C.P. No. 96
(14,238)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
CURRENT PAPERS

Graphical Solution of Multhopp's Equations for the Lift Distribution of Wings

By

F. Vandrey

LONDON : HER MAJESTY'S STATIONERY OFFICE

1952

Price 1s 9d net

ADMIRALTY RESEARCH LABORATORY

C.P.96

A.R.L./R1/G/AE/2/5

February, 1951

GRAPHICAL SOLUTION OF MULTHOPP'S EQUATIONS FOR THE LIFT-DISTRIBUTION OF WINGS

ABSTRACT

A simple graphical method is described facilitating the determination of the lift-distribution of wings. The basis is Multhopp's method of replacing approximately the integro-differential equation for the circulation by a finite system of linear equations which give the values of the circulation at certain fixed points along the span.

The values of the unknown circulations are represented by scales in a set of diagrams for the equations. The multiplication of the approximate values of the unknowns and the constant coefficients of the equations is effected by auxiliary scales in the diagrams. The corrections of the approximate values are transferred from the auxiliary scales to the main scales by a pair of dividers.

The lift-distribution of a rectangular wing is determined as a practical example.

INTRODUCTION

1. The determination of the lift-distribution of wings is usually carried out by the method of Multhopp (1), whereby Prandtl's integro-differential equation is replaced approximately by a finite system of linear equations. These equations permit a solution by iteration which converges rather quickly in all cases of practical interest.

2. The calculations are usually carried out numerically, but graphical methods are equally suitable for the procedure of iteration, and their accuracy is often sufficient for practical purposes. Multhopp himself suggested a graphical method in which the slope of straight lines represented the required values of the circulation***. The practical application of his method requires a drafting machine, for transferring the directions of these lines from the diagram of one equation of the system to that of another one.

3. A somewhat simpler graphical method is described below. The values of the circulation are represented by scales. The multiplication of these values and the constant coefficients of the equations is effected by auxiliary scales. The corrections obtained from the auxiliary scales are transferred to the main scales by a pair of dividers.

MULTHOPP'S SYSTEM OF EQUATIONS AND ITS SOLUTION BY ITERATION

4. Consider the equations of Multhopp, for example in the case of 7 points along the span of the wing, assuming symmetrical distribution of the angle of incidence with respect to the centre of the wing.

$$\begin{aligned}
\gamma_1 \left(5.23 + \frac{b}{c_1\, t_1}\right) &= \alpha_1 + 1.91\,\gamma_2 + 0.15\,\gamma_4 \\
\gamma_2 \left(2.83 + \frac{b}{c_2\, t_2}\right) &= \alpha_2 + 1.04\,\gamma_1 + 1.19\,\gamma_3 \\
\gamma_3 \left(2.16 + \frac{b}{c_3\, t_3}\right) &= \alpha_3 + 0.91\,\gamma_2 + 0.85\,\gamma_4 \\
\gamma_4 \left(2.00 + \frac{b}{c_4\, t_4}\right) &= \alpha_4 + 0.11\,\gamma_1 + 1.58\,\gamma_3
\end{aligned} \qquad (1)$$

5. Here, b denotes the span of the wing, $c_\nu = \frac{1}{2}\frac{d\,c_L}{d\,\alpha}$ ($\approx 1.1\,\pi$ in most cases) the derivative of the lift-coefficient with respect to the angle of incidence, t_ν the length of the chord of the wing-section, α_ν the angle of incidence ($\alpha = \frac{1}{57.3}\,\alpha^o$), $\gamma_\nu = \frac{c_{L\nu} \cdot t_\nu}{2b}$ the non-dimensional circulation at the point ν. The position η_ν of the points where γ_ν is calculated is shown in Figure 1.

6. In order to calculate the unknown values γ_ν from (1) for a given wing, the method of iteration is used. An estimated 0th approximation is obtained by neglecting the terms containing γ_ν on the righthand side of (1)

$$\gamma_{10} = \left(5.23 + \frac{b}{c_1\, t_1}\right)^{-1} \alpha_1 \qquad (2)$$

The first approximation is then obtained by introducing $\gamma_{\nu 0}$ into the righthand sides of (1). As γ_1 and γ_3 in (1) depend only on γ_2 and γ_4 and vice versa, a more rapid convergence is obtained by using γ_{20} and γ_{40} only in the determination of γ_{11} and γ_{31}, and using these values instead of γ_{10} and γ_{30} for the determination of γ_{21} and γ_{41}. The first step of the iteration procedure will therefore be

*** German war-time report of the Focke-Wulf Werke, Bremen. Not published.

$$\gamma_{11} = \left[5.23 + \frac{b}{c_1\ t_1}\right]^{-1} (\alpha_1 + 1.91\ \gamma_{20} + 0.15\ \gamma_{40})$$
$$\gamma_{31} = \left[2.16 + \frac{b}{c_3\ t_3}\right]^{-1} (\alpha_3 + 0.91\ \gamma_{20} + 0.85\ \gamma_{40}) \tag{3a}$$

$$\gamma_{21} = \left[2.83 + \frac{b}{c_2\ t_2}\right]^{-1} (\alpha_2 + 1.04\ \gamma_{11} + 1.19\ \gamma_{31})$$
$$\gamma_{41} = \left[2.00 + \frac{b}{c_4\ t_4}\right]^{-1} (\alpha_4 + 0.11\ \gamma_{11} + 1.58\ \gamma_{31}) \tag{3b}$$

Further approximations are calculated in the same way. After a few steps, the successive approximations $\gamma_{\nu n}$ become practically constant, i.e. they represent the solution of the system (1).

GRAPHICAL ITERATION OF MULTHOPP'S SYSTEM OF EQUATIONS

7. In order to carry out the iteration of the system (1) by graphical methods, the values $\gamma_{\nu o}$ have first to be determined, i.e. α_1, ... have to be multiplied by $\left[5.23 + \frac{b}{c_1\ t_1}\right]^{-1}$, ... This may easily be done by constructing a right-angled triangle with the sides $5.23 + \frac{b}{c_1\ t_1}$ and α_1 in suitable units (independent of each other) and a scale for γ_1 parallel to the side representing α_1 (Figure 2).

8. Let ℓ_1 be the unit length of $5.23 + \frac{b}{c_1\ t_1}$, ℓ_2 the unit length of α_1 and ℓ_3 a third unit length. A straight line *AB* is then drawn bearing a scale of unit length ℓ_1 with the origin at a distance $5.23\ \ell_1$ from *A*. Another straight line *CD* is drawn perpendicular to *AB* through the point *C* at a distance ℓ_3 from *A*.

9. Now let $\frac{b}{c_1\ t_1} = 2$ in a particular case. A second line $EF \perp Ab$ is then drawn through the point *E* which corresponds to $\frac{b}{c_1\ t_1} = 2$ on the scale of *Ab*. Further a line *GH* is drawn anywhere in the plane bearing a scale for α_1 of unit length ℓ_2. Let α_1 be 1 in a particular case. $\alpha_1 = 1$ is then transferred by a pair of dividers from *GH* to *EJ* along the line *EF*, and a straight line is drawn from *A* to *J* intersecting the line *CD* at the point *K*. The distance *CK* will then be

$$CK = JE.\frac{AC}{AE} = 5.23 + \frac{\alpha_1\ b}{c_1\ t_1}\ .\ \frac{\ell_2\ \ell_3}{\ell_1} = \gamma_{10}\frac{\ell_2\ \ell_3}{\ell_1} \tag{4}$$

The line *CD* may therefore be fitted with a scale for γ_1, of unit length $\frac{\ell_2\ \ell_3}{\ell_1}$.

10. The scales *AB*, *CD* and *GH* are obviously the same for every wing, only the lines *EF* and *AJ* (dotted in Figure 2) have to be drawn in any particular case. Almost the same system of scales can also be used for the determination of γ_{20}, γ_{30} and γ_{40}, the only difference is that the distance between *A* and the origin of the b/ct - scale must be $2.83\ \ell_1$, $2.16\ \ell_1$ or $2.00\ \ell_1$ respectively, instead of $5.23\ \ell_1$.

11. In order to obtain from γ_{10} the improved value

$$\gamma_{11} = \left[5.23 + \frac{b}{c_1\ t_1}\right]^{-1} (\alpha_1 + 1.91\ \gamma_{20} + 0.15\ \gamma_{40}) \tag{5}$$

the quantities 1.91 γ_{20} and 0.15 γ_{40} have to be added to α_1 and the result has again to be multiplied by $\left(5.23 + \frac{b}{c_1 t_1}\right)^{-1}$. Two auxiliary scales LM for 1.91 γ_2 and 0.15 γ_4 are therefore drawn in addition to the scales AB, CD and GH of Figure 2, with the unit lengths 1.91 ρ_2 for γ_2 and 0.15 ρ_2 for γ_4 (Figure 3).

12. With the aid of these auxiliary scales, the length (1.91 γ_{20} + 0.15 γ_{40}) is transferred to JN along the line JF. The line AN is then drawn in the diagram intersecting the scale of γ_1 at the point O corresponding to the improved approximate value γ_{11} of γ_1.

The improved value of γ_3, γ_{31}, is determined in exactly the same manner and then γ_{21} and γ_{41} are determined correspondingly by using the improved values γ_{11} and γ_{31} instead of γ_{10} and γ_{30}.

13. The next approximation $\gamma_{\nu 2}$ is then obtained by the same process, using γ_{21}, γ_{41}, γ_{12}, γ_{32}, instead of γ_{20}, γ_{40}, γ_{11}, γ_{31}, and correspondingly for further approximations. After a few steps, a set of values $\gamma_{\nu n}$ is obtained which remain constant if the iteration is continued. These values represent the solution of the system (1) within the range of accuracy of the method.

14. The scales AB of $\frac{b}{c_1 t_1}$, CD of γ_1, GH of α_1 and the auxiliary scales LM of γ_2 and γ_4 in the diagram of the first equation, and the corresponding scales in the diagrams of the other equations, are obviously the same for all wings. It is therefore convenient for practical applications to use blue-prints of these scales which can be kept in stock (Figure 4). The diagrams of the first and third equation of (1), and equally those of the second and fourth equation, may be designed on these blue-prints with the use of the same scale of $\frac{b}{c\ t}$ in order to obtain smaller dimensions of the whole diagram.

15. A similar combined diagram can also be designed for the case of an anti-symmetrical distribution of the angle of incidence. The equations of Multhopp are in this case

$$
\begin{aligned}
\gamma_1 \left(5.23 + \frac{b}{c_1\ t_1}\right) &= \alpha_1 + 1.85\ \gamma_2 \\
\gamma_2 \left(2.83 + \frac{b}{c_2\ t_2}\right) &= \alpha_1 + \gamma_1 + \gamma_3 \qquad (6) \\
\gamma_3 \left(2.16 + \frac{b}{c_3\ t_3}\right) &= \alpha_3 + 0.77\ \gamma_2
\end{aligned}
$$

The diagram for this case is designed in Figure 5; further explanation appears unnecessary.

16. It is of course not difficult to apply the same method to the case of 15 points along the span in order to obtain a better approximation to the original integro-differential equation of Prandtl. For a still greater number of points, however, say 31, the graphical method becomes too complicated. In such cases, the calculation should be carried out numerically.

A NUMERICAL EXAMPLE

17. A practical example will show the use of the diagrams of Figures 4 and 5.

Consider for example a rectangular wing of aspect ratio $b/t = 2\pi$. Let the value of $c = \frac{1}{2}\ \frac{d\ c_L}{d\ \alpha}$ be π (theoretical value for symmetrical wing-sections of infinitely small thickness) and the angle of incidence α be constant (e.g. = 1) along the wing in the symmetrical case, and be zero in the anti-symmetrical case at the points η_4, η_3, η_2 and 1 at the point η_1. The value of $\frac{b}{c\ t}$ will then be 2 for all points of the wing.

18. Consider first the symmetrical case. In order to obtain the 0th approximation, lines parallel to the scales of γ_ν are drawn through the points $\frac{b}{c\ t} = 2$, and the points J_ν are marked on these lines by transferring to them $\alpha_\nu = 1$ from the scale of α (Figure 4). Straight lines are then drawn through J_ν and the points (γ_ν) of the basic lines of the diagram. These lines intersect the scales of γ_ν at the points γ_{ν_0} which give the 0th approximation for the iteration process

$$\gamma_{10} = .14; \quad \gamma_{20} = .21; \quad \gamma_{30} = .25; \quad \gamma_{40} = .26$$

19. The lengths corresponding to $\gamma_{20} = .21$ and $\gamma_{40} = .26$ are now transferred from the scales (γ_1) to the dotted line from $\frac{b}{c_1\ t_1} = 2$ to J_1 in the upper diagram of Figure 4. By this means the point N_1 is obtained. The straight line connecting N_1 and (γ_1) on the basic line intersects the scale of γ_1 at a point corresponding to the improved value γ_{11}. The improved value γ_{31} is determined in the same manner and correspondingly the values γ_{21} and γ_{41} are determined from the lower part of the diagram, using in this case the values γ_{11} and γ_{31}, instead of γ_{10} and γ_{30}. The first approximation is therefore

$$\gamma_{11} = .20; \quad \gamma_{21} = .34; \quad \gamma_{31} = .34; \quad \gamma_{41} = .40.$$

Repeating the procedure once more with these improved values, the second approximation

$$\gamma_{12} = .23; \quad \gamma_{22} = .36; \quad \gamma_{32} = .40; \quad \gamma_{42} = .42$$

is obtained.

After the next step, only the value of γ_1 is slightly altered

$$\gamma_{13} = .25; \quad \gamma_{23} = .36; \quad \gamma_{33} = .40; \quad \gamma_{43} = .42$$

and further steps will give the same result, within the range of accuracy which can be achieved by a graphical method. The values γ_{ν_3} are therefore the solution of the problem. The exact solution of the equations is

$$\gamma_1 = 0.2419 \quad \gamma_2 = 0.3590 \quad \gamma_3 = 0.4042 \quad \gamma_4 = 0.4162.$$

20. The determination of the values γ_ν in the anti-symmetrical case is carried out in exactly the same way with the use of the diagram Figure 5, and therefore needs no further explanation. The result is

$$\gamma_1 = .15; \quad \gamma_2 = .03; \quad \gamma_3 = .01; \quad \gamma_4 = 0.$$

In this case only two steps of the iteration process are needed to obtain the final result. The exact solution of the equation is

$$\gamma_1 = 0.1464 \quad \gamma_2 = 0.0315 \quad \gamma_3 = 0.0058 \quad \gamma_4 = 0.$$

21. The permission of the Admiralty for the publication of this paper is acknowledged.

REFERENCE

(1) H. Multhopp: Die Berechnung der Auftriebsverteilung von Trag fluegeln. Luftfahrtforschung 15 (1938), 153-169.

F. VANDREY
F.V/EL
A.R.L., Teddington.
February, 1951.

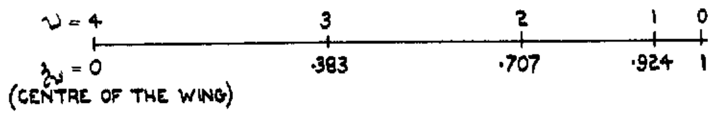

FIG.1.

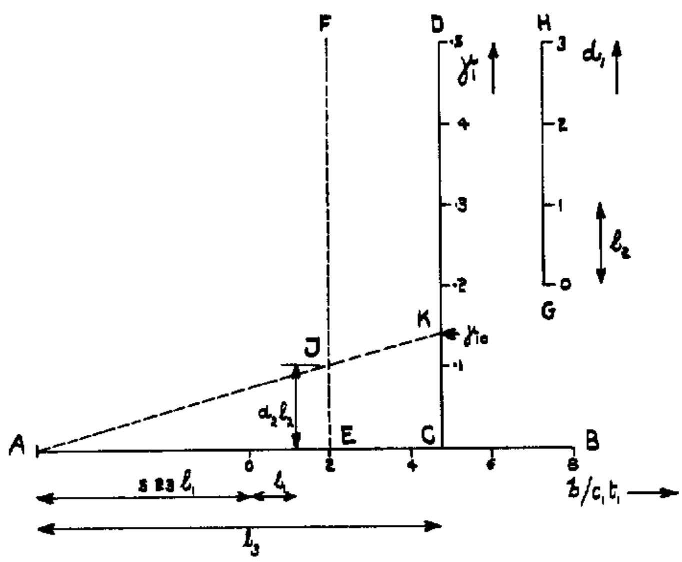

FIG.2.

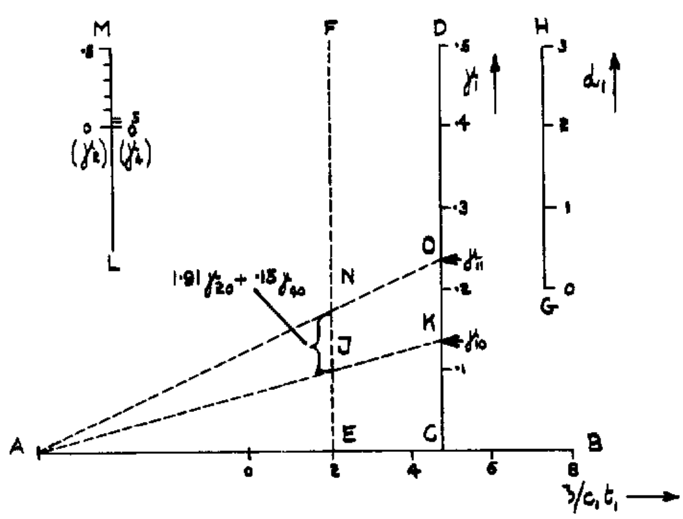

FIG.3.

LIFT – DISTRIBUTION. SYMMETRICAL CASE.
(RECTANGULAR WING, $b/t = 2\pi$, $C = \pi$, $d = 1$.)

ν	4	3	2	1	0
η_ν	0	·383	·707	·924	1
d_ν	1	1	1	1	1
$b/c_\nu t_\nu$	2	2	2	2	2
$\gamma_{\nu 0}$	·26	·25	·21	·14	0
$\gamma_{\nu 1}$	·40	·34	·34	·20	0
$\gamma_{\nu 2}$	·42	·40	·36	·23	0
$\gamma_{\nu 3}$	·42	·40	·36	·25	0

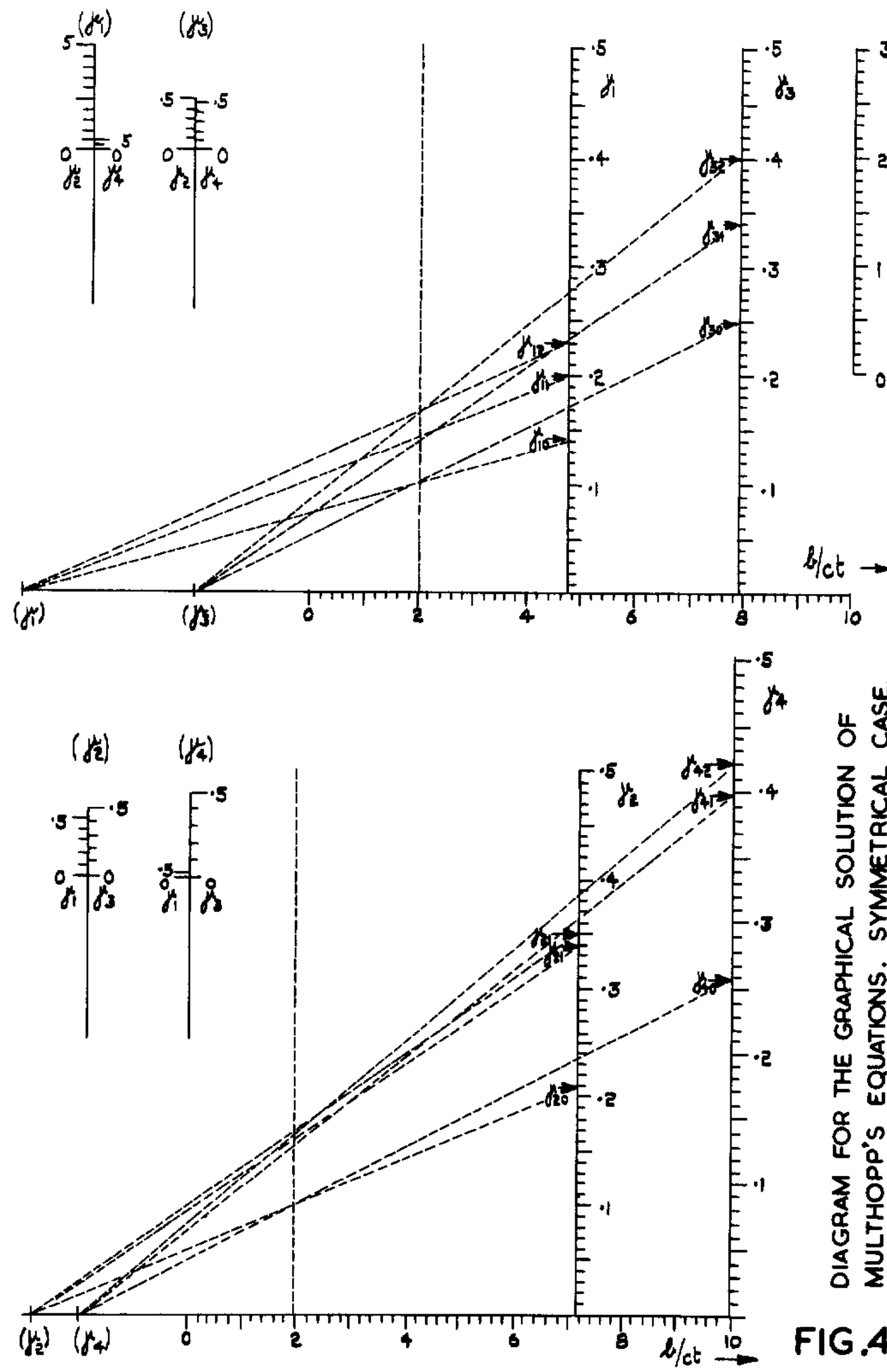

DIAGRAM FOR THE GRAPHICAL SOLUTION OF MULTHOPP'S EQUATIONS. SYMMETRICAL CASE, 7 POINTS ALONG THE SPAN.

FIG.4.

LIFT—DISTRIBUTION. ANTISYMMETRICAL CASE.
(RECTANGULAR WING, $b/t = 2\pi$, $c = \pi$, $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \alpha_4 = 0$.)

ν	4	3	2	1	0
ζ_ν	0	·383	·707	·924	1
α_ν	0	0	0	1	1
$b/c_\nu t_\nu$	2	2	2	2	2
$\gamma_{\nu 0}$	0	0	0	·14	0
$\gamma_{\nu 1}$	0	·01	·03	·15	0
$\gamma_{\nu 2}$	0	·01	·03	·15	0

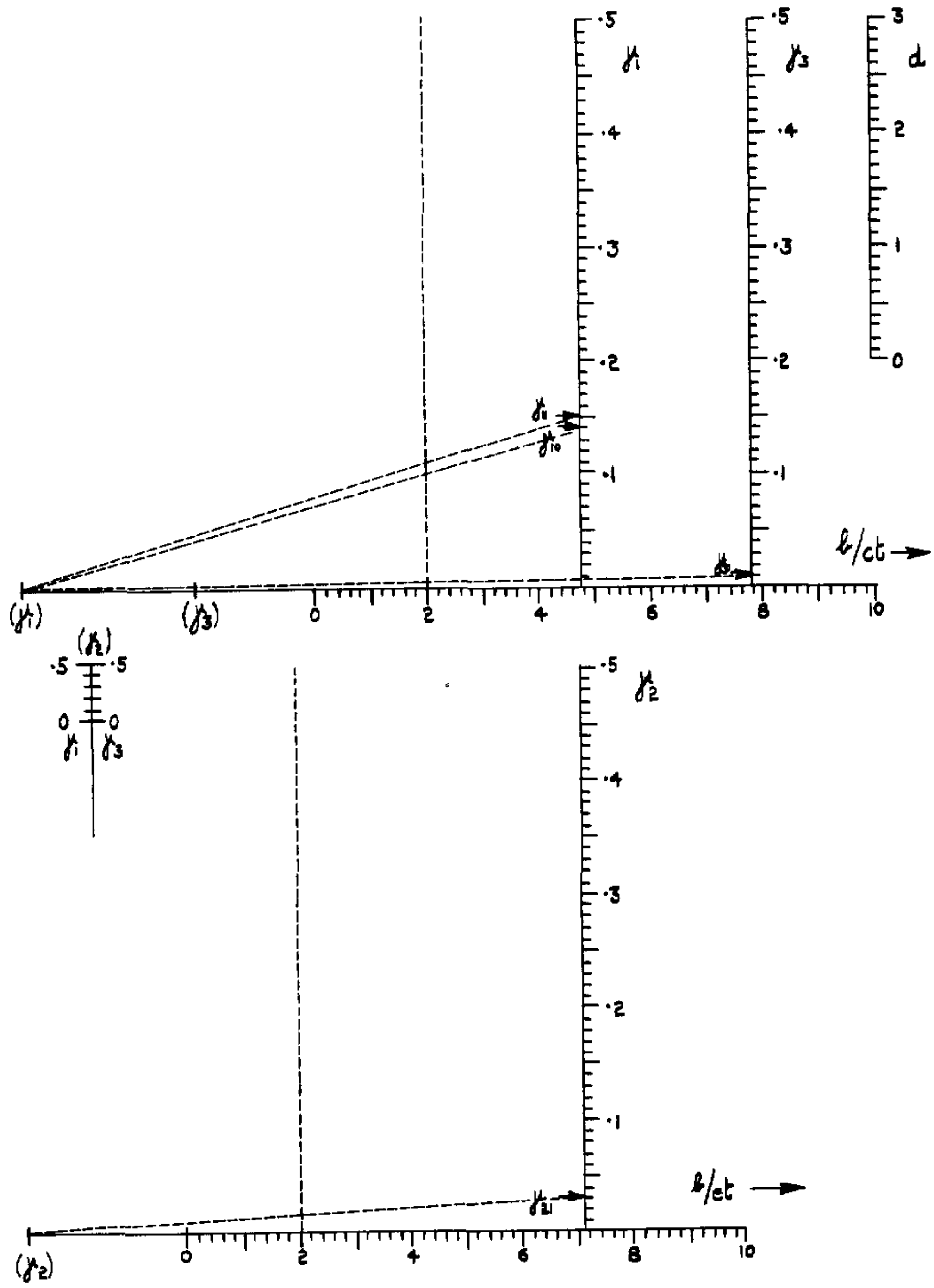

DIAGRAM FOR THE GRAPHICAL SOLUTION OF MULTHOPP'S EQUATIONS. ANTISYMMETRICAL CASE, 7 POINTS ALONG THE SPAN.

FIG.5.

C.P. No. 96
(14,238)
A.R.C. Technical Report

PRINTED AND PUBLISHED BY HER MAJESTY'S STATIONERY OFFICE
To be purchased from
York House, Kingsway, LONDON, W.C.2 423 Oxford Street, LONDON, W.1
P.O Box 569, LONDON, S E.1
13a Castle Street, EDINBURGH, 2 1 St. Andrew's Crescent, CARDIFF
39 King Street, MANCHESTER, 2 Tower Lane, BRISTOL, 1
2 Edmund Street, BIRMINGHAM, 3 80 Chichester Street, BELFAST
or from any Bookseller

1952

Price 1s 9d net

PRINTED IN GREAT BRITAIN

DS 31781/4/K3 S.O. Code No: 23-9006-96