C.P No 47
(13,682)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
CURRENT PAPERS

Dynamic Stability of the Helicopter: The Equations of Motion

By

A. H. YATES, B.Sc., B.Sc. (Eng.)

LONDON: HIS MAJESTY'S STATIONERY OFFICE
1951
ONE SHILLING NET

Current Paper No. 47

Corrigendum

February, 1954

Page 4, Equation 2a, line 2:-

$$\text{For} \left(1 + \frac{Z_q}{\mu_*}\right) \text{ read } \left(\frac{V}{\Omega R} + \frac{Z_q}{\mu_*}\right)$$

Page 4, Equation 2b, line 2:-

$$\text{For} \left(1 - \frac{y_r}{\mu_*}\frac{d}{d\tau}\right) \text{ read } \left(\frac{V}{\Omega R} - \frac{y_r}{\mu_*}\right)\frac{d}{d\tau}$$

Page 6, last three lines:-

In each line

$$\text{For} \left(1 + \frac{Z_q}{\mu_*}\right) \text{ read } \left(\frac{V}{\Omega R} + \frac{Z_q}{\mu_*}\right)$$

Page 7. In the expression for C:-

$$\text{For } \mu_* \frac{n_v}{i_c} \text{ read } \frac{V}{\Omega R} \mu_* \frac{n_v}{i_c}$$

Page 7. In the expression for D

$$\text{For } \mu_* \frac{l_v n_p - l_p n_v}{i_A i_c} \text{ read } \frac{V}{\Omega R} . \mu_* \frac{l_v n_p - l_p n_v}{i_A i_c}$$

Page 7, 1st line:-

For x_U read x_u

For x_W read x_w .

C.P. No.47

Dynamic Stability of the Helicopter - The Equations of Motion

- By -

A. H. Yates, B.Sc., B.Sc.(Eng.).

11th January, 1951

The equations of motion, as used in calculations of the dynamic stability of the helicopter, are stated here.

They are reduced to dimensionless form and the expanded forms of the coefficients of the determinantal equations are given.

Introduction/

Introduction

Several reports have recently been published on various aspects of the problem of dynamic longitudinal and lateral stability of the single-rotor helicopter but the lack of a common notation has made difficult the task of linking the various reports. This task is simplified if we write down the six equations of motion of the helicopter in terms of dimensionless derivatives in a manner analogous to that for fixed-wing aircraft.

The equations can then be simplified where necessary by neglecting certain derivatives and the characteristics of the motions of the helicopter deduced from the solutions of the equations. The main problem is, of course, the calculation of the derivatives. Some are amenable to direct calculation while others can be estimated by semi-empirical methods from the results of wind tunnel tests.

The equations of motion

The calculations apply to small disturbances of the helicopter from its equilibrium condition so that only first order derivatives need be considered.

Almost all fixed-wing aircraft have a longitudinal plane of symmetry so that, in the consideration of their stability, coupling between longitudinal and lateral motions can be ignored. Many helicopters, too, will have this plane of symmetry but the single-rotor helicopter with torque-balancing tail rotor, for example, has not and there will be a coupling between the two motions. The error caused by assuming that the motions are independent is found to be small so that the coupling is ignored in the present paper.

Some other assumptions made in deriving the normal equations of motion of a fixed-wing aeroplane cannot so easily be made for the helicopter. For the former the direction of the relative wind is assumed to be in the plane of symmetry. We shall retain this assumption in writing our equations and will thus obtain the equations of motions for small disturbances from forward or backward flight or hovering only. Another assumption - that the products of inertia $D = \sum myz$ and $F = \sum mxy$ are zero is retained since the torque balancing device does not usually involve any shift of the centre of gravity from the plane of symmetry.

In defining the motion of the helicopter we use a set of right-handed axes with origin at the centre of gravity, G, and fixed in the helicopter so that they move with it when it is disturbed. The axes GX and GZ are always taken in the plane of symmetry with the axis GX approximately forward and GZ downward. GY is taken to starboard.

If the axis GX coincides with the direction of the undisturbed motion of the helicopter then the axes are called 'wind axes' but remain fixed in the helicopter. The wind axes can be moved to coincide with their original position in undisturbed flight by rotations ψ about GZ, θ about the resulting GY and ϕ about the resulting GX.

Forces and velocities are defined in Table I and are taken positive in the directions of the axes. Angular velocities and moments are positive when they tend to rotate the helicopter in the senses $Y \rightarrow Z$, $Z \rightarrow X$, $X \rightarrow Y$.

TABLE I/

TABLE I

Description	Symbol	Unit
Axes	GX GY GZ	
Moment of Inertia	I_A I_B I_C	slugs. ft.^{2}
Product of Inertia	- I_E -	slugs. ft.^{2}
Steady velocity	V - -	ft./sec.
Disturbed velocity	V + u v w	ft./sec.
Disturbed angular velocity	p q r $(=\dot\phi)(=\dot\theta)(=\dot\psi)$	rad./sec.
Forces	X Y Z	lb.
Moments	L M N	lb.ft.

The equations of motion when 'wind axes' are adopted are as follows*.-

Longitudinal

$$\left.\begin{aligned}
&\frac{W}{g}\dot{u} - uX_u - wX_w - \dot\theta X_q + W\theta\cos\tau_c &= X_o\\
&\frac{W}{g}(\dot{w} - V\dot\theta) - uZ_u - wZ_w - \dot\theta Z_q + W\theta\sin\tau_c &= Z_o\\
&I_B\ddot\theta - uM_u - wM_w - \dot\theta M_q &= M_o
\end{aligned}\right\}\ldots(1a)$$

Lateral

$$\left.\begin{aligned}
&\frac{W}{g}(\dot{v} + V\dot\psi) - vY_v - \dot\phi Y_p - \dot\psi Y_r - W(\phi\cos\tau_c + \psi\sin\tau_c) &= Y_o\\
&I_A\ddot\phi - I_E\ddot\psi - vL_v - \dot\phi L_p - \dot\psi L_r &= L_o\\
&I_C\ddot\psi - I_E\ddot\phi - vN_v - \dot\phi N_p - \dot\psi N_r &= N_o
\end{aligned}\right\}\ldots(1b)$$

where X_o M_o etc. are forces or moments applied by control movements, etc. If we now substitute the dimensionless derivatives (Table II), these equations become:-

Longitudinal /

---

*The angle of climb of the helicopter is often denoted by τ. As aerodynamic time is also conventionally written τ we shall here use τ_c to denote the angle of climb.

Longitudinal

$$\left.\begin{aligned}
&\left(\frac{d}{d\tau} - x_u\right)\frac{u}{\Omega R} - x_w\frac{w}{\Omega R} + \left(C_L - \frac{x_q}{\mu_*}\frac{d}{d\tau}\right)\theta = x_0 \\
&- z_u\frac{u}{\Omega R} + \left(\frac{d}{d\tau} - z_w\right)\frac{w}{\Omega R} + \left[C_L \tan\tau_c - \left(1 + \frac{z_q}{\mu_*}\right)\frac{d}{d\tau}\right]\theta = z_0 \\
&- \frac{\mu_*}{i_B} m_u\frac{u}{\Omega R} - \frac{\mu_*}{i_B} m_w\frac{w}{\Omega R} + \left(\frac{d^2}{d\tau^2} - \frac{m_q}{i_B}\cdot\frac{d}{d\tau}\right)\theta = m_0
\end{aligned}\right\}\quad(2a)$$

Lateral

$$\left.\begin{aligned}
&\left(\frac{d}{d\tau} - y_v\right)\frac{v}{\Omega R} - \left(\frac{y_p}{\mu_*}\frac{d}{d\tau} + C_L\right)\phi + \left[\left(1 - \frac{y_r}{\mu_*}\frac{d}{d\tau}\right) - C_L \tan\tau_c\right]\psi = y_0 \\
&- \frac{\mu_*}{i_A}\ell_v\frac{v}{\Omega R} + \left(\frac{d^2}{d\tau^2} - \frac{\ell_p}{i_A}\frac{d}{d\tau}\right)\phi - \left(\frac{i_E}{i_A}\frac{d^2}{d\tau^2} + \frac{\ell_r}{i_A}\frac{d}{d\tau}\right)\psi = \ell_0 \\
&- \frac{\mu_*}{i_C} n_v\frac{v}{\Omega R} - \left(\frac{i_E}{i_C}\frac{d^2}{d\tau^2} + \frac{n_p}{i_C}\frac{d}{d\tau}\right)\phi + \left(\frac{d^2}{d\tau^2} - \frac{n_r}{i_C}\frac{d}{d\tau}\right)\psi = n_0
\end{aligned}\right\}\quad(2b)$$

[Note that C_L = Lift / $\rho(\Omega R)^2 S$ where S is the disc area, πR^2, and that lift, measured as usual normal to the flight path, = $W \cos\tau_c$].

The/

The dimensionless derivatives and their derivations are given in the table below.-

TABLE II

I Units of Quantities in II and III	II Quantities	III Divisors to obtain IV	IV Symbol	V Description
lb.	$X \quad Y \quad Z$ L	$\rho\,\Omega^2 R^2.S$	$C_x \quad C_y \quad C_z$ C_L	Force coefficients*
lb.ft.	$L \quad M \quad N$	$\rho\Omega^2R^2.S.R$	$C_\ell \quad C_m \quad C_n$	Moment coefficients
$\frac{\text{lb.}}{\text{ft./sec.}}$	$X_u \quad Z_u$ Y_v $X_w \quad Z_w$	$\rho.\Omega R.S$	$x_u \quad z_u$ y_v $x_w \quad z_w$	Force velocity derivatives
$\frac{\text{lb.}}{\text{rad./sec.}}$	Y_p $X_q \quad Z_q$ Y_r	$\rho.\Omega R.S.R$	y_p $x_q \quad z_q$ y_r	Force angular velocity derivatives
$\frac{\text{lb.ft.}}{\text{ft./sec.}}$	M_u $L_v \quad N_v$ M_w	$\rho.\Omega R.S.R$	m_u $\ell_v \quad n_v$ m_w	Moment velocity derivatives
$\frac{\text{lb.ft.}}{\text{rad./sec.}}$	$L_p \quad N_p$ M_q $L_r \quad N_r$	$\rho.\Omega R.S.R^2$	$\ell_p \quad n_p$ m_q $\ell_r \quad n_r$	Moment angular velocity derivatives
$\frac{\text{lb.}}{\text{rad./sec}^2}$	$Y_{\dot p}$ $X_{\dot q} \quad Z_{\dot q}$ $Y_{\dot r}$	$\rho\ .\ SR^2$	$y_{\dot p}$ $x_{\dot q} \quad z_{\dot q}$ $y_{\dot r}$	Force angular velocity derivatives**
$\frac{\text{lb.ft.}}{\text{rad./sec.}^2}$	$L_{\dot p} \quad N_{\dot p}$ $M_{\dot q}$ $L_{\dot r} \quad N_{\dot r}$	$\rho\ \ SR^3$	$\ell_{\dot p} \quad n_{\dot p}$ $m_{\dot q}$ $\ell_{\dot r} \quad n_{\dot r}$	Moment angular velocity derivatives**
slugs.ft^2	$I_A \quad I_B \quad I_C$ I_E	WR^2/g	$i_A \quad i_B \quad i_C$ i_E	Inertia coefficients

*The lift coefficient defined here is $\frac{1}{2}\left(\frac{V}{\Omega R}\right)^2$ times the lift coefficient based on $\frac{1}{2}\rho V^2$.

** These derivatives are usually neglected when considering the stability of fixed wing aircraft. There is, however, some evidence that they may affect the stability of the helicopter; they have not, however, been included in the equations given above.

The dimensionless unit of time, the airsec, is defined as

$$\hat{t} = \frac{W}{g\rho S \cdot \Omega R} \text{ secs}$$

and time measured in these units is denoted by τ so that $t_{secs} = \tau . \hat{t}$.

The relative density parameter is defined as

$$\mu_* = \frac{W}{g\rho SR}.$$

The sets of equations (2a) and (2b) are solved by assuming that the variables

$$\frac{u}{\Omega R}, \frac{w}{\Omega R}, \theta ; \frac{v}{\Omega R}, \phi, \psi$$

are functions of τ of the form $\frac{u}{\Omega R} = \left(\frac{u}{\Omega R}\right)_0 e^{\lambda\tau}$, etc.

If the motion with controls fixed is considered $(x_0 = z_0 = m_0 = y_0 = \ell_0 = n_0 = 0)$ the equation for λ becomes

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

where the coefficients are functions of the dimensionless derivatives. If i_E and the acceleration derivatives are neglected the coefficients are

Longitudinal

$$A = 1$$

$$B = -(x_u + z_w) - \frac{m_q}{i_B}$$

$$C = (x_u z_w - x_w z_u) + \frac{m_q}{i_B}(x_u + z_w) - \mu_* \frac{m_w}{i_B}\left(1 + \frac{z_q}{\mu_*}\right) - \frac{m_u}{i_B} x_q$$

$$D = -\frac{m_q}{i_B}(x_u z_w - x_w z_u) + \mu_* \frac{m_w}{i_B}\left[x_u\left(1 + \frac{z_q}{\mu_*}\right) - z_u \frac{x_q}{\mu_*} + C_L \tan \tau_0\right]$$
$$+ \mu_* \frac{m_u}{i_B}\left[C_L - x_w\left(1 + \frac{z_q}{\mu_*}\right) + z_w \frac{x_q}{\mu_*}\right]$$

E/

$$E = \mu_* \frac{m_w}{i_B} C_L \left(z_u - x_U \tan \tau_c\right) - \mu_* \frac{m_u}{i_B} C_L \left(z_w - x_W \tan \tau_c\right)$$

$$= \mu_* C_L \left(\frac{m_w z_u - m_u z_w}{i_B}\right) \text{ if } \tau_c = 0 .$$

<u>Lateral</u>

$$A = 1$$

$$B = -\frac{\ell_p}{i_A} - \frac{n_r}{i_C} - y_v$$

$$C = \frac{\ell_p n_r - \ell_r n_p}{i_A \; i_C} + \mu_* \frac{n_v}{i_C} + \frac{\ell_p y_v - \ell_v y_p}{i_A} + \frac{n_r y_v - n_v y_r}{i_C}$$

$$D = -\mu_* \frac{\ell_v}{i_A} C_L + \mu_* \frac{\ell_v n_p - \ell_p n_v}{i_A \; i_C} - y_v \left(\frac{\ell_p n_r - \ell_r n_p}{i_A i_C}\right)$$

$$+ n_v \frac{\ell_p y_r - \ell_r y_p}{i_A \; i_C} + \ell_v \left(\frac{n_r y_p - n_p y_r}{i_A \; i_C}\right) - \mu_* \frac{n_v}{i_C} C_L \tan \tau_c$$

$$E = -\mu_* C_L \left(\frac{n_v \ell_r - \ell_v n_r}{i_A \; i_C}\right) + \mu_* C_L \left(\frac{n_v}{i_C} \frac{\ell_p}{i_A} - \frac{\ell_v}{i_A} \frac{n_p}{i_C}\right) \tan \tau_c .$$

C.P. No 47
(13,682)

A R C Technical Report

PRINTED AND PUBLISHED BY HIS MAJESTY'S STATIONERY OFFICE
To be purchased from
York House, Kingsway, LONDON, W C 2 429 Oxford Street, LONDON, W 1
P O Box 569, LONDON, S E 1
13a Castle Street, EDINBURGH, 2 1 St. Andrew's Crescent, CARDIFF
39 King Street, MANCHESTER, 2 Tower Lane, BRISTOL, 1
2 Edmund Street, BIRMINGHAM, 3 80 Chichester Street, BELFAST
or from any Bookseller

1951

Price 1s od net

PRINTED IN GREAT BRITAIN