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A Relaxation Treatment of Shock Waves

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A Relaxation Treatment of Shock Waves

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Presented by Professor A. Thom

8th July, 1950.

SUMMARY.

Emmons[1] has given a relaxation method of dealing with shock waves when the compressible stream function is the dependent variable. This paper briefly outlines a procedure to adopt when $\log \frac{1}{q}$ (q = velocity magnitude) is taken as the dependent variable.* A method of allowing for the presence of vorticity behind the shock wave is also given.

Introduction

Nomenclature:

(x,y)	Physical plane, in which $z = x + iy$.
(ϕ,ψ)	The transformed incompressible flow plane in which the aerofoil is represented by a slit on $\psi = 0$.
w	$= \phi + i\psi$.
(q,θ)	Compressible flow velocity vector in polar coordinates.
L	$= \log(1/q)$
α	Angle between the compressible and incompressible velocity vectors.
n	Interval of the square mesh.
X	Residual of the relaxation process.
M	Local Mach number.

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* Thom adopted the dependent variable $\log \frac{1}{q}$ because of its special suitability for shock wave calculations.

M_0 Undisturbed stream Mach number.

a Local velocity of sound.

a_0 Stagnation point sound velocity.

It has been shown in reference 2 that ignoring α which is usually small, L satisfies the equation

$$\frac{\partial^2 L}{\partial \phi^2} + \frac{\partial^2 L}{\partial \psi^2} = \frac{\partial}{\partial \phi}\left(M^2 \frac{\partial L}{\partial \phi}\right) \qquad (1)$$

Figure 1 shows a typical square of the mesh in the (ϕ, ψ) plane. The value of L at the point "3" say, will be denoted by "L_3", etc.

Points 6 and 7 bisect the intervals 35 and 51 respectively. Then a suitable difference equation representing (1) is[2]

$$X_5 = (1-M_5^2)L_1 + L_2 + (1-M_5^2)L_3 + L_4 - 2(2-M_5^2)L_5 + \tfrac{1}{2}M_5^2\left(\frac{a_0}{a}\right)^2 (L_1 - L_3)^2, \qquad (2)$$

in which the last term is usually negligible. While (2) is the most suitable form from which to calculate the residuals, it does not lead to a very suitable relaxation pattern. It is easily shown[2] that an appropriate relaxation pattern is that given in Figure 2.

Emmons[1] found that at times he was unable to eliminate by relaxation all the residuals in a supersonic patch in the field. The author has experienced the same difficulty. Quoting from reference 3:-

"Relaxation in the supersonic patches is still possible, but somewhat less convergent than in the elliptical region of the differential equation. An essential requirement of relaxation is that the elimination of a residual at one mesh point should not involve the appearance of larger residuals at neighbouring mesh points (not including residuals already at these points). Examination of the relaxation pattern of Figure 2 for $M > 1$, reveals that for this requirement to be fulfilled it may be necessary to eliminate a residual at one point by altering L at a neighbouring mesh point. This procedure works for a time but it has been discovered that when M reaches a certain value it becomes impossible to find a continuous solution for L., i.e. it is not possible to eliminate all the residuals. It is however possible to arrange the unrelaxed residuals in pairs of opposite sign along the lines in the field, and to deduce from these the existence of a discontinuity in L lying between them. The magnitude and position of this shock wave can also be deduced from the size of the residuals."

The unrelaxed residuals are arranged in pairs of opposite sign along two equipotential lines of the mesh, starting from the aerofoil boundary and finishing a short distance out in the field. There is no unique arrangement, but it is shown below that having decided upon the location of the foot of the shock, we can determine the position and shape of the rest of the shock wave. Actually, in the relaxation, the difficulty occurs when and where the gradient become very large, and so it seems natural to select the position of the foot of the shock to coincide with the point on the boundary at which difficulty with the residuals first occurs. It may be true that the shock assumes the

position/

position in which the average strength is a minimum, but in any case it is known[5] that the boundary layer plays an important role in the actual location of the shock. Of course these remarks only apply to the shocks springing from the curved surface of the aerofoil, typical of transonic flow. The method given in this paper will apply to the highly oblique shock of supersonic flow but better methods involving the use of the theory of characteristics exist. However the method of characteristics cannot be applied to transonic or "mixed" flow.

Another important point is that relaxation will not work if the mesh is too coarse in a region in which the higher derivatives of the dependent variable are large. This is certainly true in the region where a shock is about to appear, and so there is no certainty whether the inability of the computor to eliminate the residuals is due to the presence of a weak shock, or due to having too coarse a mesh in the region. This is only important from a theoretical point of view since infinite rates of change of L never occur in practice. It is sufficient experimentally to define a shock wave as existing when $\frac{\partial L}{\partial s}$ exceeds a certain value. The mesh size must be selected so that relaxation is still just possible when this gradient is achieved.

1. The Shock Wave Equations.

These equations can be found in references 5 or 6. In Figure 3 we have a shock wave CD at an angle β to the velocity vector q_b.

All quantities on the upstream side of the shock will be denoted by a suffix "b", and those on the downstream side by a suffix "a". q_b is resolved into N_b normal to the shock, and T_b tangential to the shock, which deflects the streamlines through an angle η.

The conditions at the shock wave are

$$T_a = T_b, \quad \text{i.e.} \quad q_b \cos\beta = q_a \cos(\beta - \eta), \tag{3}$$

and

$$N_a N_b = a_*^2 - \mu^2 T^2 , \tag{4}$$

where $\mu^2 = \frac{\gamma - 1}{\gamma + 1}$, and a_* is the "critical speed".

Now $a_* = q_*$, where q_* is the "limit speed", which occurs in Bernoulli's law

$$\tfrac{1}{2}q_*^2 = \tfrac{1}{2}q^2 + \frac{a^2}{\gamma - 1} . \tag{5}$$

From these equations it follows that

$$a_*^2 = \mu q^2\left(1 + \frac{2}{(\gamma - 1)M^2}\right)$$

Inserting/

Inserting the free stream values, $q = 1$, $M = M_0$, into the right hand side of this equation, we have

$$a_*^2 = \mu\left(1 + \frac{2}{(\gamma - 1)M_0^2}\right), \quad \text{or writing } \gamma = 1.4 ,$$

$$a_*^2 = \frac{1}{6}\left(1 + \frac{1}{5M_0^2}\right). \tag{6}$$

Now $N_b = q_b \sin\beta$, $N_a = q_a \sin(\beta - \eta)$, and so from (3) and (4) it follows that

$$q_a q_b = \frac{(\gamma + 1)\, a_*^2}{\gamma \cos\eta - \cos(2\beta - \eta)} \tag{7}$$

For a normal shock wave $\beta = \pi/2$, $\eta = 0$, and so

$$q_a q_b = a_*^2 . \tag{8}$$

In this paper we also make use of the equation

$$q\zeta = T_* \frac{\partial S}{\partial n} - \frac{\partial(\frac{1}{2}q_*^2)}{\partial n}, \quad \text{(reference 5, § 14)} \tag{9}$$

relating the vorticity ζ, velocity q and temperature T_* to the rate of change of entropy S, and the Bernoulli constant $\frac{1}{2}q_*^2$, at right angles to the direction of flow. "$\frac{1}{2}q_*^2$" remains constant through the shock, and constant on any given streamline behind the shock, and therefore, since it is constant in the region of isentropic flow in front of the shock, it must be constant throughout the isentropic region behind the shock. The entropy, however, is constant on each streamline, except at the shock when it increases an amount ΔS proportional to the third power of the shock strength[6]. Since the entropy is constant in front of the shock,

$$\frac{\partial S}{\partial n} = \frac{\partial(\Delta S)}{\partial n},$$

and using the gas equation, i.e. $p = R\rho T_*$, and the equation $a^2 = \gamma p/\rho$, (9) can be written

$$\zeta q = \frac{p}{R\rho}\frac{\partial(\Delta S)}{\partial n} = \frac{a^2}{\gamma R}\frac{\partial(\Delta S)}{\partial n} \tag{10}$$

One further result required is an expression for ΔS. For the relatively weak shocks of transonic flow, this can be written (reference 6, §4.1),

$$\Delta S \doteqdot \frac{2\gamma R}{(\gamma + 1)^2}\,\frac{(M-1)^3}{3} . \tag{11}$$

2/

2. Calculation of the Shock Waves from Residuals.

Taking logarithms we deduce from (7) that

$$L_b + L_a = Y , \tag{12}$$

where Y is a constant for a normal shock, and a function of β and η for an oblique shock. The value of Y for a normal shock follows from equation (4). Figure 4 shows a shock wave crossing a section of the mesh, which is assumed to be sufficiently refined (see comments in the introduction). The value of L at "1b" is that which would occur at point "1" if the "b" region were extended continuously beyond the shock wave. "0a" is the result of a similar extension of the "a" region. We shall assume that the residuals are zero throughout the two regions except at mesh points neighbouring the shock wave, i.e. the process of collecting residuals in pairs mentioned in the introduction has been carried out. From the values of the residuals X_0 and X_1 at points

0 and 1 , it is required to fix the position and strength of the shock wave lying between points 0 and 1 . The shock strength D , say, can be measured by

$$L_a - L_b = D. \tag{13}$$

Ignoring the last term of equation (2) , which is usually negligible, we have

$$\begin{cases} X_0 = L_2 + L_4 + (1-M_0^2)L_3 + (1-M_0^2)L_1 - (4-2M_0^2)L_0 , \\ X_1 = L_6 + L_7 + (1-M_1^2)L_5 + (1-M_1^2)L_0 - (4-2M_1^2)L_1 , \quad \text{and} \\ 0 = L_2 + L_4 + (1-M_0^2)L_3 + (1-M_0^2)L_{1b} - (4-2M_0^2)L_0 , \\ 0 = L_6 + L_7 + (1-M_1^2)L_5 + (1-M_1^2)L_{0a} - (4-2M_1^2)L_1 , \end{cases}$$

i.e. $L_{1b} = L_1 - \dfrac{X_0}{1-M_0^2} = L_1 - X_0'$, say, where $X_0' = \dfrac{X_0}{1-M_0^2}$,

which is permissible since $M_0 \neq 1$, otherwise no shock would occur. Similarly

$$L_{0a} = L_0 = X_1' .$$

Now $L_b \doteq L_0 + \varepsilon (L_{1b} - L_0) = L_0 + \varepsilon (L_1 - L_0 - X_0')$,

and $L_a \doteq L_1 + (1-\varepsilon)(L_{0a} - L_1) = L_1 + (1-\varepsilon)(L_0 - L_1 - X_1')$,

and so from (12) and (13)

$$Y = 2L_0 + 2\varepsilon (L_1 - L_0) + \varepsilon (X_1 - X_0) - X_1 , \tag{14}$$

$$D = \varepsilon (X_1' + X_0') - X_1' . \tag{15}$$

From (14) we have
$$\varepsilon = \frac{Y + Y_1' - 2L_0}{2L_1 + X_1' - (2L_0 + X_0')} \tag{16}$$

Now/

Now α_0, and a_1 can be determined by an integration carried out along opposite sides of the shock wave (see equation 8, reference 3), starting as close as possible to the shock on the surface; extrapolation will yield α_a and α_b, and hence $\eta = \alpha_a - \alpha_b$, can be found.

From equations (3) and (13) we can deduce that

$$\beta = \cot^{-1}\left(\frac{\sin\eta}{e^{-D} - \cos\eta}\right) \tag{17}$$

The values of ε and β deduced from (16) and (17) will probably be inconsistent, for knowing the position of the foot of the shock and either ε or β enables us to completely define the position of the rest of the shock. It follows that, if the foot of the shock is fixed, there is only one arrangement of the residuals along the parallel equipotential lines that will define ε and β consistently. The author has not investigated this further, but believes that it would not be difficult by trial and error to arrange the L field in the neighbourhood of the shock so that (16) and (17) lead to consistent results. All this can be avoided, of course, by assuming that a normal shock wave sufficiently represents the actual situation.

3. The Entropy Gradient Behind the Shock Wave.

The effect of the entropy gradient can be ignored in a first approximation (reference 5, §74), but it may be of interest to investigate the minor effects of allowing for this gradient on a solution calculated by the methods of the previous section.

We shall assume that α can be neglected, i.e. that

$\dfrac{\partial}{\partial n} = q_0 \dfrac{\partial}{\partial\psi}$, where q_0 is the incompressible velocity, and so

equations (10) and (11) yield

$$\begin{aligned}\frac{\partial}{\partial\psi}\left(\frac{\zeta}{qq_0}\right) &= \frac{1}{\gamma R}\frac{\partial}{\partial\psi}\left(\frac{1}{M^2}\frac{\partial S}{\partial\psi}\right)\\ &= \frac{2}{(\gamma+1)^2}\frac{\partial}{\partial\psi}\left\{\frac{(M^2-1)^2}{M^2}\frac{\partial M^2}{\partial\psi}\right\}\\ &= \frac{-4}{(\gamma+1)^2}\frac{\partial}{\partial\psi}\left\{(M^2-1)^2\left(\frac{a_0}{a}\right)^2\frac{\partial L}{\partial\psi}\right\},\end{aligned}$$

or putting $\gamma = 1.4$, we have finally

$$\frac{\partial}{\partial\psi}\left(\frac{\zeta}{qq_0}\right) \doteq -0.70\frac{\partial}{\partial\psi}\left\{(M^2-1)^2\left(\frac{a_0}{a}\right)^2\frac{\partial L}{\partial\psi}\right\} \tag{18}$$

Now/

Now in the Appendix of reference 2 it is shown that the presence of vorticity would add a term

$$\frac{\partial}{\partial\psi}\left(\frac{\zeta}{qq_0}\right)$$ to the right hand side of equation (1).

Equation (18) shows how this could be computed in the L field. It would make a small contribution to the residual which could be relaxed by the usual pattern since the effect is very small.

4. Conclusions.

Until at least a normal shock wave has been treated by the method of this paper no definite conclusions can be drawn. The author experienced the difficulty mentioned in the introduction with the cylinder problem in reference 4 at a Mach number of 0.5, but the calculations were carried out only far enough to verify the practicability of the method of this paper.

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FIG 1

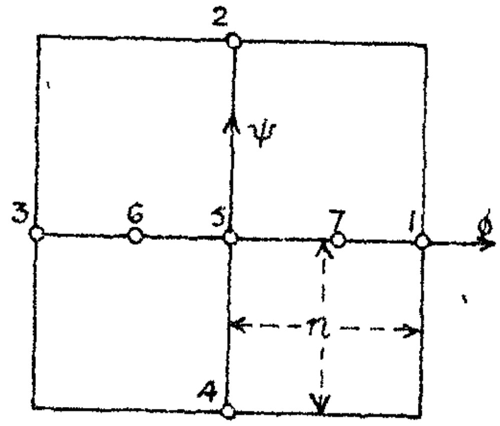
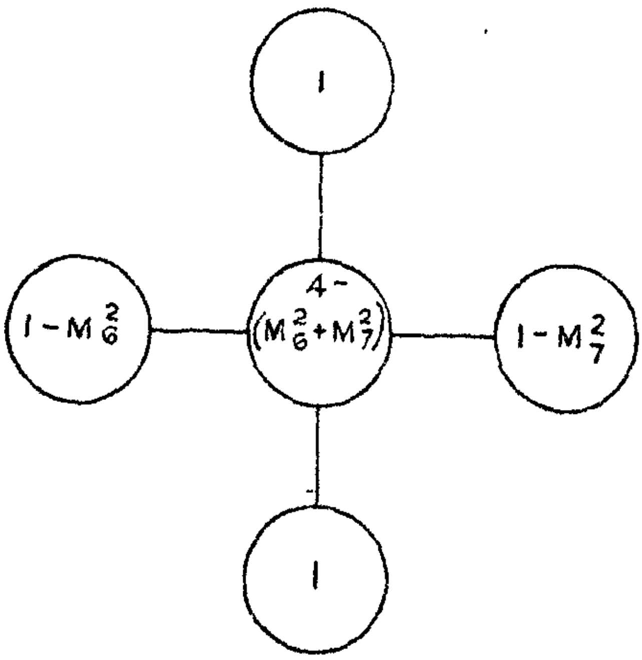


FIG 2



EPS

FIG. 3.

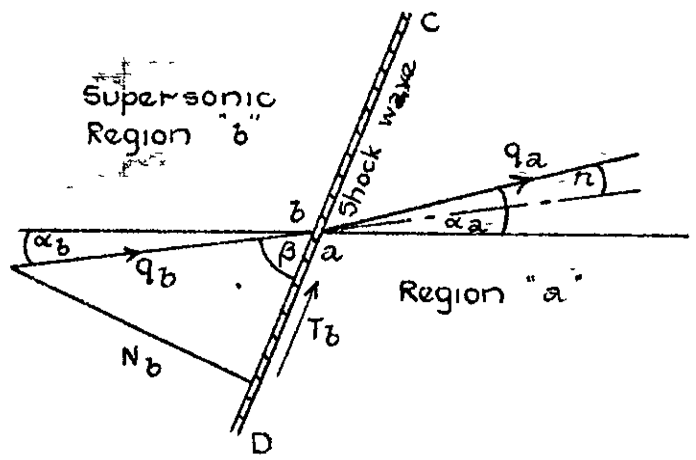
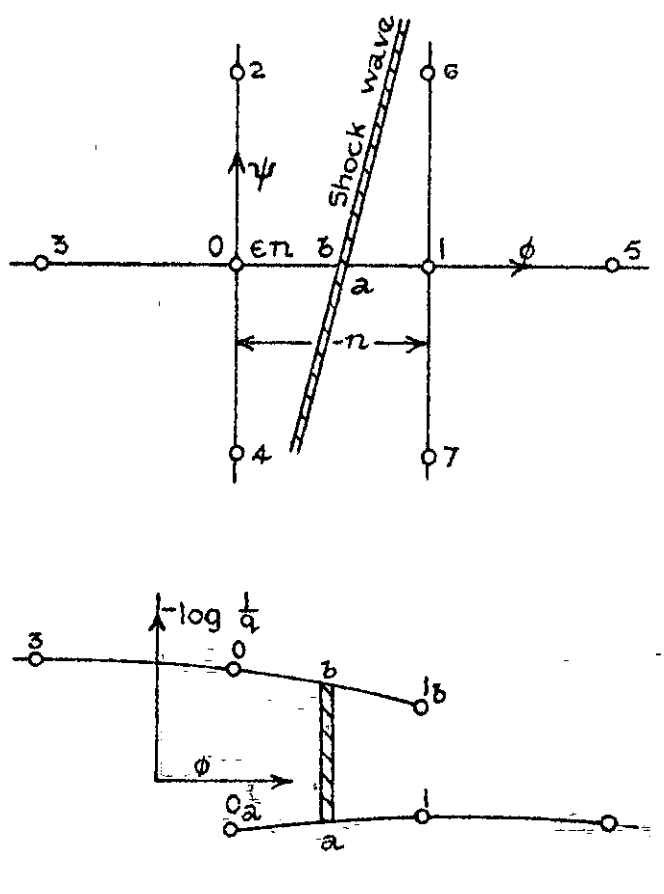

C
Shock wave
Supersonic
Region "b"
q_a
n
b
α_a
α_b
β
a
q_b
Region "a"
T_b
N_b
D

FIG 4


2
6
Shock wave
ψ
3
0
ϵn
b
1
ϕ
5
a
n
4
7
$-\log \frac{1}{q}$
3
0
b
1_b
ϕ
0_a
1
a

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