c \\
-\cos \beta_2 c & -\beta_2 \sin \beta_2 c & t_2^3 (\varepsilon-1+\nu) \cos \beta_2 c & t_2^3 (\varepsilon+1-\nu)\, \beta_2 \sin \beta_2 c
\end{vmatrix} = 0$$

......(22)

The four rows of this determinant are now divided by $(\sinh \alpha_1 b)$, $(\sin \beta_1 b)$, $(-\cosh \alpha_2 c)$ and $(-\cos \beta_2 c)$ respectively, and the columns by (1), (K), (t_1^3) and $(-Kt_1^3)$ respectively, which is permissible, as none of these qualities is in general equal to zero when the determinant vanishes.

Eliminating b and c by using the relations

$$b = s/2\ (1+\zeta) \qquad (23)$$

and

$$c = s\zeta/2\ (1+\zeta), \qquad (24)$$

we have

$$\Delta = \begin{vmatrix} 1 & a_1 & b_1 & a_2 b_1 \\ 1 & a_2 & b_2 & a_1 b_2 \\ 1 & a_3 & b_3 & a_4 b_3 \\ 1 & a_4 & b_4 & a_3 b_4 \end{vmatrix} = 0 \qquad (25)$$

where

$$\left.\begin{aligned} a_1 &= 1 - \nu + \gamma\varepsilon \\ a_2 &= 1 - \nu - \gamma\varepsilon \\ a_3 &= (1-\nu+\varepsilon)\,\gamma^3 \\ a_4 &= (1-\nu-\varepsilon)\,\gamma^3 \end{aligned}\right\} \qquad (26)$$

and

$$\left.\begin{aligned} b_1 &= \sqrt{\gamma\varepsilon + 1}\,\coth\left\{\frac{\pi\sqrt{\gamma\varepsilon + 1}}{2\lambda\,(1+\zeta)}\right\} \\ b_2 &= \sqrt{\gamma\varepsilon - 1}\,\cot\left\{\frac{\pi\sqrt{\gamma\varepsilon - 1}}{2\lambda\,(1+\zeta)}\right\} \\ b_3 &= -\sqrt{\varepsilon + 1}\,\tanh\left\{\frac{\pi\zeta\sqrt{\varepsilon + 1}}{2\lambda\,(1+\zeta)}\right\} \\ b_4 &= \sqrt{\varepsilon - 1}\,\tan\left\{\frac{\pi\zeta\sqrt{\varepsilon - 1}}{2\lambda\,(1+\zeta)}\right\} . \end{aligned}\right\} \qquad (27)$$

Evaluation of stress

It is now necessary to find an expression for the buckling stress. It follows from equations (6) and (16) that

$$\sigma_{cr} = \frac{E}{12\,(1-\nu^2)} K^2\, t_2^2\, \varepsilon^2 . \qquad (28)$$

Let $\bar{\sigma}$ be the stress required to produce buckling of a uniform strip of the same width s and cross-sectional area st_o. Timoshenko has shown[3] that

$$\bar{\sigma} = \frac{4\pi^2 E}{12\,(1-\nu^2)} \frac{t_o^2}{s^2} \tag{29}$$

so that

$$\mu = \frac{\sigma_{cr}}{\bar{\sigma}} = \left(\frac{K\,\varepsilon\, s\, t_2}{2\,\pi\, t_o}\right)^2 . \tag{30}$$

This can be put in terms of ζ, γ, λ and ε, giving

$$\mu = \left\{\frac{\varepsilon\,(1+\zeta)\,\gamma}{2\lambda\,(1+\gamma\zeta)}\right\}^2 . \tag{31}$$

Method of Computation

The problem now is to evaluate

$$\mu = \mu\,(\zeta, \gamma, \lambda, \varepsilon) \tag{32}$$

given that $\nu = 0.3$ and that

$$\Delta = \Delta(\zeta, \gamma, \lambda, \varepsilon) = 0 . \tag{33}$$

ζ and γ are chosen, thus fixing the shape of the panel; then a number of values of λ are chosen, for each of which the value of ε for vanishing of Δ is determined by trial and error, and hence μ can be calculated. In this way μ is obtained as a function of ζ, γ and λ.

For a long panel there is no restriction on the possible values of the wavelength due to boundary conditions, so a long panel buckles with that wavelength which gives the minimum value of the buckling stress. Therefore, for a given ζ and γ, μ is plotted against λ, and the minimum of the curve represents the actual wavelength of buckle and buckling stress. By this method μ and λ are obtained as functions of γ and ζ. The results of these computations are plotted in Figs.4 to 6.

Special Cases

$\gamma = 1$. This is the case of a uniform plate. It follows from its definition that $\mu = 1$. Timoshenko has shown[3] that $\lambda = 1$.

$\gamma \to 0$. The central part of the plate is very thin, so buckles like a clamped plate of width 2c and thickness t_2. For this Timoshenko has obtained[4]

$$\sigma_{cr} = \frac{k\pi^2\, E t_2^2}{12\,(1-\nu^2)(2c)^2} , \tag{34}$$

where k is 6.967; and the plate buckles with a half-wavelength of 0.668(2c), from which it follows that

$$\lambda = 0.668 \frac{\zeta}{1 + \zeta} . \tag{35}$$

Using equations (29) and (34):

$$\mu = \frac{\sigma_{cr}}{\bar{\sigma}} = k \left\{ \frac{(1+\zeta)^2}{2 \, (1+\gamma\zeta)} \frac{\gamma}{\zeta} \right\}^2 . \tag{36}$$

When $\gamma \to 0$ this becomes

$$\mu = k \frac{(1+\zeta)^4}{4\zeta^2} \gamma^2 . \tag{37}$$

Equation (37) was used to plot μ for small values of γ in Figs.4 to 6.

APPENDIX III

Calculation of the Stress Required to Produce Buckling of a Clamped Panel

Boundary Conditions

As in the simply supported case equations (15) apply, and A_2 and B_2 vanish due to symmetry. As the boundary $y_1 = 0$ is clamped, the conditions there are that

$$\left.\begin{aligned} w_1 &= 0 \\ \text{and} \qquad \frac{\partial w_1}{\partial y_1} &= 0 \,. \end{aligned}\right\} \qquad (38)$$

Equations (15) and (38) give

$$\left.\begin{aligned} (w_1)_o &= \sin Kx\,(C_1 + D_1) = 0 \\ \text{and} \qquad \left(\frac{\partial w_1}{\partial y_1}\right)_o &= \sin Kx(\alpha_1 A_1 + \beta_1 B_1) = 0 \,. \end{aligned}\right\} \qquad (39)$$

On substitution of equations (39), equations (15) become:

$$\left.\begin{aligned} w_1 &= \sin Kx \left\{H\left(\frac{\sinh \alpha_1 y_1}{\alpha} - \frac{\sin \beta_1 y_1}{\beta}\right) + C_1\,(\cosh \alpha_1 y_1 - \cos \beta_1 y_1)\right\} \\ w_2 &= \sin Kx\,(C_2 \cosh \alpha_2 y_2 + D_2 \cos \beta_2 y_2) \,. \end{aligned}\right\} \qquad (40)$$

Junction Boundary Conditions

The same boundary conditions, equations (20), apply at the junctions between the two thicknesses as in the simply supported case. On differentiation of equations (40) and substitution of the differential coefficients, these become:

/(41)

$$\left.\begin{aligned}
&H\left(\frac{\sinh\alpha_1 b}{\alpha_1}-\frac{\sin\beta_1 b}{\beta_1}\right)+C_1(\cosh\alpha_1 b-\cos\beta_1 b)-C_2\cosh\alpha_2 c-D_2\cos\beta_2 c = 0\\
&H(\cosh\alpha_1 b-\cos\beta_1 b)+C_1(\alpha_1\sinh\alpha_1 b+\beta_1\sin\beta_1 b)\\
&\qquad +C_2\alpha_2\sinh\alpha_2 c-D_2\beta_2\sin\beta_2 c = 0\\
&t_1^3 H\left\{(\gamma\varepsilon+1-\nu)\frac{\sinh\alpha_1 b}{\alpha_1}+(\gamma\varepsilon-1+\nu)\frac{\sin\beta_1 b}{\beta_1}\right\}+t_1^3 C_1\{(\gamma\varepsilon+1-\nu)\cosh\alpha_1 b\\
&\qquad +(\gamma\varepsilon-1+\nu)\cos\beta_1 b\}-t_2^3 C_2(\varepsilon+1-\nu)\cosh\alpha_2 c\\
&\qquad +t_2^3 D_2(\varepsilon-1+\nu)\cos\beta_2 c = 0\\
&t_1^3 H\{(\gamma\varepsilon-1+\nu)\cosh\alpha_1 b+(\gamma\varepsilon+1-\nu)\cos\beta_1 b\}+t_1^3 C_1\{(\gamma\varepsilon-1+\nu)\alpha_1\sinh\alpha_1 b\\
&\qquad -(\gamma\varepsilon+1-\nu)\beta_1\sin\beta_1 b\}+t_2^3 C_2\alpha_2(\varepsilon-1+\nu)\sinh\alpha_2 c\\
&\qquad +t_2^3 D_2\beta_2(\varepsilon+1-\nu)\sin\beta_2 c = 0\,.
\end{aligned}\right\}\quad(41)$$

<u>Condition of Buckling</u>

The condition of buckling is that the determinant of the coefficients of H, C_1, C_2 and D_2 vanishes. On dividing the rows and columns of this determinant by various functions, as in the simply supported case, the condition becomes: $\Delta' =$

$\dfrac{\cosh\alpha_1 b}{\sqrt{\gamma\varepsilon+1}}-\dfrac{\sin\beta_1 b}{\sqrt{\gamma\varepsilon-1}}$	$\cosh\alpha_1 b - \cos\beta_1 b$	$\dfrac{a_1\sinh\alpha_1 b}{\sqrt{\gamma\varepsilon+1}}-\dfrac{a_2\sin\beta_1 b}{\sqrt{\gamma\varepsilon-1}}$	$a_2\cosh\alpha_1 b - a_1\cos\beta_1 b$
$\cosh\alpha_1 b - \cos\beta_1 b$	$\sqrt{\gamma\varepsilon+1}\sinh\alpha_1 b + \sqrt{\gamma\varepsilon-1}\sin\beta_1 b$	$a_1\cosh\alpha_1 b - a_2\cos\beta_1 b$	$a_2\sqrt{\gamma\varepsilon+1}\sinh\alpha_1 b + a_1\sqrt{\gamma\varepsilon-1}\sin\beta_1 b$
1	b_3	a_3	$a_4 b_3$
1	b_4	a_4	$a_3 b_4$

$= 0\,. \quad \ldots\ldots(42)$

This problem can be solved numerically by the same method as was employed in the simply supported case.

Special Cases

$\gamma = 1$ This is a clamped plate of uniform thickness, for which Timoshenko[4] has shown

$$\sigma_{cr} = \frac{k\pi^2 E}{12\,(1-\nu^2)} \frac{t_o^2}{s^2} \tag{43}$$

where k is 6.967, and the half-wavelength is 0.668s.

Using equation (29)

$$\left.\begin{aligned} \mu &= \frac{\sigma_{cr}}{\bar{\sigma}} = \frac{k}{4} = 1.742 \\ \text{while} \quad \lambda &= 0.668 . \end{aligned}\right\} \tag{44}$$

$\gamma \to 0$ The situation is the same as in the simply supported case.

APPENDIX IV

Determination of the Shear Stiffness of the Panel

When the panel is subjected to a shearing force N_{xy} per unit length, the shearing stress is

$$\tau_{xy} = N_{xy}/t = G\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) . \tag{45}$$

As there is no displacement in the y-direction, v vanishes, leaving

$$\frac{du}{dy} = \frac{N_{xy}}{Gt} .$$

The shear stiffness of the panel is the shearing force per unit length divided by the average shear strain, which is

$$\frac{N_{xy}}{\frac{1}{s}\int \frac{du}{dy}\,dy} = \frac{Gs}{\int \frac{dy}{t}} = Gs\left(\frac{2b}{t_1} + \frac{2c}{t_2}\right)^{-1} , \tag{46}$$

which can be put in the form:

$$\text{Shear stiffness} = Gt_o\left\{\frac{(1+\zeta)^2\gamma}{(\gamma+\zeta)(1+\gamma\zeta)}\right\} . \tag{47}$$

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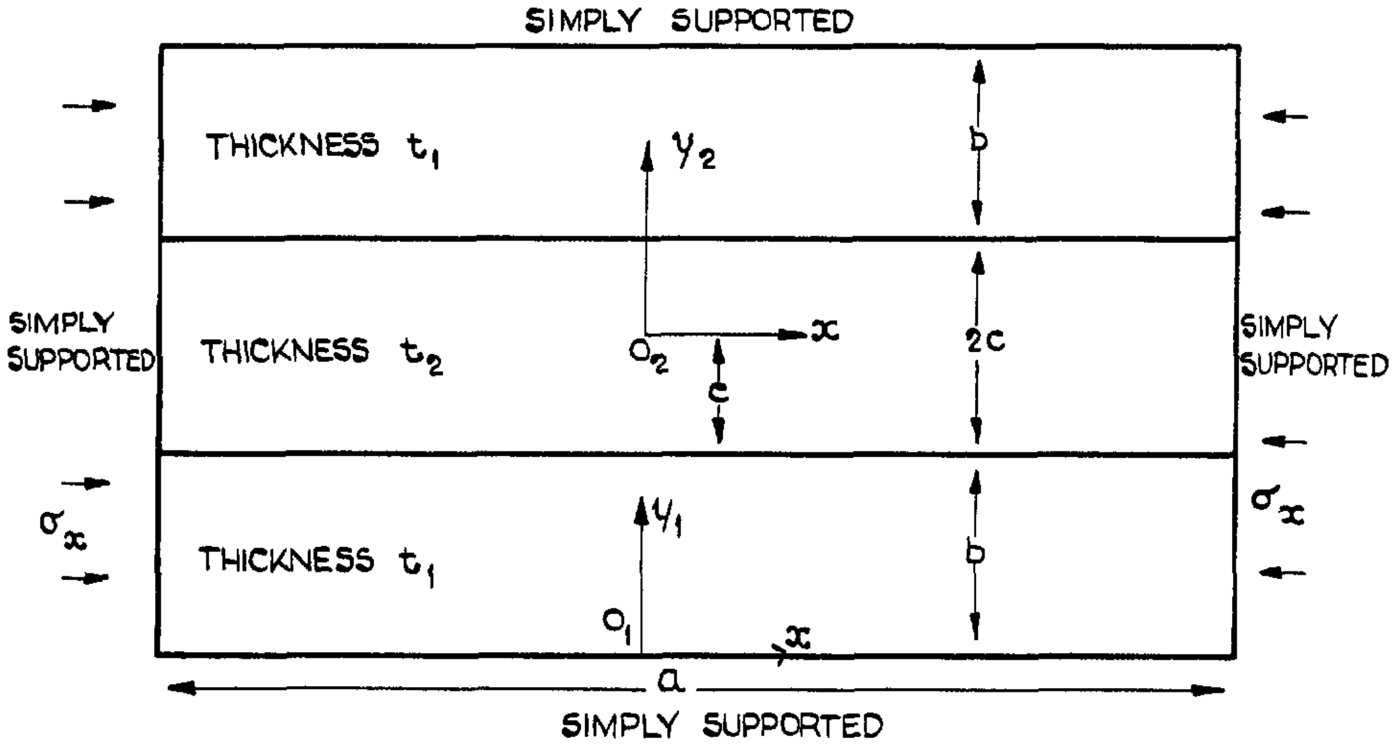

FIG. 3 FIGURE SHOWING NOTATION.

FIG. 4

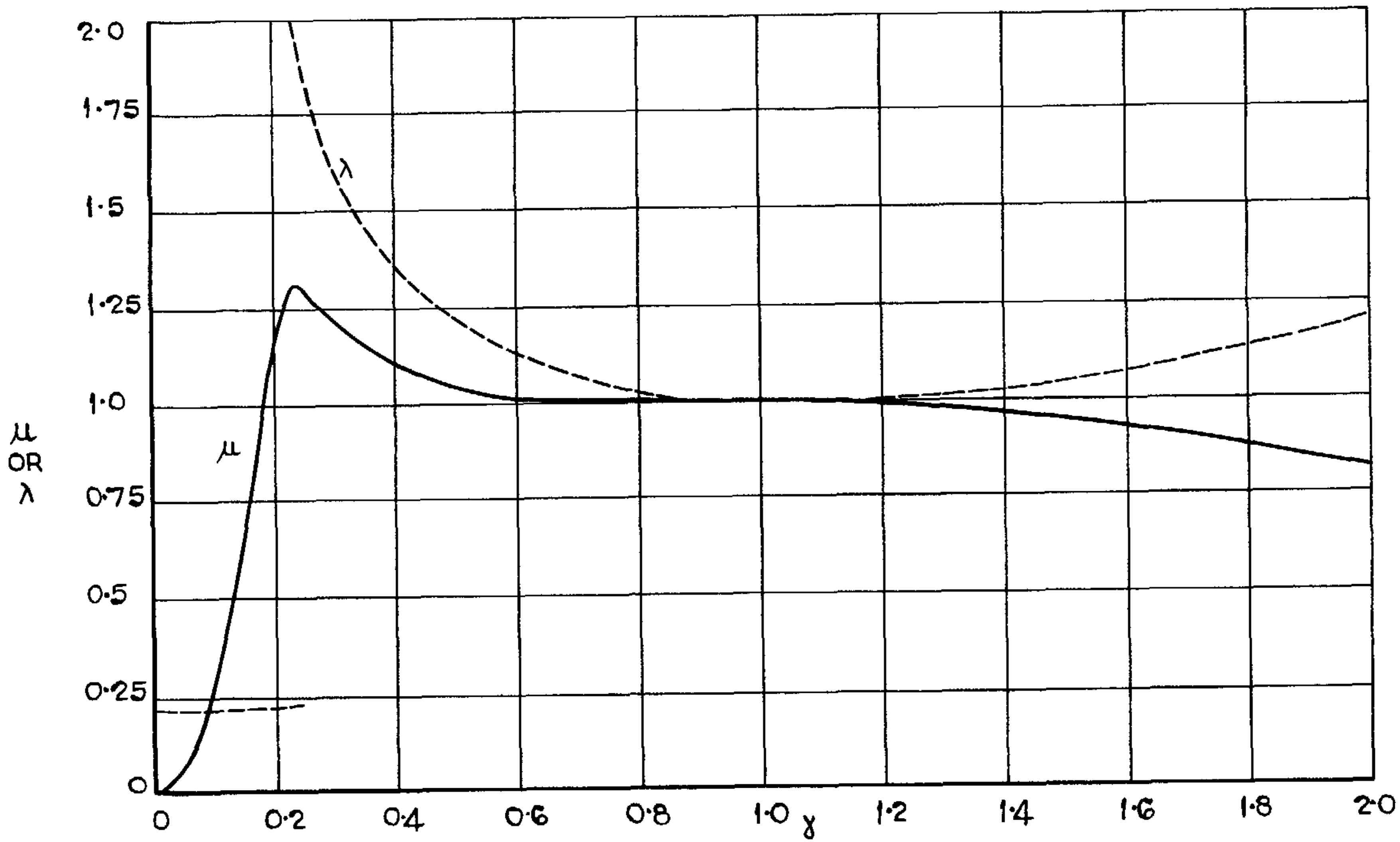

FIG. 4 BUCKLING STRESS AND WAVELENGTH OF BUCKLE OF A SIMPLY SUPPORTED PANEL $\varphi = \frac{1}{2}$.

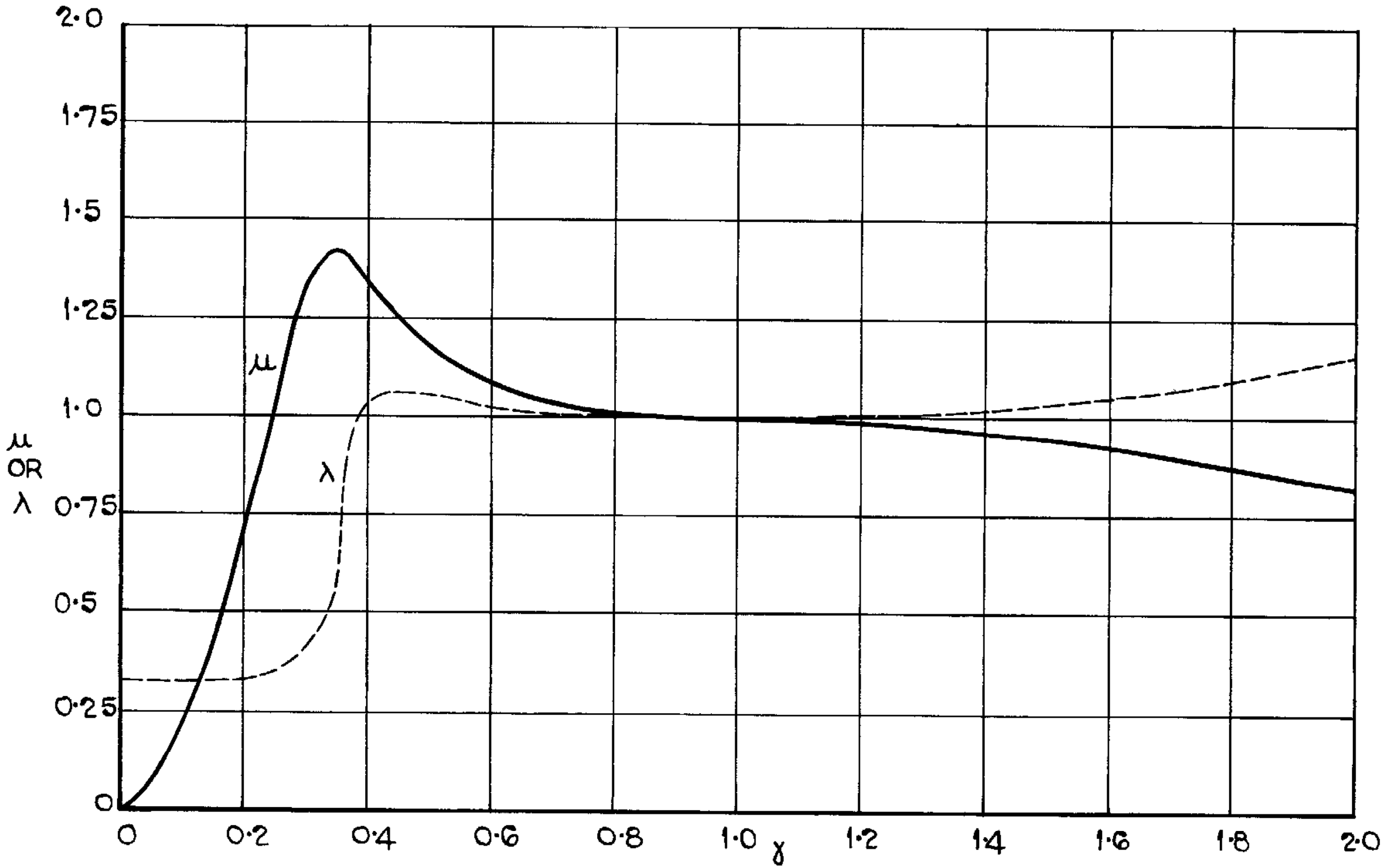

FIG. 5 BUCKLING STRESS AND WAVELENGTH OF BUCKLE OF A SIMPLY SUPPORTED PANEL $\varphi = 1$.

FIG. 5

FIG. 6

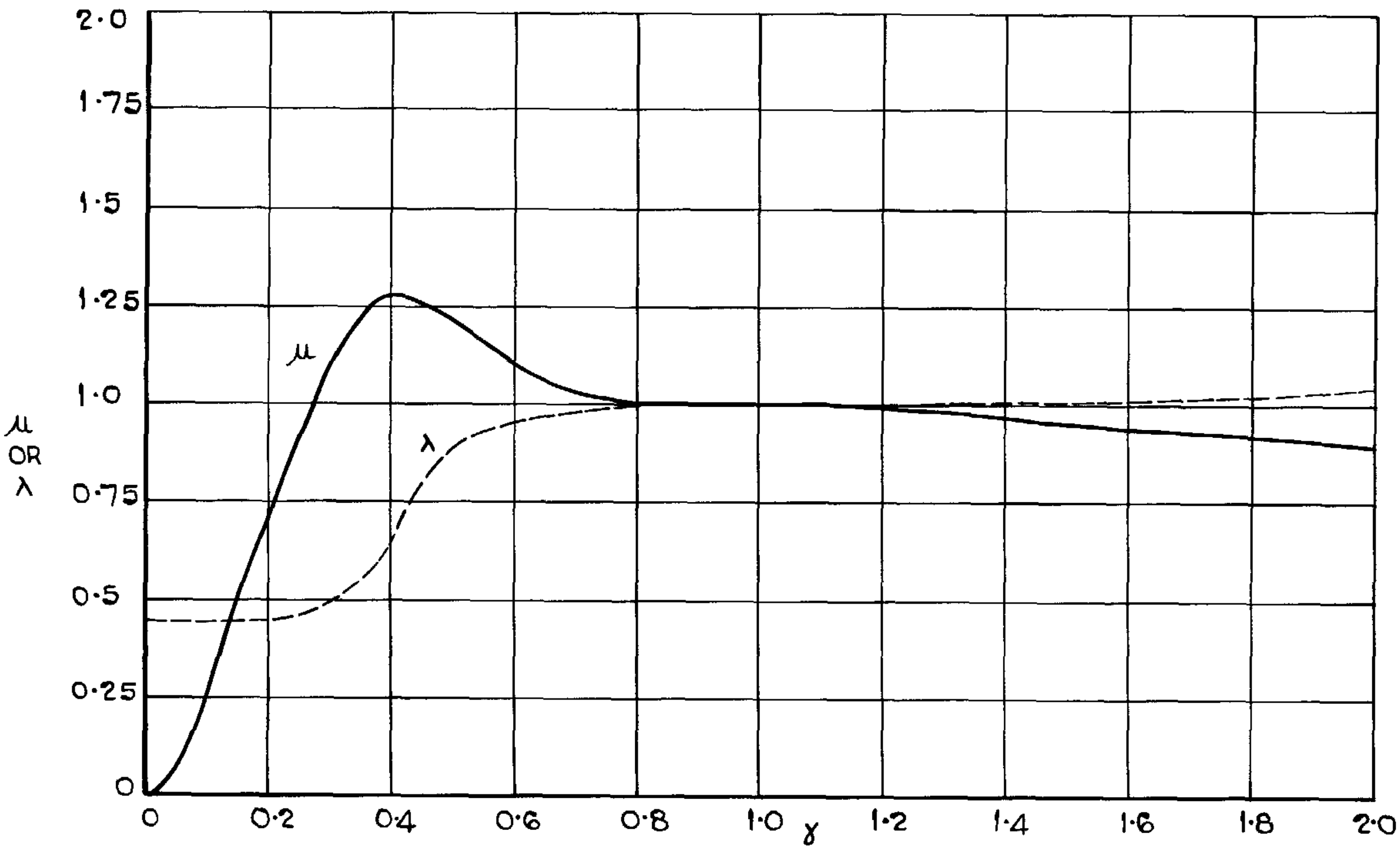

FIG. 6 BUCKLING STRESS AND WAVELENGTH OF BUCKLE OF A SIMPLY SUPPORTED PANEL $\psi = 2$.

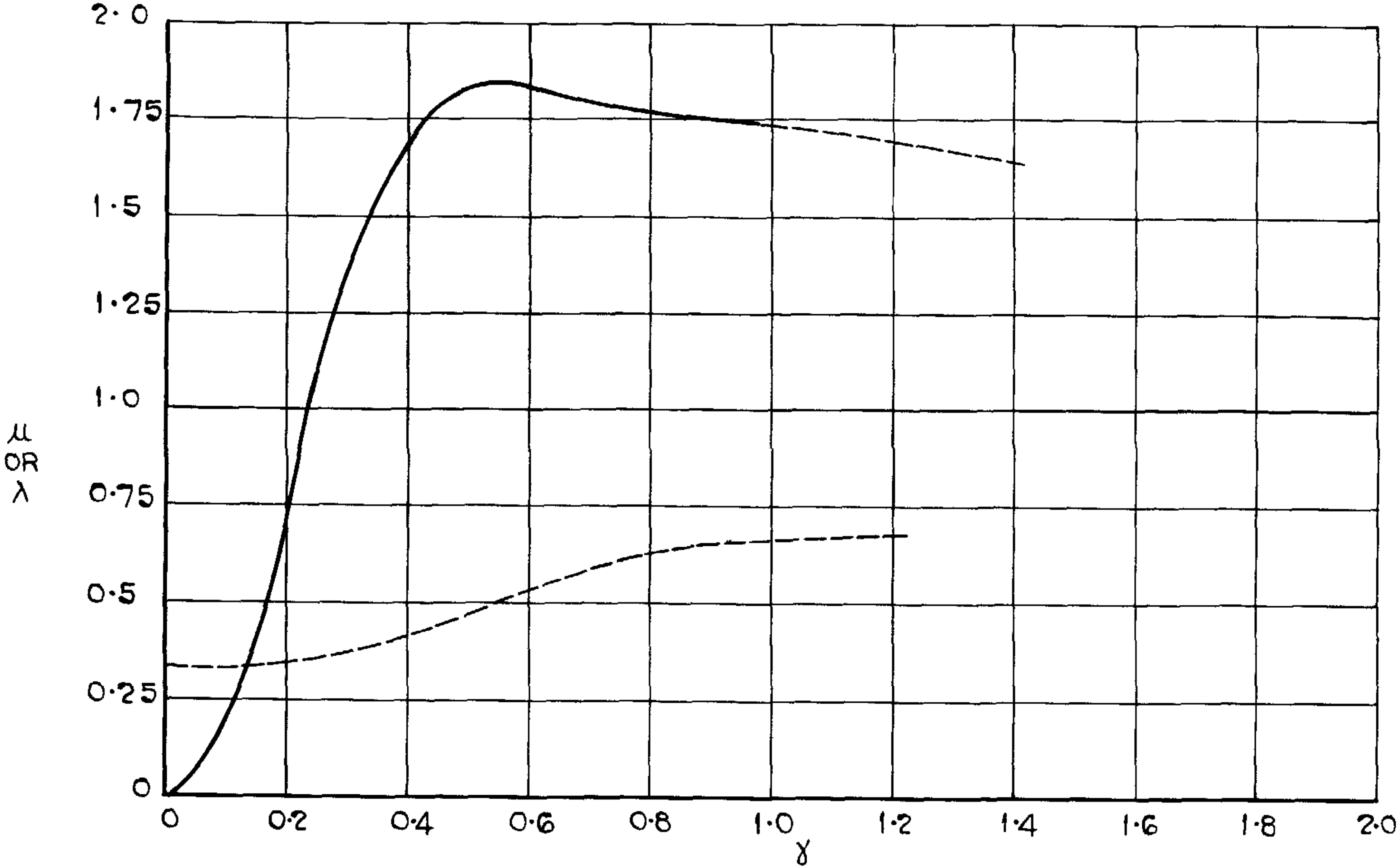

FIG. 7 BUCKLING STRESS AND WAVELENGTH OF BUCKLE OF A CLAMPED PANEL $\varphi = 1$.

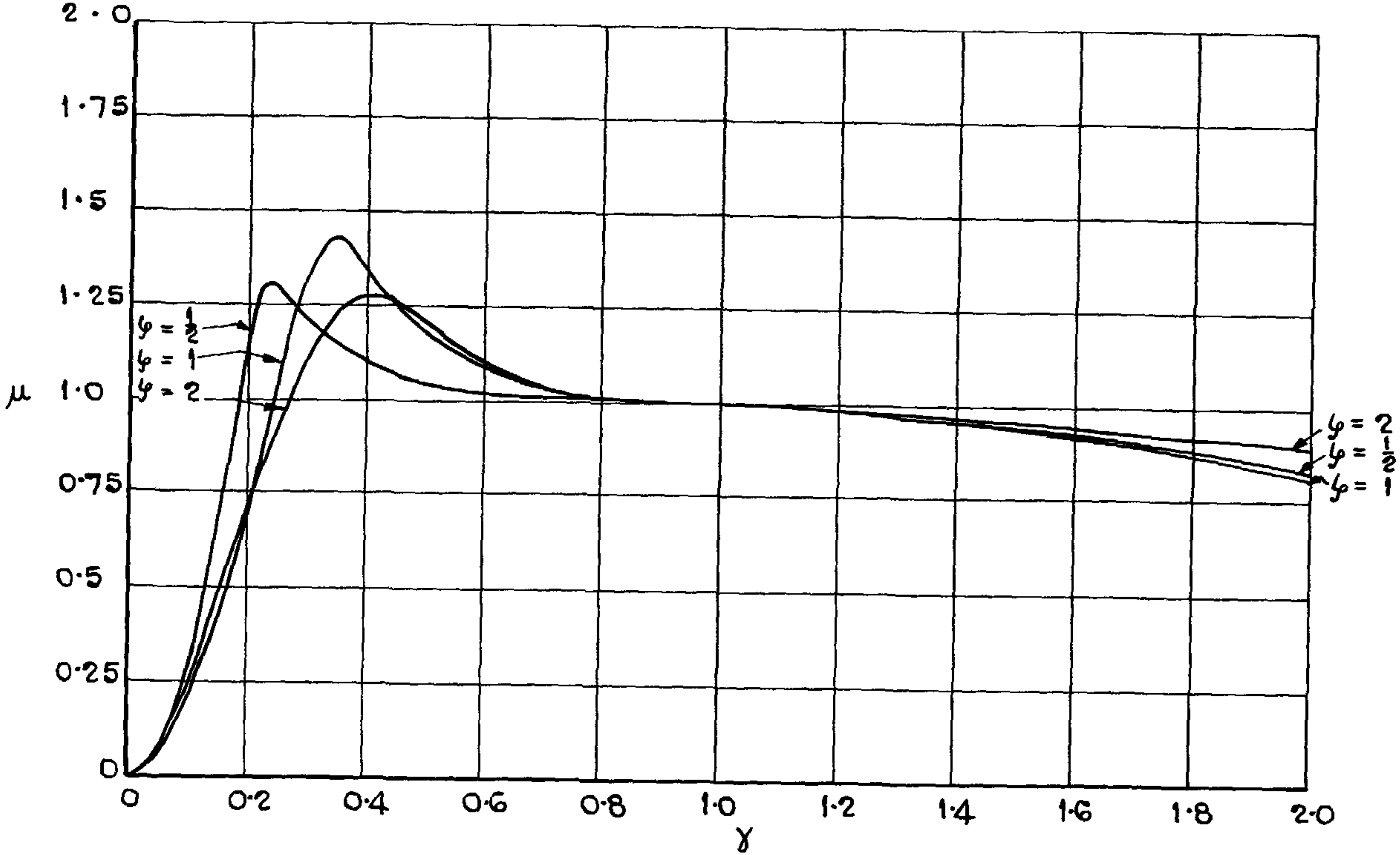

FIG. 8 BUCKLING STRESS OF SIMPLY SUPPORTED PANELS.

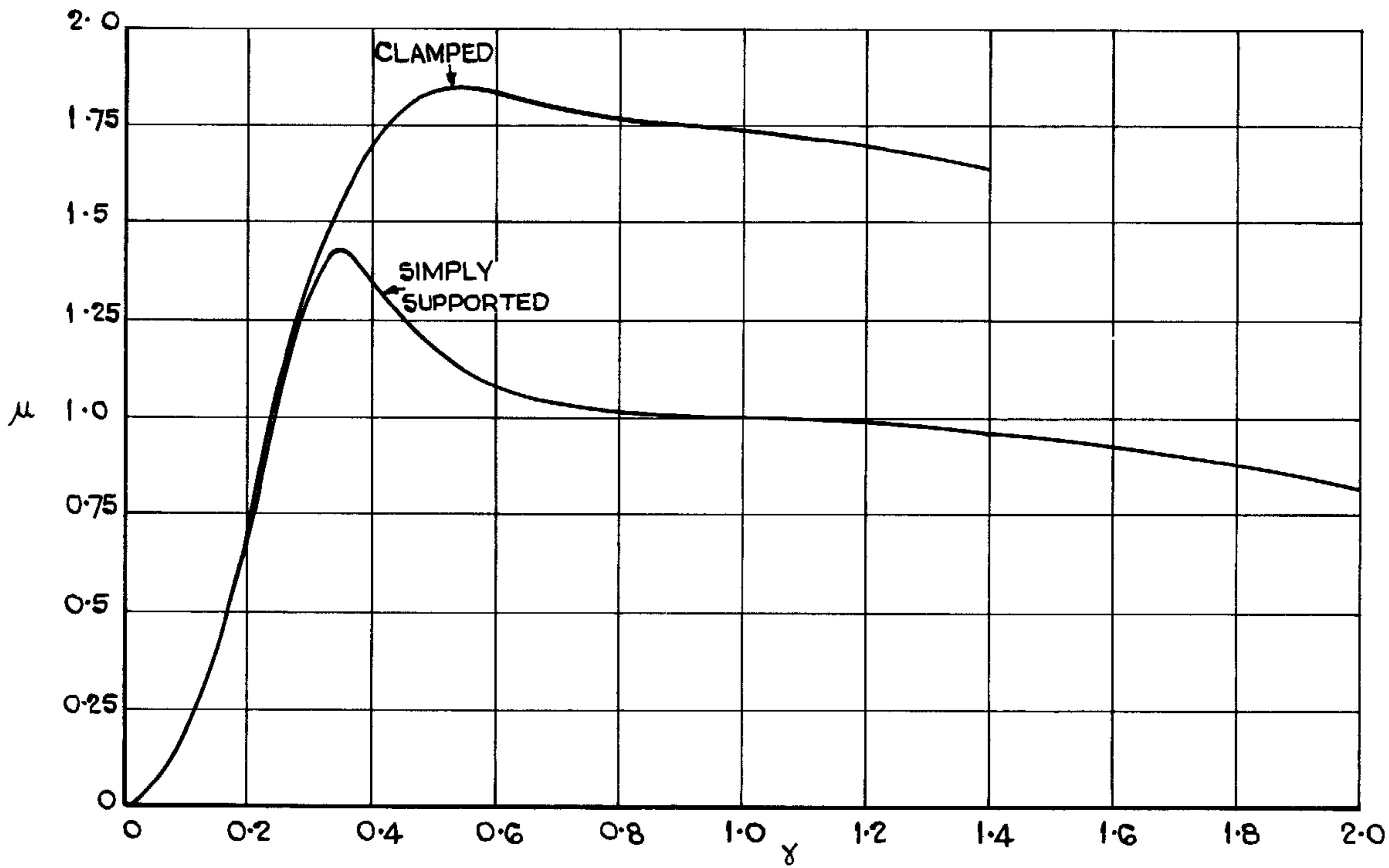

FIG. 9 BUCKLING STRESS OF SIMPLY SUPPORTED AND CLAMPED PANELS $\varphi = 1$.

FIG. 9

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