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Notes on the Derivation of True Air Temperature from Aircraft Observations

By

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Notes on the derivation of True Air Temperature from Aircraft Observations.

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Definition of terms used.

T_s = true air temperature
T_i = the indicated temperature
P_s = the static air pressure in the absence of position error (the true air pressure)
P_t = the total air pressure or the pressure of the air if brought to rest relative to the aircraft
q = $P_t - P_s$ the pressure rise
V = the true air speed
V_i = the indicated air speed if referred to slow aircraft but now called the 'equivalent air speed' if the compressibilty of the air is significant.
V_r = the reading of the air speed indicator in the absence of position error.
P_o = the M.S.L. pressure, 1013.2 mb. ICAN
T_o = the M.S.L. temperature, 288°A ICAN
ρ_o = the M.S.L. density, 1.226×10^{-3} gm/cm³ ICAN
ρ = air density at the level considered.
α = the air speed correction factor

$$T_i - T_s = \frac{\alpha}{K^2}\left(\frac{V}{100}\right)^2$$

where K is the factor converting from knots to cm/sec.

λ defined in the equation

$$T_i - T_s = \lambda \frac{V^2}{2c_p}, \quad 0 < \lambda < 1$$

a = speed of sound; $a_o = 3.40 \times 10^4$ cm/sec.
m = Mach Number = V/a
k^2 = convertion factor knots² to (cm/sec)² = 2.65×10^3
γ = ratio of specific heat at constant pressure to specific heat at constant volume for air = 1.402

Accentuation of a symbol indicates that it has the value corresponding to the immediate neighbourhood of the static source, e.g. P_s' is the static pressure before the application of position error correction. All units unless stated are given in cm. gm. sec.

Review of existing practice.

If we start with the simple relation connecting the temperature rise with the air speed

$$T_1 - T_s = \frac{\lambda V^2}{2c_p} \quad \text{.........................(1)}$$

or its equivalent form

$$T_1 - T_s = \frac{\alpha}{K^2}\left(\frac{V}{100}\right)^2 \quad \text{....................(2)}$$

then, given T_i and V, T_s can be found.

T_1 is read off directly from the temperature indicator corrected for instrumental error, and, for slow aircraft, V is derived from the equation

$$V = V_i\sqrt{\frac{\rho_o}{\rho}} \quad \text{......................(3)}$$

$$= V_1\sqrt{\frac{P_o}{T_o}}\sqrt{\frac{T_s}{P_s}} \quad \text{..................(4)}$$

as described in the Meteorological Air Observer's Handbook p.242. In this case V_i is taken to be identical with the air speed indicator reading corrected for instrument and position errors. It is not necessary, however, to extract V separately and T_s is obtained by combining (1) and (4) in the form

$$T_i - T_s = \frac{\lambda}{2c_p}\frac{V_i^2 P_o T_s}{T_o P_s} \quad \text{...................(5)}$$

$$\text{or} \qquad T_s = \frac{T_i}{\left(1 + \frac{\lambda}{2c_p}\frac{V_i^2 P_o}{T_o P_s}\right)} \quad \text{.............(6)}$$

As speeds increase these simple relationships no longer hold and it is necessary to distinguish between V_i, sometimes called the 'equivalent air speed' and defined as V_1 in (3) or (4), and V_r which is the reading on the air speed indicator, corrected for instrumental error.

The energy equation under adiabatic conditions can be written

$$\frac{q}{P_s} = \left\{1 + \frac{\gamma-1}{2}\left(\frac{V}{a}\right)^2\right\}^{\frac{\gamma}{\gamma-1}} - 1 \quad \text{.....(7)}$$

or using (3) where V_i now stands for the 'equivalent air speed'

$$\frac{q}{P_s} = \left\{1 + \frac{\gamma-1}{2}\frac{P_o}{P_s}\left(\frac{V_i}{a_o}\right)^2\right\}^{\frac{\gamma}{\gamma-1}} - 1 \quad \text{...(8)}$$

Expanding the R.H.S. of(8) and remembering that $a_o = \gamma P_o/\rho_o$ we get

$$q = \frac{\rho_o V_i^2}{2}\left\{1 + \frac{P_o}{4P_s}\left(\frac{V_i}{a_o}\right)^2 + \frac{2-\gamma}{24}\left(\frac{P_o}{P_s}\right)^2\left(\frac{V_i}{a_o}\right)^4 + \ldots\right\} \quad \text{...(9)}$$

Now, in U.K. air speed indicators are calibrated according to the formula

$$q = \frac{\rho_o V_r^2}{2}\left\{1 + \frac{1}{4}\left(\frac{V_r}{a_o}\right)^2\right\} \quad \text{.....................(10)}$$

In the absence of position error, when V_r would refer to the reading of the air speed indicator in the free air stream, (9) and (10) together give the relation between V_i and V_r, namely,

$$V_r^2 \left\{ 1 + \frac{V_r}{4a_o^2} \right\} = V_i^2 \left\{ 1 + \frac{P_o}{4P_s} \left(\frac{V_i}{a_o}\right)^2 + \frac{2-\gamma}{24} \left(\frac{P_o}{P_s}\right)^2 \left(\frac{V_i}{a_o}\right)^4 + \ldots \right\} \quad \ldots(11)$$

Many sets of tables are available giving the values of V_i corresponding to values of V_r, and the conversion from V_r to V_i is termed 'correcting for the compressibility error'.

If in (4), V_r were used as V_i, then the value of V which would be obtained would be too large by an amount ΔV where, if all terms containing $(V_i/a_o)^6$ or smaller be neglected,

$$\Delta V = 0.285\sqrt{\frac{\rho_o}{\rho}} \left\{ \frac{P_o}{P} - 1 \right\} \left\{ \frac{V_i}{100} \right\}^3 \quad \ldots\ldots\ldots(12)$$

$$= 0.285 \frac{P_s T_o}{P_o T_s} \left\{ \frac{P_o}{P} - 1 \right\} \left\{ \frac{V}{100} \right\}^3 \quad \ldots\ldots\ldots\ldots(13)$$

In table 1 are given some of the values of ΔV from (13) for three selected air speeds and three altitudes, T_s being assumed to have the values corresponding to an ICAN atmosphere. The table also shows the errors in T_s which would arise from using V_r for V_i in (4) assuming $\lambda = 10^{-4} \times 2c_p/K^2$ (i.e. $\alpha = 1$). The temperature errors have been converted to °F for convenience.

Table 1.

Pressure Altitude	True air speed in knots					
	200		300		400	
mb.	error in air speed knots	error in temperature °F	error in air speed knots	error in temp. °F	error in air speed knots	error in temp. °F
500	1.28	0.09	4.3	0.46	10.2	1.5
300	2.0	0.14	6.7	0.66	16.0	2.3
150	2.6	0.19	8.6	0.87	20.5	2.9

When the position error has to be taken into account then, assuming the position error to be in the form of a static pressure correction ΔP_s the value of V_r should first be adjusted by applying the correction ΔV_r corresponding to ΔP_s obtained by differentiation of (10)

$$\Delta V_r = - \frac{\Delta P_s}{\rho_o V_r'(1 + V_r'^2/2a_o^2)} \quad \ldots\ldots\ldots\ldots(14)$$

In this expression accents indicate values referred to the neighbourhood of the static source, i.e. before the position error has been applied. If desired, the tables or graphs giving V_i in terms of V_r from (11) can be modified with the aid of (14) to give V_i directly from V_r' thus combining the two steps $V_r' \rightarrow V_r$, $V_r \rightarrow V_i$ into one.

Charnley and Fleming (Ref. 1) however prefer to insert the correction for position error after the correction for compressibility which means working with elaborate correction formulae but which, since the two steps are finally combined, comes to the same in the end.

In the method of approach to the true air temperature outlined above, the introduction of such terms as the 'indicated air speed' and the 'equivalent air speed' produces artificial errors which have thenceforth to be eliminated by manipulating awkward mathematical formulae. It is therefore more logical and much simpler not to employ the terms V_r and V_i but to use instead the pressure terms from which they are derived, namely q and P_s. In the method which will now be described this course has been adopted and a much neater analysis is obtained which leads to a more rapid derivation of the true air temperature.

Re-writing (1)

$$T_i - T_s = \frac{\lambda V^2}{2c_p}$$

and putting $a^2 = \gamma(c_p - c_v)T_s$ we get

$$T_i - T_s = \frac{\lambda V^2 (\gamma - 1) T_s}{2a^2} \quad ....(15)$$

or

$$T_s = \frac{T_i}{1 + \lambda(\gamma - 1)m^2/2} \quad ....(16)$$

It will be seen that (15) and (5) and (16) and (6) are equivalent forms. Now from (7) remembering that $m^2 = V^2/a^2 = V^2\rho/\gamma P$

$$\frac{\gamma - 1}{2} m^2 = (1 + \frac{q}{P})^{\frac{\gamma - 1}{\gamma}} - 1 \quad ......(17)$$

For purposes of reference denote either side of this equation by F

i.e. $$F\left(\frac{q}{P}\right) = (1 + \frac{q}{P})^{\frac{\gamma - 1}{\gamma}} - 1$$

Equation (16) can now be written

$$T_s = \frac{T_i}{1 + \lambda \left[(1 + \frac{q}{P_s})^{\frac{\gamma - 1}{\gamma}} - 1\right]} \quad ....(18)$$

Equation (18) gives a direct expression for T_s in terms of the known quantities q, P_s and T_i for a thermometer of known λ, and incorporates all the steps represented by equations (5) to (11). If, in addition, q/P_s be treated as one variable, T_s could be obtained from one set of tables involving T_i and q/P_s, given λ.

It follows from what is discussed in the Appendix that, if interpolation is allowed, tables of this type could be constructed having some eighty-four entries of q/P_s against, say, two hundred entries of temperature in degrees Fahrenheit, and that a separate set of such tables would be required for each value of λ. If these figures were represented graphically then several curves correspnding to several values of the parameter λ could be drawn and the whole set of information would be contained on one graph.

To give the required degree of accuracy, however, without interpolation, the tables would have to contain at least ten times as many entries of q/P_s and at least twice as many entries of T_i which would make them unwieldy for normal observing duty. It is better therefore, in computing T_s from equation (18), first to form the function F(q/P) by the aid of tables and then to complete the calculation by slide rule. Table II, which is described in detail in the Appendix, gives values of F(q/P) in terms of q/P and Figure 1 shows F(q/P) graphically.

The use of q in these calculations raises the question of whether it is worth having instruments calibrated in terms of q instead of V_r as at present. Meanwhile, however, q can be obtained readily by a straight conversion from V_r using equation (10). The pressure equivalents, q, of values of V_r over the range 200-600 knots are given in table III.

The successive steps in both methods are compared below, it being assumed that all instrumental readings have been corrected for instrumental errors.

<u>Existing method using V_i</u>

(i) Apply the position error correction.
(ii) Convert V_r to V_i by applying the compressibility correction.
(iii) Using (5) or (6) obtain T_s by slide rule

OR

(i) Apply the compressibility correction to V_r.
(ii) Apply the position error correction to the result of (i) to get V_i

(iii) As before.
As already mentioned, steps (i) and (ii) may be combined but, as this means the manipulation of three variables V_r, P_s and the position error to give V_i, the combination can only be rendered graphically.

Alternative method using q.

(i) Apply position error corrections to q' and P_s'.
(ii) Form the function F(q/P) from tables.
(iii) Use (18) and obtain T_s by slide rule.

Discussion.

In the first method, the application of the position error (if we do not wish to use the multitudinous graphs which result from combining steps (i) and (ii),) entails making some side calculations or using subsidiary graphs or tables based on equation (14) or similar. Also, in the step to V_i, the pressure occurs as an additional parameter which, even if the applications of position error and compressibility error are separated, means taking account simultaneously of P_s and V_r, whereas, in the alternative method, the replacement of V_r by q, eliminating V_r from the argument, allows the ratio q/P_s to appear as an independent variable, thus simplifying the analysis.

Furthermore step (iii) in the first method still contains the extra variable P_s, in addition to V_i, and is therefore longer than the corresponding step (ii) in the alternative method which contains only F(q/P).

Reference.

(1) CHARNLEY W. J. and FLEMING I, Corrections applied to air-speed indicator and altimeter readings for position error and compressibility effects.
R.A.E. Report No. Aero 2299, February 1949. A.R.C. 12,365.

Appendix.

The pressure rise which we have called q could be obtained either directly from the airspeed indicator or else from a second aneroid measuring the total pressure P_t. The proportional error in the first case, (obtained by logarithmic differentiation of (18), assuming $\lambda = 1$ and $\Delta T_i = 0$ to simplify the formulae) is given by

$$\left(\frac{\Delta T_s}{T_s}\right)_q = \frac{\gamma - 1}{\gamma} \frac{\frac{q}{P_s}}{1 + \frac{q}{P_s}} \left(\frac{\Delta P_s}{P_s} - \frac{\Delta q}{q}\right) \quad \text{.........(19)}$$

i.e.

$$\left|\left(\frac{\Delta T_s}{T_s}\right)_q\right|_{max} = \frac{\gamma - 1}{\gamma} \frac{\frac{q}{P}}{1 + \frac{q}{P}} \left(\left|\frac{\Delta P_s}{P_s}\right| + \left|\frac{\Delta q}{q}\right|\right) \quad \text{.........(20)}$$

Likewise in the second case from (18) putting $1 + \frac{q}{P_s} = \frac{P_t}{P_s}$

$$\left(\frac{\Delta T_s}{T_s}\right)_{P_t} = \frac{\gamma - 1}{\gamma} \left(\frac{\Delta P_s}{P_s} - \frac{\Delta P_t}{P_t}\right) \quad \text{.................(21)}$$

i.e.

$$\left|\left(\frac{\Delta T_s}{T_s}\right)_{P_t}\right|_{max} = \frac{\gamma - 1}{\gamma} \left(\left|\frac{\Delta P_s}{P_s}\right| + \left|\frac{\Delta P_t}{P_t}\right|\right) \quad \text{...............(22)}$$

If ΔP represents a position error then $\Delta P_s = -\Delta q$, $\Delta P_t = 0$ which makes (19) and (21) identical. If, however, ΔP is an instrumental error then $\left|\frac{\Delta P_s}{P_s}\right| = \left|\frac{\Delta P_t}{P_t}\right|$ and to the same order also $= \left|\frac{\Delta q}{q}\right|$ so that the ratio

$$\left|\left(\frac{\Delta T_s}{T_s}\right)_q\right|_{max} \Big/ \left|\left(\frac{\Delta T_s}{T_s}\right)_{P_t}\right|_{max} = \frac{\frac{q}{P}}{1 + \frac{q}{P}} < 1$$

From which it follows that it is advantageous to measure q directly in preference to P_t.

Present types of aneroid altimeters and airspeed indicators after correction for instrumental errors are accurate to one per cent and if great care is taken and corrections applied also for changes with temperature then one half per cent can be obtained. For our purpose we will take the figure one per cent as representing the accuracy of altimeters and air-speed indicators.

Now for any small variation in q/P of value $\Delta(q/P)$ the corresponding proportional variation in T, namely $\Delta T_s/T_s$ is, from (18) by differentiation, assuming $\lambda = 1$, and omitting any variation in T_i for the moment,

$$\frac{\Delta T}{T} = -\frac{\gamma - 1}{\gamma} \frac{\Delta\left(\frac{q}{P}\right)}{1 + \frac{q}{P}} \quad \ldots\ldots\ldots\ldots\ldots(23)$$

Also

$$\Delta\left(\frac{q}{P}\right) = \frac{q}{P}\left(\frac{\Delta q}{q} - \frac{\Delta P}{P}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

So that

$$\left|\Delta\left(\frac{q}{P}\right)\right|_{max} = \frac{q}{P}\left(\left|\frac{\Delta q}{q}\right|_{max} + \left|\frac{\Delta P}{P}\right|_{max}\right) \quad \ldots\ldots\ldots\ldots(25)$$

If we take

$$\left|\frac{\Delta q}{q}\right|_{max} = \left|\frac{\Delta P}{P}\right|_{max} = 0.01$$

then from (24)

$$\left|\Delta\left(\frac{q}{P}\right)\right|_{max} = \frac{q}{P}(0.02)$$

and hence

$$\left|\frac{\Delta T}{T}\right|_{max} = 0.02\,\frac{\gamma - 1}{\gamma} \cdot \frac{\frac{q}{P}}{1 + \frac{q}{P}}$$

$$= 0.00572\,\frac{\frac{q}{P}}{1 + \frac{q}{P}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

The values of $|\Delta(q/P)|_{max}$ and $|\Delta T/T|_{max}$ for different values of q/P are given below.

q/P	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\|\Delta(q/P)\|_{max}$	.002	.004	.006	.008	.010	.012	.014	.016	.018
$\|\Delta T/T\|_{max} \times 10^3$	.52	.95	1.32	1.63	1.90	2.14	2.35	2.54	2.71

As would be expected the accuracy falls off at the higher speeds and higher altitudes which correspond to larger values of q/P. At a M.S.L. temperature of 15°C (288°A) and q/P = 0.1, ΔT = 0.15°C (0.27°F) and at - 53°C (220°A) and q/P = 0.8, ΔT = 0.56°C (1°F).

The values for $|\Delta(q/P)|_{max}$ in the second line suggest that if a table for F(q/P) were being constructed it would be sufficient for entries of q/P to be made at intervals of 0.001 which would then provide values of T to within the limits of instrumental accuracy.

For a pressure range of 1000 - 100 mb. approx. and indicated air speeds between 200 and 600 knots, but for true air speed not exceeding the speed of sound, the values of q/P vary between 0.06 and 0.893, tables for F(q/P) for values of q/P, spaces at intervals of 0.001 would contain 833 entries but would give the desired precision without the need for interpolation. A coarser table with steps in q/P of 0.01 would contain 84 entries, but, to get the required degree of accuracy, intermediate values would have to be interpolated which would not be practical when converting numerous temperature readings.

The limits of accuracy in the derived value of the true air temperature also depend, of course, on the accuracy in the reading of T_i. With a good aircraft thermometer using a null-reading type indicator the reading can be made to 0.1°F, and, with pointer indicators or with recorders, the best accuracy that can be expected from existing equipment is ± 0.5°F. Adding these errors to the others arising from the pressure instruments gives, with null reading type indicators, for q/P = 0.1 and T - 15°C (59°F) an accuracy to about ± 0.4°F, and at the other end of the scale for q/P = 0.8 and T = - 53°C (-63°F) an accuracy to about ± 1.1°F while with pointer indicators or recorders these figures would be larger by 0.5°F. In all cases it is assumed that λ is known and that no error arises from variations in λ.

TABLE II

$$F\left(\frac{q}{p}\right) = \left(1 + \frac{q}{p}\right)^{\frac{\gamma - 1}{\gamma}} - 1$$

q/p	0	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0							.017	.020	.022	.025
.1	.028	.030	.033	.035	.039	.041	.043	.046	.048	.052
.2	.054	.056	.059	.061	.064	.066	.069	.071	.074	.076
.3	.078	.080	.082	.085	.087	.090	.092	.094	.096	.099
.4	.101	.104	.106	.108	.110	.113	.114	.117	.119	.121
.5	.123	.126	.127	.129	.132	.134	.136	.139	.140	.142
.6	.144	.147	.148	.150	.153	.154	.156	.158	.161	.162
.7	.165	.166	.168	.170	.171	.174	.176	.178	.180	.181
.8	.184	.185	.188	.190	.191	.193	.195	.196	.199	.200

TABLE III

V_r Knots	q mb.	V_r Knots	q mb.
200	66.466	400	283.694
210	73.443	410	299.312
220	80.801	420	315.495
230	88.529	430	332.199
240	96.647	440	349.402
250	105.153	450	367.109
260	114.052	460	385.325
270		470	404.055
280	133.062	480	423.677
290	143.187	490	443.465
300	153.715	500	464.187
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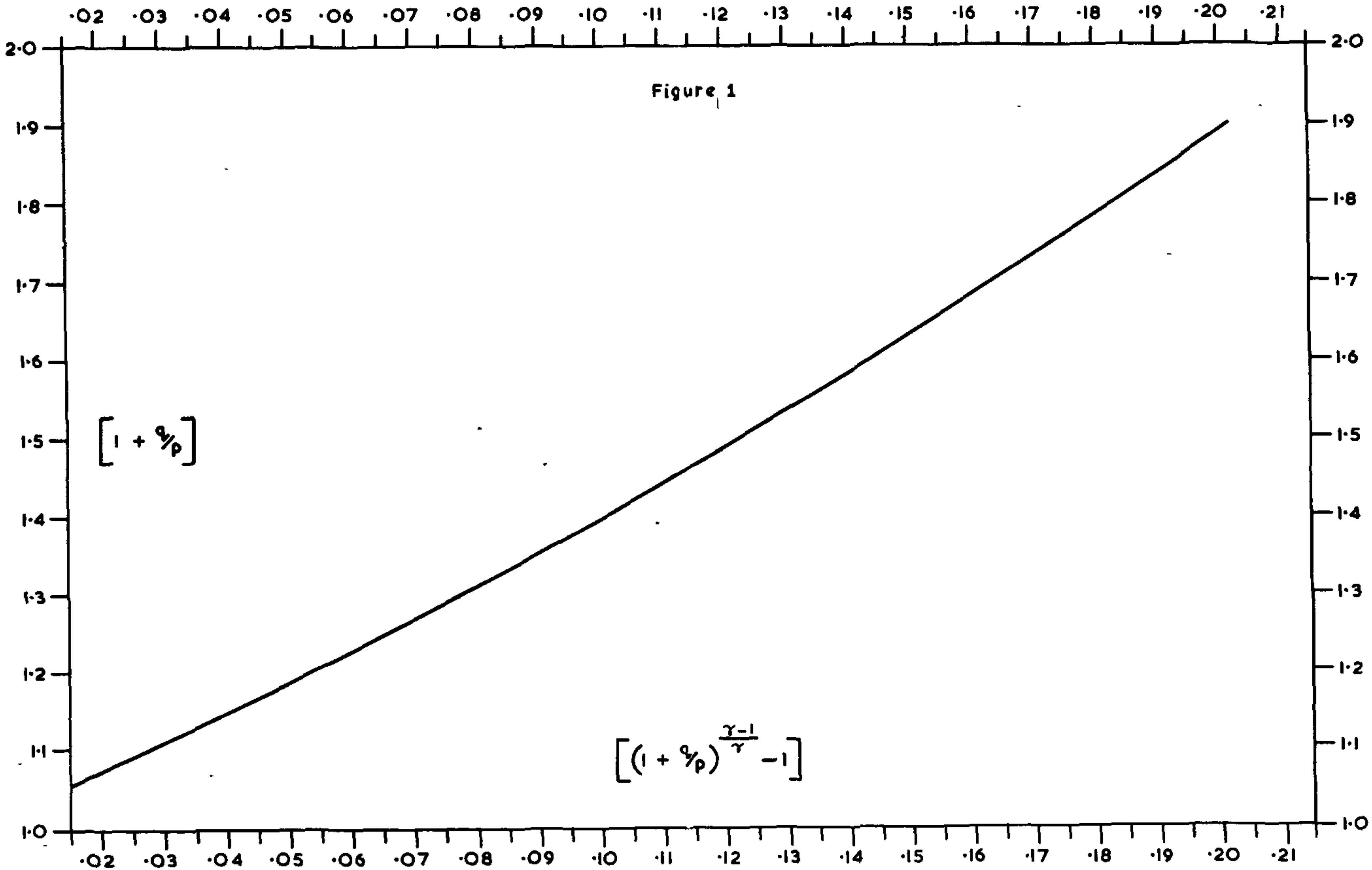
Figure 1
·02 ·03 ·04 ·05 ·06 ·07 ·08 ·09 ·10 ·11 ·12 ·13 ·14 ·15 ·16 ·17 ·18 ·19 ·20 ·21
2·0 1·9 1·8 1·7 1·6 1·5 1·4 1·3 1·2 1·1 1·0
$\left[1 + q/p\right]$
$\left[\left(1 + q/p\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$

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