C.P. No. 149
(15,750)
A.R.C. Technical Report

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Unsteady Cavitating Flow Past Curved Obstacles

By

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(New Zealand Scientific Defence Corps at present seconded to the Aerodynamics Division of the N.P.L.)

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1954

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26th March, 1953

Summary

The plane incompressible flow past two symmetric curved obstacles, between which is a finite constant pressure cavity, is calculated for the case when the cavity length and pressure are functions of time.

1. Introduction

In spite of its obvious applications to the theory of underwater cavity formation the subject of unsteady cavitating flow has only recently received the attention of applied mathematicians. von Karman[1] calculated a particular symmetric unsteady flow with a finite cusped cavity behind a flat plate, while Gilbarg[2] has obtained a formula giving all such flows not only for a flat plate but also for any symmetric polygon. Unfortunately cusped cavities have the series defect of being physically unrealistic in the limit as the flow becomes steady. In steady flow physical cavities almost invariably have positive values of the cavitation number Q, (defined in equation (5)), but for cusped cavities, $Q < 0$. A second unrealistic feature of these cavities is that the flow separates from the obstacle behind the point of minimum pressure - a defect which applies, in any case, to all flows past polygons. A full discussion of these points (for the steady case) occurs in Ref.3. By continuity these defects must also appear in the unsteady flows for which the rate of change of cavity length is small, but in general it is only these slowly varying flows that are likely to be adequately represented by the theory. The reason for this is that the free boundaries enclosing the cavity are actually material lines, and generally not the streamlines it is necessary to assume they are in the mathematical theory. As pointed out by Gilbarg[2], it is physically reasonable that errors from this approximation can be neglected for slowly varying flows.

The introduction of a second body at the rear of the cavity, as in Riabouchinsky's model of steady cavitating flow[3], overcomes the above mentioned defects. In the next section we shall obtain a formula for the symmetrical unsteady flow about two curved obstacles between which extends a cavity for which the pressure is constant at any given time. Fig. 1, where B is moving relative to A along the axis of symmetry, illustrates such a flow. Gilbarg's result is the special case when A is a polygon and B degenerates to a point.

2. General Theory

One form of the differential equation of the flow is

$$\nabla^2 f = 0, \qquad \ldots(1)$$

where

$$f = \log(U/q) + i\theta, \qquad \ldots(2)$$

i/

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$i = \sqrt{-1}$, (q,θ) is the velocity vector in polar co-ordinates and U is the velocity at infinity. Equation (2) can be written in the alternative form

$$f = \log\left(U\frac{dz}{dw}\right), \qquad \ldots(3)$$

where z is the physical plane, $z = x + iy$, and w is the complex potential, $w = \phi + i\psi$. Equation (1) holds in the z-plane and in any other plane derived from it by a conformal transformation.

The upper halves of the w and z-planes are shown in Fig. 2 where the dotted line representing the free boundary is assumed in the following analysis to be the streamline $\psi = 0$. The boundary conditions are that θ vanishes in $A_\infty B$, EF_∞ and is known in BC and DE, while on CD the velocity satisfies Bernoulli's equation

$$\left(\frac{q}{U}\right)^2 = 1 + Q - \rho\frac{\partial}{\partial t}\phi, \qquad \ldots(4)$$

where

$$Q = (p_\infty - p_o)/\tfrac{1}{2}\rho U^2, \qquad \ldots(5)$$

is the cavitation number, p_c and p_∞ are the pressures in the cavity and at infinity respectively, ρ is the density and t is the time. In addition if θ is measured from the flow direction at infinity, then from (2)

$$\lim_{z=\infty} f(z) = 0\ , \qquad \ldots(6)$$

while the absence of circulation about the obstacles requires that

$$\lim_{z=\infty} zf(z) = 0\ . \qquad \ldots(7)$$

The solution of (1) in the ϵ-plane, shown in Fig. 1 and defined by

$$w = a(1 - \coth\epsilon), \quad \epsilon = \delta + i\xi, \quad \delta,\xi \text{ real}, \qquad \ldots(8)$$

has already been calculated[4] for the boundary conditions, θ known on $\xi = 0$ and $\log(U/q)$ known on $\xi = \tfrac{1}{2}\pi$. The result is

$$f = f_o + \frac{2}{\pi}\int_{\delta^*=-\infty}^{\infty} \tanh^{-1}\exp(\delta^* - \epsilon)\, d\theta(\delta^*) - \frac{2}{\pi}\int_{\delta^*=-\infty}^{\infty} \tan^{-1}\exp(\delta^* - \epsilon)\, dr(\delta^*)\ , \qquad \ldots(9)$$

where θ is the flow direction on $\xi = 0$, r is the value of $\log(U/q)$ on $\xi = \tfrac{1}{2}\pi$, and f_o is the value of f at $w = 0$ (point C in Fig. 2). On the free streamlines, $\xi = \tfrac{1}{2}\pi$, and (8) becomes

Q/

$$Q = a(1 - \tanh\delta),$$

so that equation (4) can be written

$$r = -\tfrac{1}{2}\log\left(1 + Q - \frac{2a'}{U^2} + \frac{2a'}{U^2}\tanh\delta\right), \qquad \ldots(10)$$

the dash denoting the time rate of change. From (9) and (10)

$$f(\epsilon) = f_o + \frac{2}{\pi}\int_{\delta^*=-\infty}^{\infty} \tanh^{-1}\exp(\delta^* - \epsilon)\,d\theta(\delta^*) + \frac{2a'}{\pi U^2}\int_{-\infty}^{\infty}\frac{\operatorname{sech}^2\delta^*\tan^{-1} e^{\delta^*-\epsilon}\,d\delta^*}{\left(1 + Q - \frac{2a'}{U^2} + \frac{2a'}{U^2}\tanh\delta^*\right)} \qquad \ldots(11)$$

Two distinct cases now arise, one of which, it is shown below, is physically unrealistic.

<u>Case I</u>:- If $1 + Q \geqslant \frac{4a'}{U^2}$, equation (11) becomes

$$f(\epsilon) = f_o + \frac{2}{\pi}\int_{\delta^*=-\infty}^{\infty} \tanh^{-1}\exp(\delta^* - \epsilon)\,d\theta(\delta^*) + \log\left(\frac{1 + e^{-\epsilon}}{1 + \lambda e^{-\epsilon}}\right), \qquad \ldots(12)$$

where

$$\lambda = \sqrt{\left(\frac{1 + Q - 4a'/U^2}{1 + Q}\right)}. \qquad \ldots(13)$$

<u>Case II</u>:- If $1 + Q \leqslant \frac{4a'}{U^2}$, equation (11) becomes

$$f(\epsilon) = f_o + \frac{2}{\pi}\int_{\delta^*=-\infty}^{\infty} \tanh^{-1}\exp(\delta^* - \epsilon)\,d\theta(\delta^*) + \tfrac{1}{2}\log\left\{\frac{(1 + e^{-\epsilon})^2}{1 - \lambda^2 e^{-2\epsilon}}\right\}, \qquad \ldots(14)$$

where λ^2 is now a negative number. Now $(1 - \lambda^2 e^{-2\epsilon})$ vanishes at some point on the free streamline $\epsilon = \delta + \frac{1}{2}i\pi$, where, from (14), the direction of the free streamline is discontinuous. This is physically unrealistic, so Case II will not be considered further.

It is convenient to transform the solution (12) into the ζ-plane shown in Fig. 2 and defined by

$$w = a\left\{1 - \tfrac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)\right\}, \quad \zeta = \eta + iy, \ \eta, y \ \text{real} \qquad \ldots(15)$$

It is easily verified that this equation transforms the upper half of the w-plane into the upper half of the unit circle in the ζ-plane. From (8) and (15) equation (12) then becomes

$f(\zeta)$/

$$f(\zeta) = f_0 + \frac{1}{\pi}\int_{\eta^*=-1}^{1} \log\left(\frac{1 - \zeta\eta^*}{\zeta - \eta^*}\right) d\theta(\eta^*) - \log(1 - \sigma\zeta) , \qquad \ldots(16)$$

where

$$\sigma = \frac{\lambda - 1}{\lambda + 1} . \qquad \ldots(17)$$

From (6) and (7), $w = Uz + O(z^{-2})$, so that from (15) the two conditions at infinity are equivalent to

$$\lim_{\zeta=0} f(\zeta) = \lim_{\zeta=0} \frac{1}{\zeta} f(\zeta) = 0 .$$

Equation (16) and these two limits yield

$$f_0 = \frac{1}{\pi}\int_{\eta^*=-1}^{1} \log(-\eta^*)\, d\theta(\eta^*) , \qquad \ldots(18)$$

and

$$\sigma = \frac{1}{\pi}\int_{\eta^*=-1}^{1} \left(\eta^* - \frac{1}{\eta^*}\right) d\theta(\eta^*) . \qquad \ldots(19)$$

At $w = 0$, $\delta = \infty$, and (10) yields

$$r_0 = -\tfrac{1}{2}\log(1 + Q) ;$$

whence (18) can be written in the more useful form

$$Q = \exp\left\{-\frac{2}{\pi}\int_{\eta^*=-1}^{1} \log(\eta^*)\, d\theta(\eta^*)\right\} - 1 . \qquad \ldots(20)$$

Combining (16) and (18) we have

$$\frac{dw}{dz} = U(1 - \sigma\zeta)\exp\left\{-\frac{1}{\pi}\int_{\eta^*=-1}^{1} \log\left(\frac{1 - \zeta\eta^*}{1 - \zeta/\eta^*}\right) d\theta(\eta^*)\right\} , \qquad \ldots(21)$$

or if $F(\zeta)$ is defined by

$$F(\zeta) = \int^{\zeta} \left\{\frac{1 - \zeta^2}{\zeta^2(1 - \sigma\zeta)} \exp\left[\frac{1}{\pi}\int_{\eta^*=-1}^{1} \log\left(\frac{1 - \zeta\eta^*}{1 - \zeta/\eta^*}\right) d\theta(\eta^*)\right]\right\} d\zeta , \qquad \ldots(22)$$

then/

then the required solution is given parametrically by

$$z = \frac{a}{2U} F(\zeta) ,$$

and equation (15). One final point in this general theory is that since the positive square root must be taken in (13) it follows from (17) that

$$\sigma \geqslant -1 . \qquad \ldots(23)$$

3. Special Cases

The following special cases of the above theory are of some interest.

(a) In the case of a cusped cavity $\theta(\eta^*) = 0$ in $-1 \leqslant \eta^* \leqslant k,\ k > 0$.

(b) If in addition $\theta(\eta^*)$ is a step-function, i.e., the obstacle is a polygon, the Stieltjes integrals in (19) and (21) degenerate to

$$\frac{dw}{dz} = U \left\{ 1 - \zeta \sum_{i=0}^{n-1} (1 - \alpha_i)\left(k_i - \frac{1}{k_i}\right)\right\} \prod_{i=0}^{n-1} \left(\frac{1 - \zeta/k_i}{1 - \zeta k_i}\right)^{1-\alpha_i} ,\ k_i > 0 , \qquad \ldots(24)$$

where the angle between the x axis and the first side of the polygon is $\alpha_0\pi$ and the vertex angle of the polygon are $\alpha_i\pi$, $i = 1, 2, \ldots n-1$.

(c) In the case of a flat plate $\alpha_0 = \frac{1}{2}$, $\alpha_i = 0$, so that

$$\frac{dw}{dz} = U \left\{ 1 - \frac{\zeta}{2}\left(k_0 - \frac{1}{k_0}\right)\right\} \left(\frac{1 - \zeta/k_0}{1 - \zeta k_0}\right)^{\frac{1}{2}} \qquad \ldots(25)$$

Equations (24) and (25) are due to Gilbarg[2].

(d) Cavities of constant shape are obtained when $\theta(\eta^*)$ is independent of time, and in this case the theory is exact, since the free boundaries are also streamlines. For the case of a flat plate the constant shape cavities are thus given by the equation $k_0 = \text{constant}$.

(e) Von Karman[1] obtained the particular case of a constant shape cavity behind a flat plate for which $\sigma = -1$, (cf. (23)), i.e., $k_0 = \sqrt{2} - 1$.

(f) The case $\sigma = 0$ is the theory of steady Riabouchinsky flow about curved obstacles.

For the reasons given in the Introduction the cusped cavities described in (a) - (e) are not satisfactory representations of physical cavities.

(g) The simplest possible model of cavitating flow without a cusped cavity is shown in Fig. 3. There are discontinuities of $\frac{1}{2}\pi$ in θ at $\eta^* = k_0$ and $\eta^* = -k_1$, say, where $k_0, k_1 > 0$. Equations (19) - (22) yield

$$\sigma = \frac{(1 + k_o k_1)(k_o - k_1)}{2k_1 k_o} , \qquad \ldots(26)$$

$$Q = \frac{1}{k_o k_1} - 1 ., \qquad \ldots(27)$$

$$\frac{dw}{dz} = U(1 - \sigma\zeta)\left(\frac{1 - \zeta/k_o}{1 - \zeta k_o}\,\frac{1 + \zeta/k_1}{1 + \zeta k_1}\right)^{\frac{1}{2}} , \qquad \ldots(28)$$

$$h = \frac{a}{2U}\{F(1) - F(k_o)\} , \qquad \ldots(29)$$

and

$$l = \frac{a}{2U}\{F(k_o) - F(-k_1)\} , \qquad \ldots(30)$$

where

$$F(\zeta) = \int^{\zeta} \frac{1 - \zeta^2}{\zeta^2(1 - \sigma\zeta)}\left(\frac{1 - \zeta k_o}{1 - \zeta/k_o}\,\frac{1 + \zeta k_1}{1 + \zeta/k_1}\right)^{\frac{1}{2}} d\zeta , \qquad \ldots(31)$$

$2h$ is the width of the front plate, and l is the cavity length. $F(\zeta)$ is expressible in terms of elementary functions.

Suppose U, h, a' and Q are known, then (13), (17), (26), (27) and (29) determine the corresponding values of k_o, k_1 and a. The length of the cavity, which cannot be assigned independently, follows from (30), while (28) completes the description of the flow.

The drag coefficient on the front plate is given by

$$C_D = \frac{1}{h}\int_{\eta=k_o}^{1} (C_p + Q)\, dy(\eta) ,$$

where C_p is the pressure coefficient on the "wetted" surface of the front plate. From (4) and (15)

$$C_D = 1 + Q - \frac{a}{2hU}\int_{k_o}^{1} \frac{q}{U}\left(\frac{1}{\eta^2} - 1\right) d\eta + \frac{a'a}{2hU^3}\int_{k_o}^{1} \frac{U}{q}\,\frac{1 - \eta^4}{\eta^3}\, d\eta , \qquad \ldots(32)$$

where from (28)

$$\frac{q}{U} = (1 - \sigma\eta)\left\{\frac{\eta/k_o - 1}{1 - \eta k_o}\,\frac{1 + \eta/k_1}{1 + \eta k_1}\right\}^{\frac{1}{2}} . \qquad \ldots(33)$$

4./

4. Curved Obstacles

To fix ideas consider the case of a continuously curved obstacle, the cavity behind which is completed by a flat plate as shown in Fig. 4. Where $\theta(\eta^*)$ is continuous, i.e., in $k_0 < \eta^* \leqslant 1$,

$$d\theta(\eta^*) = \frac{d\theta}{ds}\frac{ds}{d\phi}\frac{d\phi}{d\eta^*}d\eta^* ,$$

i.e.,

$$d\theta(\eta^*) = -\frac{a}{2Rq}\frac{1-\eta^{*2}}{\eta^{*2}}d\eta^* , \qquad \ldots(34)$$

where $R = -ds/d\theta$, is the radius of curvature of the surface. From equations (19), (20) and (34)

$$\sigma = \frac{(1+k_0k_1)(k_0-k_1)}{2k_0k_1} + \frac{a}{2\pi U}\int_{k_0}^{1}\frac{U(1-\eta^2)^2}{Rq\,\eta^3}d\eta , \qquad \ldots(35)$$

and

$$Q = \frac{1}{k_0k_1}\exp\left\{\frac{a}{\pi U}\int_{k_0}^{1}\frac{U}{Rq}\frac{1-\eta^2}{\eta^2}\log\eta\, d\eta\right\} - 1 , \qquad \ldots(36)$$

where we have used the same notation for the discontinuities in θ as in section 3(g). On $\zeta = \eta$, $k_0 \leqslant \eta \leqslant 1$, equation (21) becomes

$$q(\eta) = U(1-\sigma\eta)\left(\frac{\eta/k_0 - 1}{1-\eta k_0}\,\frac{1+\eta/k_1}{1+\eta k_1}\right)^{\frac{1}{2}}\exp\left[\frac{a}{2\pi U}\int_{k_0}^{1}\frac{U}{Rq}\frac{1-\eta^{*2}}{\eta^{*2}}\log\frac{1-\eta\eta^*}{1-\eta/\eta^*}d\eta^*\right] . \qquad \ldots(37)$$

This integral equation can be solved by the following iterative process. A distribution $q(\eta)$ is assumed and used in

$$Us(\eta) = \int_{k_0}^{\eta}\frac{U}{q}d\phi(\eta) = \frac{a}{2}\int_{k_0}^{\eta}\frac{1-\eta^2}{\eta^2}\frac{U}{q}d\eta,\ k_0 \leqslant \eta \leqslant 1 , \qquad \ldots(38)$$

to find the perimeter distance, s, from the front stagnation point. This enables the known $R(s)$ relation to be changed to a $R(\eta)$ relation, which together with the assumed $q(\eta)$ can be used under the integral sign in (37) to obtain a new $q(\eta)$ relation, thus completing the first iteration. The process is now repeated until $q(\eta)$ is unchanged by an iteration. At each stage k_0, k_1 and "a" should be modified to satisfy (35), (36) and

$$US = \frac{a}{2}\int_{k_0}^{1}\frac{1-\eta^2}{\eta^2}\frac{U}{q}d\eta ,$$

where/

where 2S is the total wetted perimeter distance. In general the integrations have to be performed numerically. When the solution on $\zeta = \eta$ has been found, the solution for general values of ζ follows from (21) and (34) without further iteration. The question of the convergence of the iterations still remains, but experience with similar integral equations[5] leads the author to expect practical convergence in most cases.

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No.	Author(s)	Title, etc.
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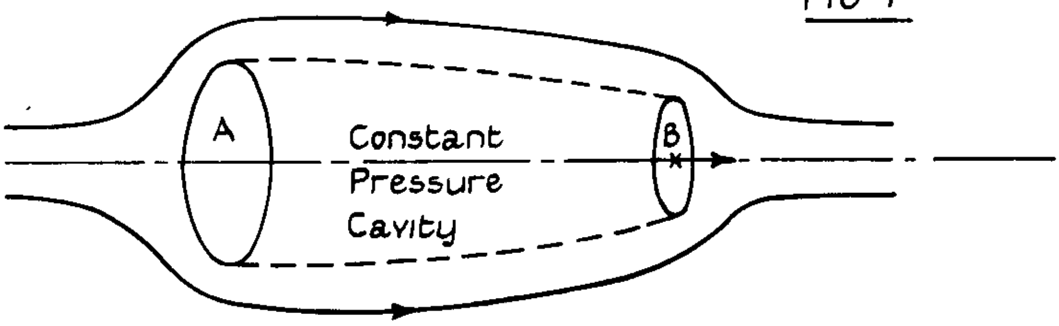
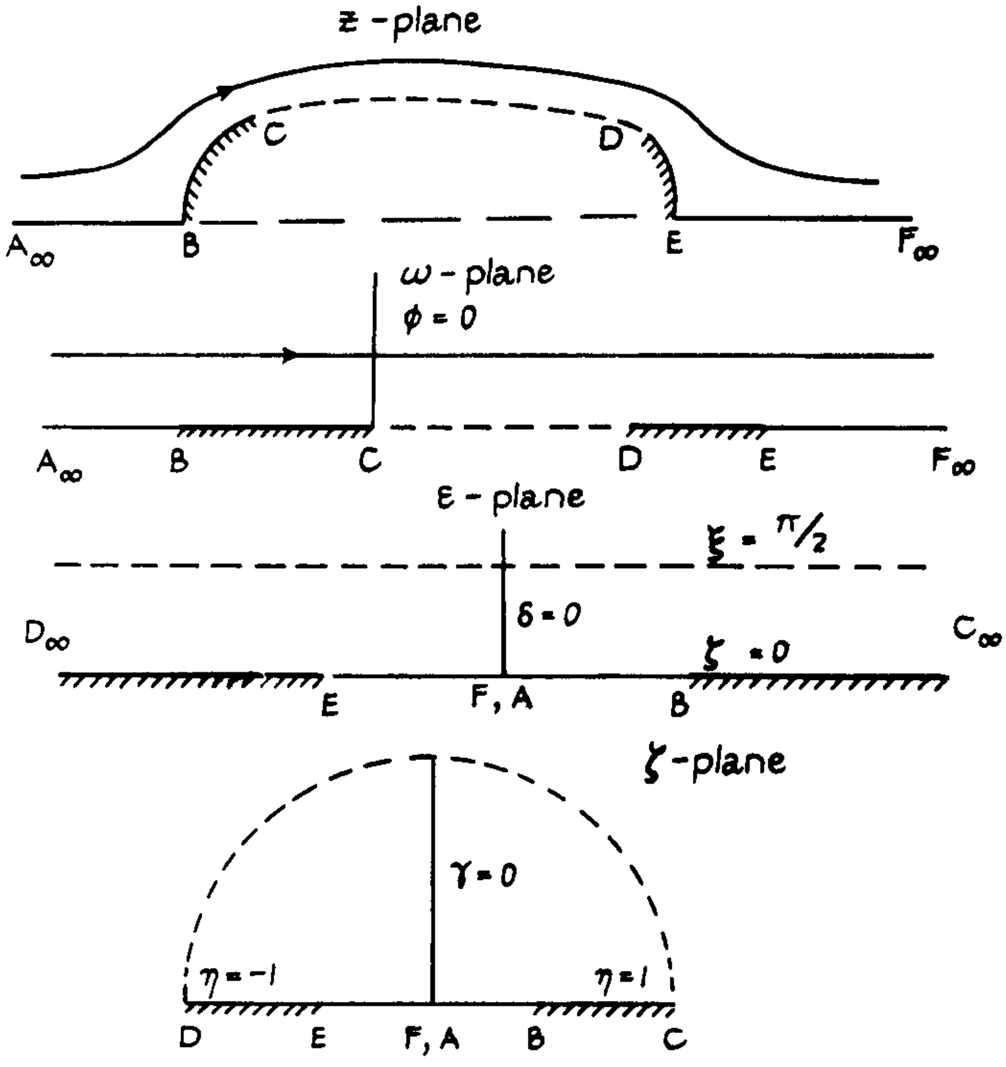
FIGS 1-4.

FIG 1



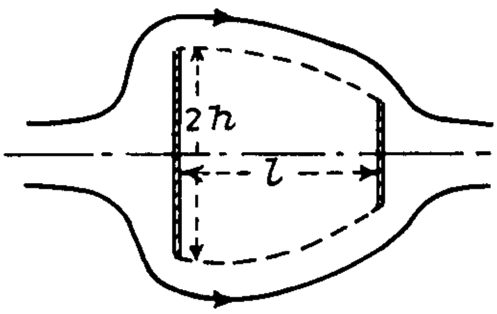
Unsteady Riabouchinsky Flow.

FIG. 2.



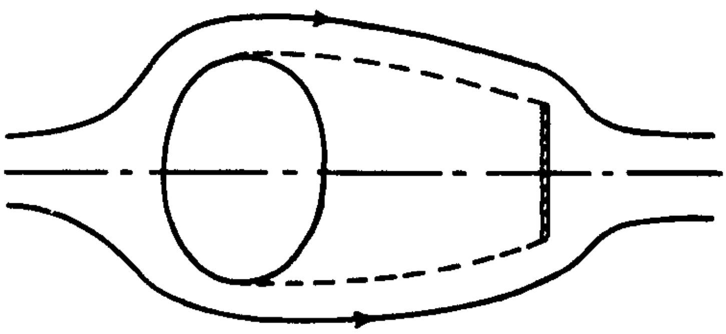
Planes of the Independent Variables

FIG 3



Simplest Case of a Non-cusped Cavity.

FIG 4.



Curved Obstacle.

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1954

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PRINTED IN GREAT BRITAIN

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