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The Vibrations of a Swept Wing

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The Vibrations of a Swept Wing

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SUMMARY

The vibrations of a swept wing with ribs parallel to the direction of flight are considered theoretically. The couplings of torsion and flexure due to the skewness of the ribs and the building-in of the root section are investigated.

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1 Introduction

The modes and frequencies of a swept wing with ribs parallel to the direction of flight are considered. The skewness of the ribs and the building-in of the skew root section each give rise to a coupling of torsion and flexure. The effects of each coupling on the frequencies of vibration are considered. In addition, the introduction of skew ribs into the structural geometry changes the torsional and flexural stiffnesses. The effects of these changes on the frequencies of vibration are considered.

The basic differential equations are deduced using the generalised approximate theory for torsion and bending as developed by Mansfield[1].

Attention is mainly centred upon the fundamental modes and frequencies.

2 Assumptions and structural characteristics

(i) The wing is represented by a 4-boom thin-walled cylindrical box of singly symmetrical rectangular cross-section, reinforced with stringers and ribs.

(ii) The ribs are parallel to the direction of flight.

(iii) The generalised approximate theory for torsion and bending as developed by Mansfield[1] may be used.

(iv) Load-deflection relations are linear.

(v) The shear and inertia centres of a cross-section coincide.

(vi) Shear deflection, and shear lag may be ignored.

3 The differential equations

Mansfield has shown that the deformation of a swept wing may be determined from the relations:

$$\left.\begin{aligned} M_x &= S_x \frac{d\theta}{dx} + S_{xy} \frac{d^2 z}{dx^2} , \\ M_y &= S_{xy} \frac{d\theta}{dx} + S_y \frac{d^2 z}{dx^2} . \end{aligned}\right\} \quad (1)$$

These relations correspond to the approximate theory for torsion and bending, there is the cross-stiffness term S_{xy} because the ribs are not at right angles to the spars. Expressions for S_x, S_y, and S_{xy}, corresponding to a 4-boom, thin-walled, cylindrical box of singly symmetrical rectangular cross-section, are given in Appendix I.

Since it is assumed that the shear and inertia centres coincide, the equilibrium equations of the wing are:

$$\left.\begin{aligned} \frac{dM_x}{dx} - J\ddot{\theta} &= 0 , \\ \frac{d^2M_y}{dx^2} + m\ddot{z} &= 0 . \end{aligned}\right\} \quad (2)$$

In considering a mode of vibration, θ and z vary sinusoidally with time, and therefore the time element may be eliminated from equation (2) to give:

$$\left.\begin{aligned} \frac{dM_x}{dx} + Jp^2\theta &= 0 \, , \\ \frac{d^2M_y}{dx^2} - mp^2 z &= 0 \, . \end{aligned}\right\} \qquad (3)$$

The differential equations for θ and z can be obtained from equations (1) and (3), and can be expressed in non-dimensional form as follows:

$$\left[\left(\frac{d^2}{d\xi^2} + \lambda^2\right)\left(\frac{d^2}{d\xi^2} + \mu^2\right)\left(\frac{d^2}{d\xi^2} - \mu^2\right) - \kappa^2 \frac{d^6}{d\xi^6}\right][\theta \text{ or } z] = 0 \, . \qquad (4)$$

The solutions of equation (4), apart from the time variation, may be written:

$$\theta = A_1 \sin\alpha\xi + A_2 \cos\alpha\xi + A_3 \sin\beta\xi + A_4 \cos\beta\xi + A_5 \sinh\gamma\xi + A_6 \cosh\gamma\xi,$$

$$\left(\frac{S_y}{S_x}\right)^{\frac{1}{2}} \frac{\pi z}{L} = B_1 \sin\alpha\xi + B_2 \cos\alpha\xi + B_3 \sin\beta\xi + B_4 \cos\beta\xi + B_5 \sinh\gamma\xi + B_6 \cosh\gamma\xi,$$

(5)

where α, β, γ, are real, and are such that

$$\pm\, i\alpha, \ \pm\, i\beta, \ \pm\, \gamma,$$

are the roots of:

$$(r^2 + \lambda^2)(r^2 + \mu^2)(r^2 - \mu^2) - \kappa^2 r^6 = 0 \, , \qquad (6)$$

and by virtue of equations (1) and (3),

$$\left.\begin{aligned} \frac{A_1}{B_2} = -\frac{A_2}{B_1} &= \frac{1}{\kappa}\,\alpha\left[1 - \left(\frac{\mu}{\alpha}\right)^4\right], \\ \frac{A_3}{B_4} = -\frac{A_4}{B_3} &= \kappa\beta\left[1 - \left(\frac{\lambda}{\beta}\right)^2\right]^{-1}, \\ \frac{A_5}{B_6} = \frac{A_6}{B_5} &= -\kappa\gamma\left[1 + \left(\frac{\lambda}{\gamma}\right)^2\right]^{-1}. \end{aligned}\right\} \qquad (7)$$

Further relations between the constants A and B in equation (5) are to be determined from the boundary conditions, introduced in para. 4.

4 The modes and frequencies

The modes and frequencies for some special cases will now be considered. In para. 4.1, attention is concentrated upon the influence of the cross-stiffness coupling (κ) due to the skewness of the ribs. In para. 4.2, the effects on the frequencies of vibration of changes in the torsional and flexural stiffnesses due to the introduction of skew ribs into the structure are considered. In para. 4.3, attention is concentrated upon the influence of the coupling due to the building-in of the skew root section.

4.1 The coupling due to the skewness of the ribs

The modes and frequencies for two special cases are now considered, taking into account the skew rib coupling (κ). The case of a wing vibrating in sinusoidal modes is considered first in para. 4.11. In para. 4.12 the vibrations of a wing built-in along the normal section D'OC' (see Fig.1) are considered. The analysis of the modes and frequencies in para. 4.11 is rigorous, the analysis in para. 4.12 (continued in Appendix II) is more complicated and gives only approximate results.

4.11 The wing vibrating in sinusoidal modes

A simple sinusoidal mode of vibration, with a wavelength of 2L, which satisfies equation (5), is given by:

$$\left.\begin{aligned} \theta &= A \sin \xi , \\ z &= B \cos \xi . \end{aligned}\right\} \qquad (8)$$

The frequency equation, obtained by substituting equation (8) in equation (4) is:

$$(1 - \lambda^2)(1 - \mu^2)(1 + \mu^2) - \kappa^2 = 0 , \qquad (9)$$

and it is possible to obtain the solutions of equation (9) in the form:

$$\left.\begin{aligned} \lambda &= \left\{\frac{\nu^2 + 1}{2} - \frac{(\nu^2 - 1)}{2} \sqrt{1 + \left(\frac{2\nu\kappa}{\nu^2 - 1}\right)^2}\right\}^{\frac{1}{2}} , \\ \mu^2 &= \frac{1}{\nu} \left\{\frac{\nu^2 + 1}{2} + \frac{(\nu^2 - 1)}{2} \sqrt{1 + \left(\frac{2\nu\kappa}{\nu^2 - 1}\right)^2}\right\}^{\frac{1}{2}} . \end{aligned}\right\} \qquad (10)$$

Since κ is small, λ and μ are approximately equal to unity for all values of ν (they are exactly equal to unity if κ is zero) and therefore the corresponding modes of vibration may, in general, be associated with a predominantly torsional and a predominantly flexural mode respectively.

The fractional changes in frequency due to the introduction of the skew rib coupling, for these predominantly torsional and predominantly flexural modes, are obtained from equation (10) and have been plotted against ν in Fig.3 for various values of κ. The greatest and least values of these fractional changes occur when $\nu = 1$, and are given by:

$$\left.\begin{aligned}(\lambda - 1)_{\nu = 1} &= (1 \pm \kappa)^{\frac{1}{2}} - 1 , \\ (\mu^2 - 1)_{\nu = 1} &= (1 \mp \kappa)^{\frac{1}{2}} - 1 .\end{aligned}\right\} \qquad (11)$$

A useful and non-dimensional measure of the proportion of flexure to torsion in a mode is afforded by the ratio $\frac{dz}{dx}/\theta$. Using equations (7), (8), and (10) it may be shown that, for the predominantly torsional mode,

$$\frac{dz}{dx}/\theta = - \frac{S_{xy}}{S_y} \left[\frac{\dfrac{2\nu^2}{(\nu^2 - 1)}}{1 + \sqrt{1 + \left(\dfrac{2\nu\kappa}{\nu^2 - 1}\right)^2}} \right] , \qquad (12)$$

and for the predominantly flexural mode,

$$\theta/\frac{dz}{dx} = \frac{S_{xy}}{S_x} \left[\frac{\dfrac{2}{(\nu^2 - 1)}}{1 + \sqrt{1 + \left(\dfrac{2\nu\kappa}{\nu^2 - 1}\right)^2}} \right] . \qquad (13)$$

These ratios have been plotted against ν for various values of κ in Fig.2.

The jump discontinuities at $\nu = 1$ shown in Figs. 2 and 3 occur because as the value of ν passes through the region surrounding $\nu = 1$ the predominantly torsional mode, due to its increasing flexural component, gradually changes into a predominantly flexural mode and, conversely, the predominantly flexural mode, due to its increasing torsional component, gradually changes into a predominantly torsional mode. At $\nu = 1$ the modes cannot be recognised as being either predominantly torsional or predominantly flexural.

4.12 General modes of vibration

The modes and frequencies of a wing built-in along the normal section D'OC' (see Fig.1) are now considered.

The boundary conditions at the root $(\xi = 0)$ and at the free end $(\xi = \pi)$ are:

$$\left.\begin{array}{llll} \text{(i)} & (\theta)_{\xi=0} = 0 , & \text{(iv)} & (M_x)_{\xi=\pi} = 0 , \\ \text{(ii)} & (z)_{\xi=0} = 0 , & \text{(v)} & (M_y)_{\xi=\pi} = 0 , \\ \text{(iii)} & \left(\frac{dz}{dx}\right)_{\xi=0} = 0 , & \text{(vi)} & \left(\frac{dM_y}{dx}\right)_{\xi=\pi} = 0 . \end{array}\right\} \qquad (14)$$

The modes and frequencies may be determined by combining these conditions with the general expressions for θ and z given in equation (5). The analysis is, however, complicated, an outline of it is given in Appendix II. Results which are obtained indicate that the effect of the skew rib coupling on the fundamental frequencies of vibration is negligibly small.

4.2 The effects on the frequencies of changes in the torsional and flexural stiffnesses due to the introduction of skew ribs

The stiffening of a swept wing with ribs parallel to the direction of flight introduces a degree of skewness into the structure, which, besides coupling the torsional and flexural vibrations, also changes the torsional and flexural stiffnesses. Owing to these changes in stiffness, the fractional change in frequency associated with the torsional modes is:

$$\left(\frac{S_x}{S_{x,o}}\right)^{\frac{1}{2}} - 1 = \left\{\frac{h\left(\frac{1}{W_1} + \frac{1}{W_2}\right) + 2b}{h\left(\frac{1}{W_1} + \frac{1}{W_2}\right) + \frac{2b(2\delta Q - \varepsilon P)}{\varepsilon R}}\right\}^{\frac{1}{2}} - 1 , \qquad (15)$$

and the fractional change in frequency associated with the flexural modes is:

$$\left(\frac{S_y}{S_{y,o}}\right)^{\frac{1}{2}} - 1 = \left\{\frac{I_1 + I_2 + 4\,bth^2 \dfrac{R\left\{P - \frac{hR}{2b}\left(\frac{1}{W_1} + \frac{1}{W_2}\right)\right\}}{\varepsilon\left\{P - \frac{hR}{2b}\left(\frac{1}{W_1} + \frac{1}{W_2}\right)\right\} - 2\delta Q}}{I_1 + I_2 + 4\,bth^2\left\{\dfrac{1 + X + Y + XY(1 - \sigma^2)}{1 + Y(1 - \sigma^2)}\right\}}\right\}^{\frac{1}{2}} - 1. \qquad (16)$$

These fractional changes have been plotted against the sweepback angle (η) in Fig.4. Details of the particular wing cross-section which is considered are given in Appendix IV. Fig.4 shows that, when $\eta \approx 45^\circ$, there is a maximum fractional increase in frequency for the torsional modes of 0.06; the fractional change in frequency for the flexural modes is small for $\eta < 50^\circ$.

4.3 The coupling due to the building-in of the skew root section

Attention is now concentrated upon the coupling due to the building-in of the skew root section DOC (see Fig.1), the skew rib coupling is ignored. The skew root effect is considered by comparing the modes and frequencies of the swept wing, with the modes and frequencies of a similar wing built-in along the normal section D'OC'. Root boundary conditions are formulated by which the positions C and D are fixed; the two ideal conditions that the slopes $\left(\frac{dz}{dx}\right)$ at C and D are each zero, are replaced by the single condition that the algebraic sum of those slopes is zero.

The root conditions are now,

$$\left.\begin{aligned} &\text{(i)} && (z + b\theta)_{\xi = \zeta} = 0 , \\ &\text{(ii)} && (z - b\theta)_{\xi = -\zeta} = 0 , \\ &\text{(iii)} && \left(\frac{dz}{d\xi}\right)_{\xi = \zeta} + \left(\frac{dz}{d\xi}\right)_{\xi = -\zeta} = 0 , \end{aligned}\right\} \tag{17}$$

and the conditions at the free end of the wing $(\xi = \pi)$ are the same as those given in equation (14).

Since the rib coupling is ignored, the constants A_3, A_4, A_5, A_6, B_1, B_2, are zero, so that a mode of vibration is given by:

$$\left.\begin{aligned} \theta &= A_1 \sin \lambda\xi + A_2 \cos \lambda\xi, \\ \left(\frac{S_y}{S_x}\right)^{\frac{1}{2}} \frac{\pi z}{L} &= B_3 \sin \mu\xi + B_4 \cos \mu\xi + B_5 \sinh \mu\xi + B_6 \cosh \mu\xi. \end{aligned}\right\} \tag{18}$$

The constants A and B may now be eliminated to give the frequency equation:

$$\left.\begin{aligned} &\cos \lambda\pi \, (1 + \cos \mu\pi \cosh \mu\pi) + \frac{\cos \lambda\pi \, (\cos \mu\zeta - \cosh \mu\zeta)^2}{2 \cos \mu\zeta \cosh \mu\zeta} \\ &+ \tfrac{1}{2} \sin \lambda\pi \tan \lambda\zeta \, (\tanh \mu\zeta - \tan \mu\zeta)(\sin \mu\pi \cosh \mu\pi + \cos \mu\pi \sinh \mu\pi) = 0. \end{aligned}\right\} \tag{19}$$

Using equation (19), the fractional changes in frequency due to the introduction of the skew root coupling may be determined. In Fig.5 the fractional changes in frequency for the fundamental predominantly torsional

FIG.1 (a, b & c).

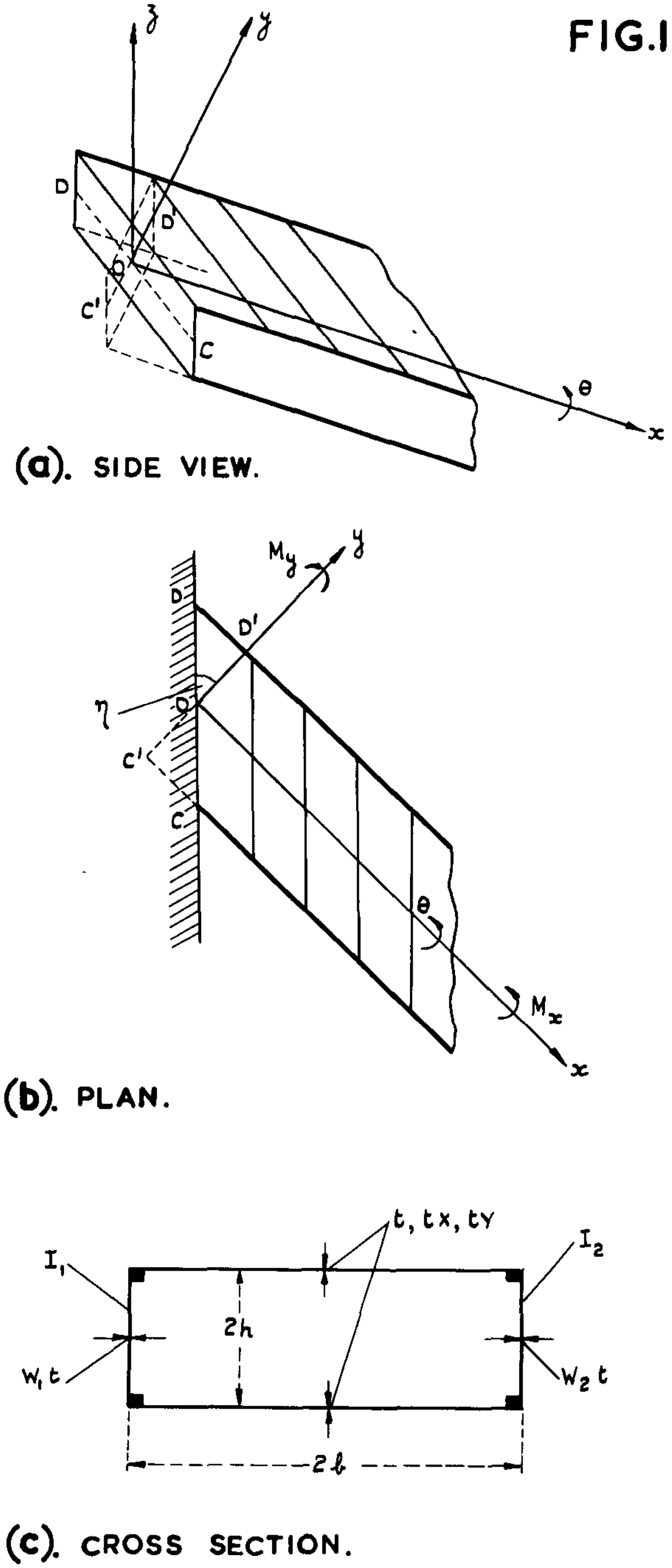
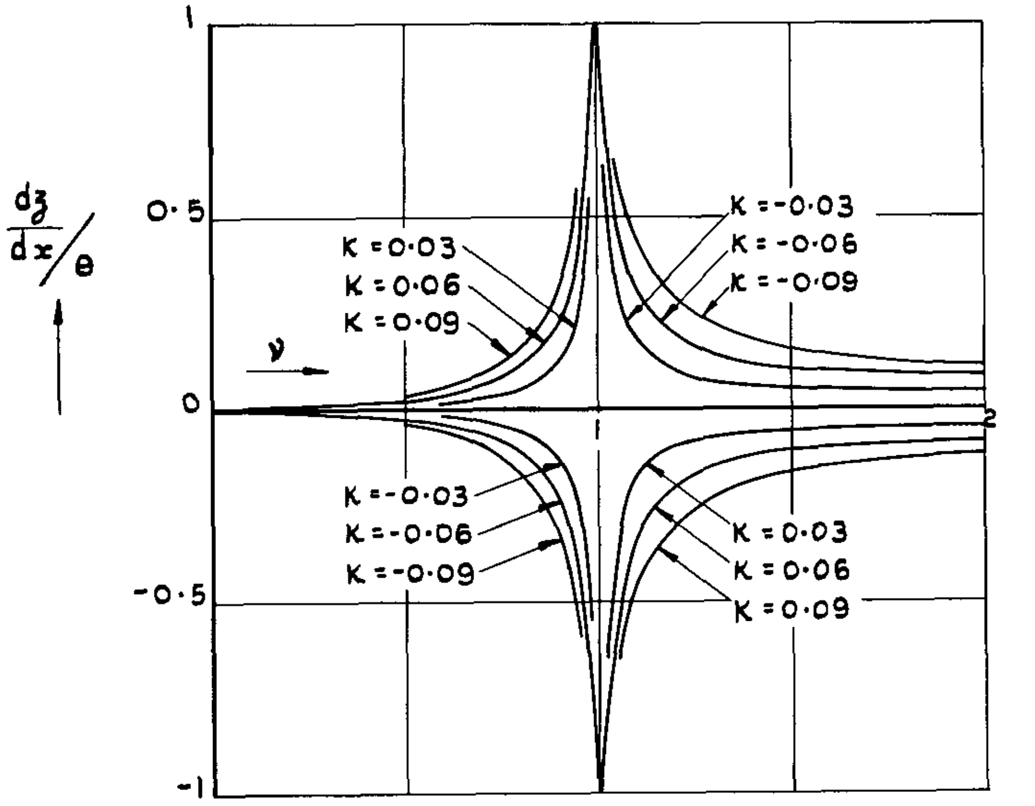
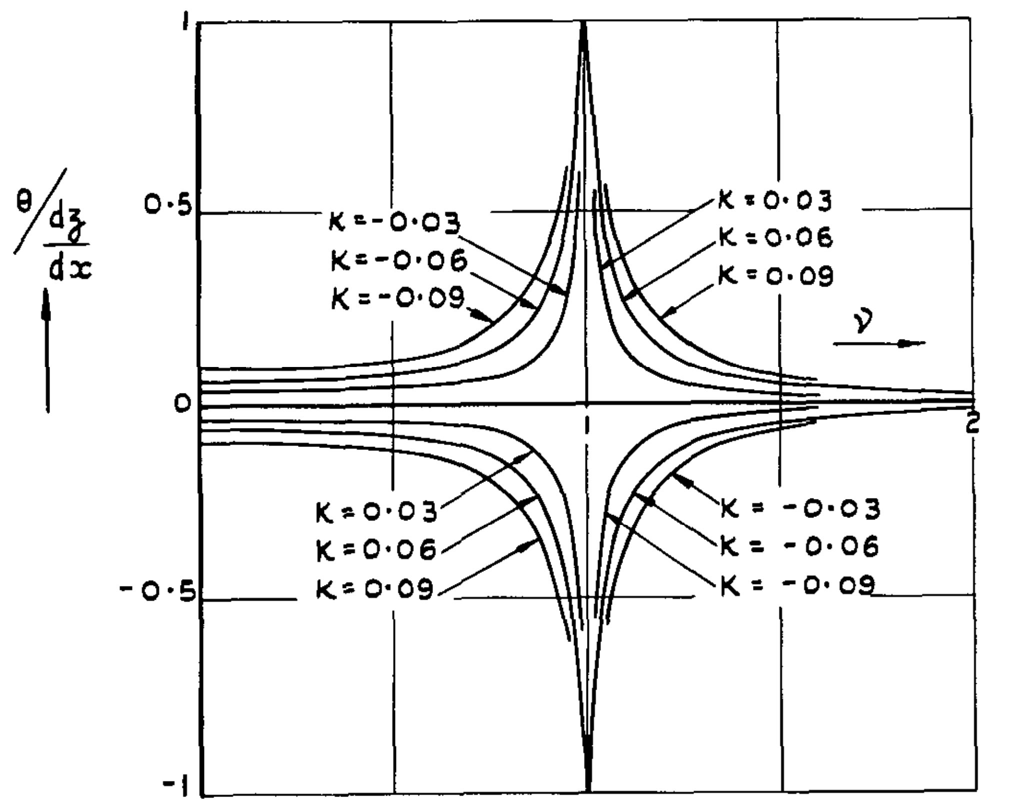


FIG.1 (a,b & c). THE WING, SHOWING NOTATION.

FIG 2 (a & b).



(a). PREDOMINANTLY TORSIONAL MODE.



(b). PREDOMINANTLY FLEXURAL MODE.

FIG. 2 (a & b). THE WING VIBRATING IN SINUSOIDAL MODES $(S_x = S_y)$.

FIG.3(a & b).

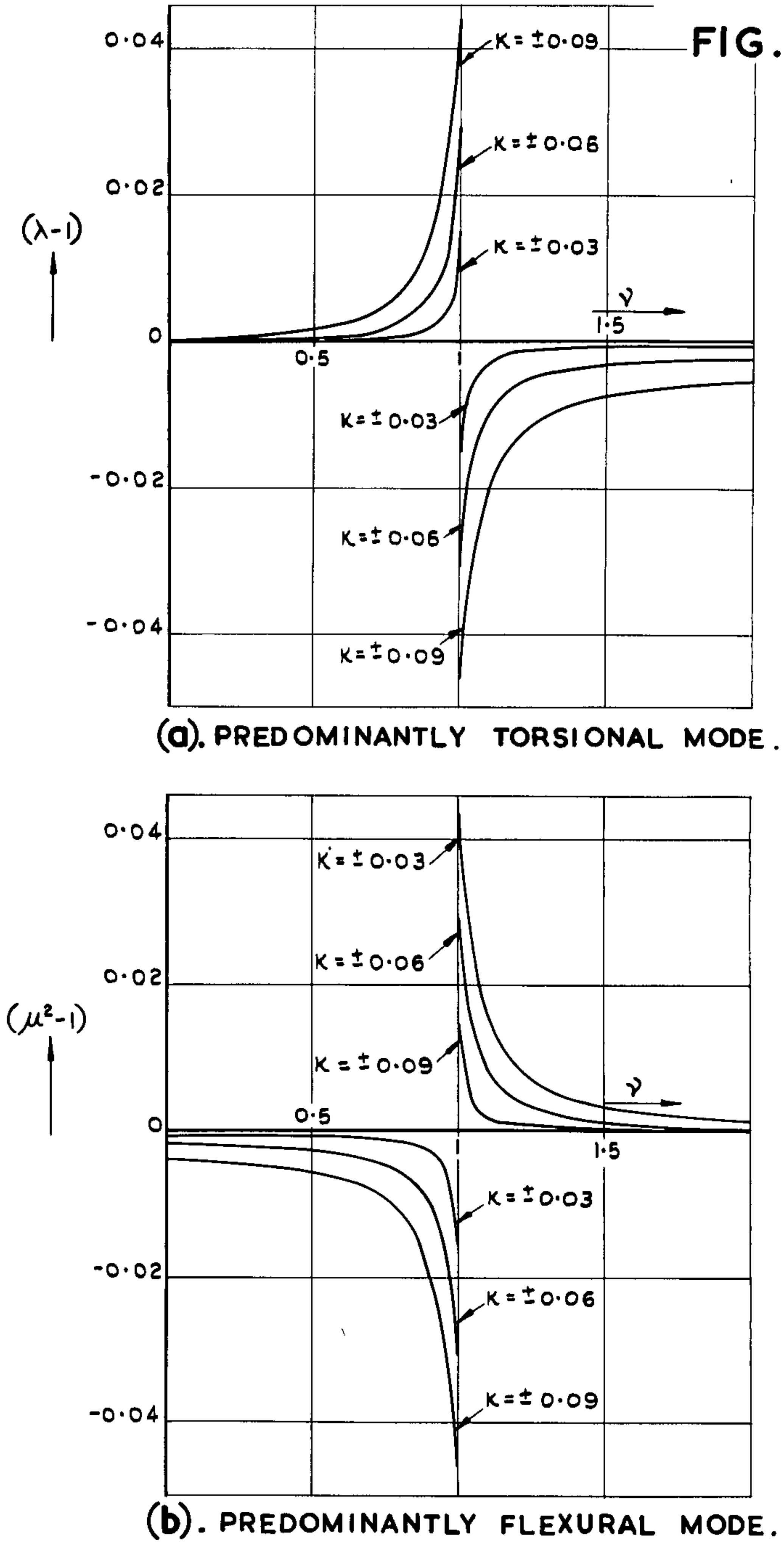
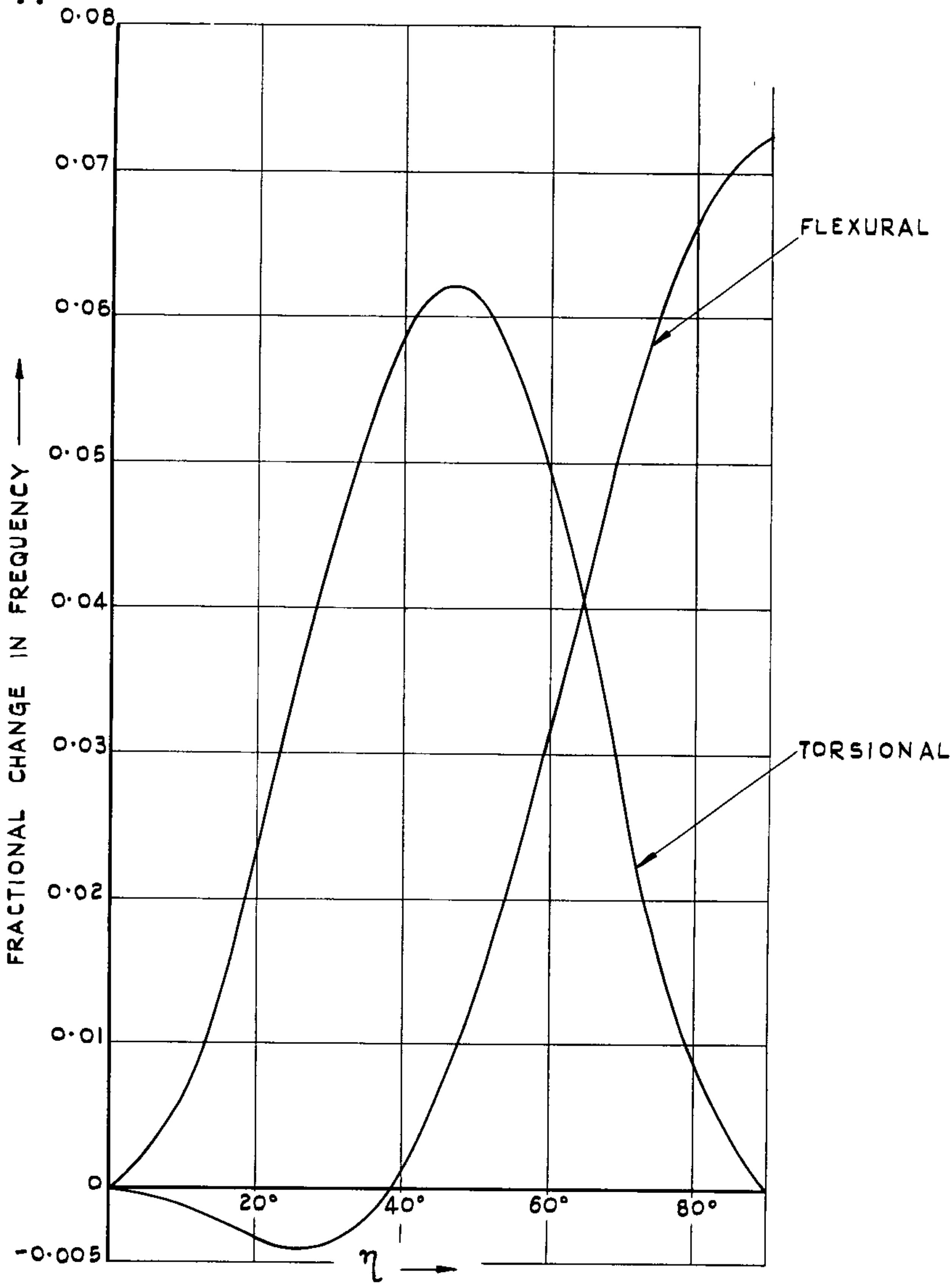


FIG. 3 (a & b). FRACTIONAL CHANGES IN FREQUENCY DUE TO THE SKEW RIB COUPLING – THE WING VIBRATING IN SINUSOIDAL MODES.

FIG.4.



X = 0·3
Y = 0·25
σ = 0·25
$I_1 + I_2 = 86{\cdot}4$ (IN.)^4
l = 22 IN.
h = 6 IN.
t = 0·1 IN.
$W_1 = W_2 = 2$.

FIG. 4. FRACTIONAL CHANGES IN FREQUENCY DUE TO STIFFNESS CHANGES ASSOCIATED WITH THE INTRODUCTION OF SKEW RIBS.

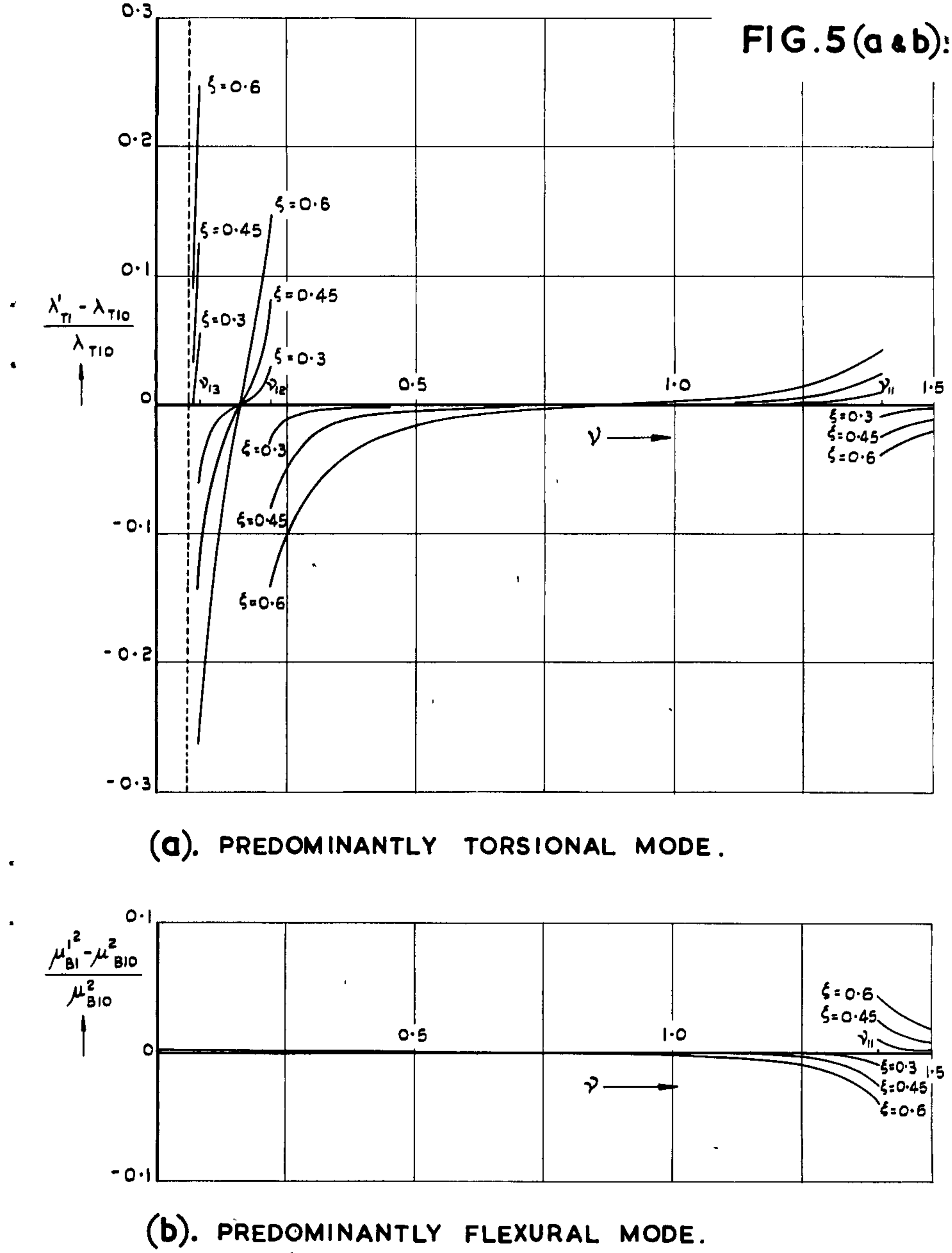


FIG. 5 (a & b). FRACTIONAL CHANGES IN FREQUENCY DUE TO THE SKEW ROOT COUPLING FOR THE FUNDAMENTAL MODES OF VIBRATION.

FIG.6.

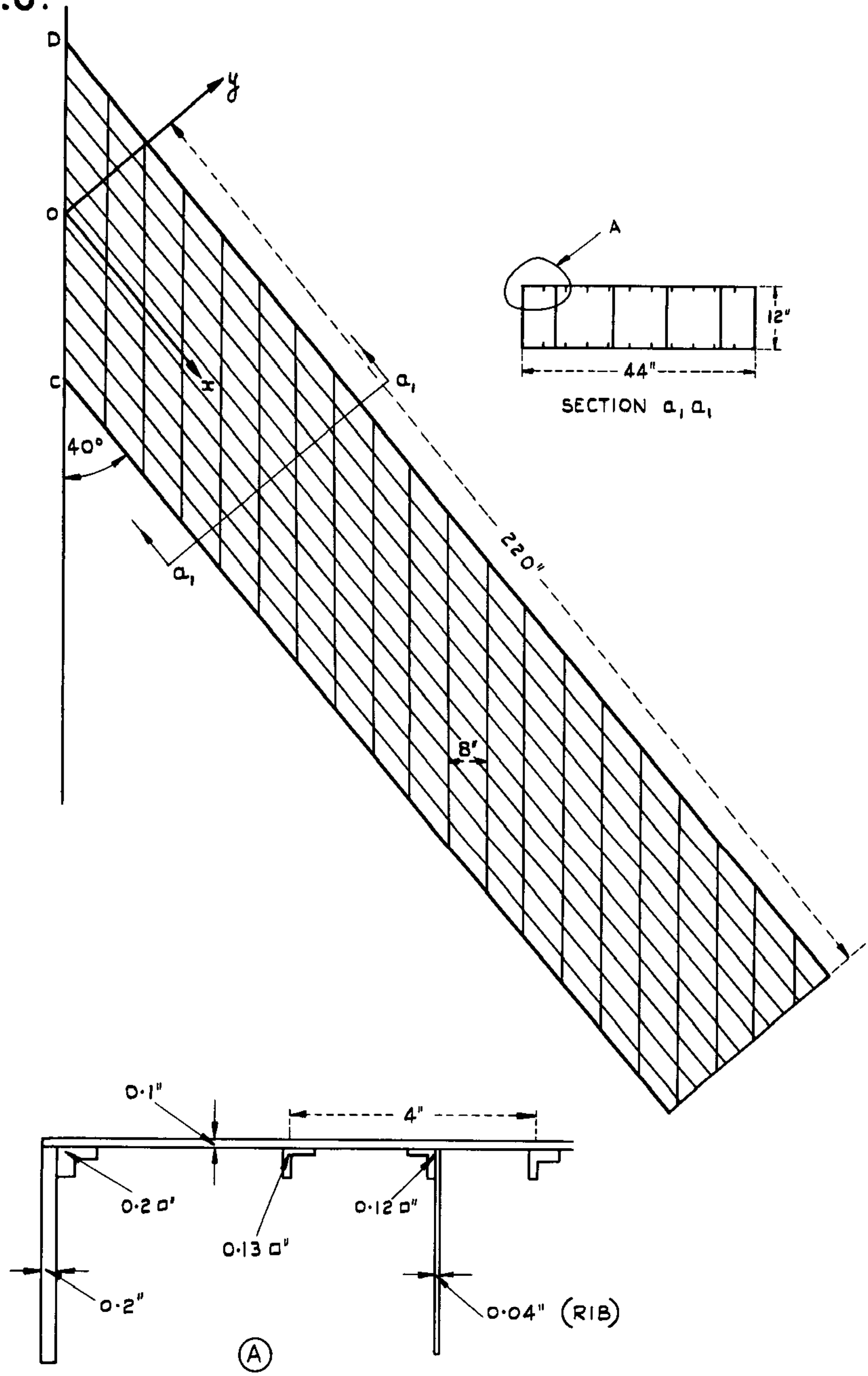


FIG. 6. DETAILS OF THE SWEPT WING USED AS EXAMPLE.