NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2774

A METHOD FOR FINDING A LEAST-SQUARES POLYNOMIAL THAT PASSES THROUGH A SPECIFIED POINT WITH SPECIFIED DERIVATIVES

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A METHOD FOR FINDING A LEAST-SQUARES POLYNOMIAL THAT PASSES THROUGH A SPECIFIED POINT WITH SPECIFIED DERIVATIVES

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SUMMARY

A method is presented for finding an mth-degree polynomial that passes through a specified point with $(m - j)$ specified derivatives $(1 \leqq j \leqq m)$ and is a least-squares polynomial for other points spaced at unequal intervals of the independent variable.

INTRODUCTION

Recently it became necessary to be able to find the velocity and the rate of change of velocity along the surface at any point on the forward portion of a symmetrical airfoil at zero angle of attack. The given data consisted of values of the velocity at unequal intervals along the surface and of the first derivative of the velocity at the stagnation point. The given velocity at one of the points seemed to be slightly in error.

In the development of a method for finding the velocity and its first derivative, a procedure was needed for obtaining a third-degree polynomial that passes through a specified point with a specified slope and is a least-squares curve for other points spaced at unequal intervals of the independent variable. A search of the available literature (refs. 1 to 3) in which the least-squares method is discussed failed, however, to disclose a procedure that combines the least-squares method with the requirement that the polynomial pass through a specified point with a specified first derivative. Such a procedure, therefore, had to be developed.

In the course of the analysis, it became evident that its scope could easily be expanded to provide a method for finding an mth-degree polynomial that passes through a specified point with $(m - j)$ specified derivatives $(1 \leqq j \leqq m)$ and, moreover, is a least-squares polynomial for the remaining points. The specified $(m - j)$ derivatives are not necessarily the first $(m - j)$ derivatives.

The method presented should be useful when the value of a function or the value of a function and some of its derivatives are known from theory at one value of the independent variable and experimentally obtained values of the function are available for other values of the independent variable. An example is the fitting of a polynomial to the portion of a laminar-boundary-layer velocity profile near the surface. In this case the value of the velocity and some of the derivatives of the velocity are known at the surface from theory and measured values of the velocity are available at a number of points through the boundary layer. The procedure should also be useful when it is necessary to divide the total range of the independent variable into convenient intervals and to find a least-squares polynomial for the points in each interval. The polynomial can then be made to pass through the point given by the preceding polynomial at the end of the preceding interval with the required derivatives. Examples are the fitting of an airfoil velocity distribution by a number of polynomials and the fitting of a boundary-layer velocity profile by polynomials.

SYMBOLS

$N + 1$	number of points
m	degree of polynomial
x	independent variable
y	dependent variable
x_0,y_0	specified point through which curve passes
$z = x - x_0$	
$z_n = x_n - x_0$	
$C_0,C_1,C_2,\ldots C_m$	coefficients in polynomial
$C_{u_1},C_{u_2},C_{u_3},\ldots C_{u_j}$	coefficients to be calculated by least-squares method
$C_{k_1},C_{k_2},C_{k_3},\ldots C_{k_{m-j}}$	coefficients to be calculated from given derivatives
C_p	pth coefficient in polynomial

j	number of coefficients to be calculated by least-squares method
$m - j$	number of coefficients to be calculated from given derivatives
$n = 0, 1, 2, 3, \ldots N$	
S	sum

ANALYSIS

Let there be given $N + 1$ points (x_0,y_0), (x_1,y_1), . . . (x_N,y_N). It is required to find an mth-degree polynomial that passes through the point (x_0,y_0) with specified $(m - j)$ derivatives and is also a least-squares polynomial for the remaining N points. Some of the remaining N points can lie on one side of x_0 and some on the other side of x_0, or all the remaining N points can lie on one side of x_0.

The polynomial is written in the form

$$y = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + \ldots + C_m z^m \tag{1}$$

where

$$z = x - x_0$$

The requirement that the curve determined by the polynomial (eq. (1)) pass through the point (x_0,y_0) makes $C_0 = y_0$. Additional $(m - j)$ coefficients in the polynomial (eq. (1)) are found from the known $(m - j)$ derivatives at point (x_0,y_0) by placing $z = 0$ in the relation for the pth derivative:

$$\frac{d^p y}{dz^p} = p!C_p + (p + 1)!C_{p+1}z + \frac{(p + 2)!}{2!} C_{p+2}z^2 + \frac{(p + 3)!}{3!} C_{p+3}z^3 + \ldots + \frac{(p + q)!}{q!} C_{p+q}z^q + \ldots + \frac{m!}{(m - p)!} C_m z^{m-p} \tag{2}$$

Thus

$$C_p = \frac{1}{p!}\left(\frac{d^p y}{dz^p}\right)_{z=0} \tag{3}$$

There are thus $(m - j + 1)$ known coefficients. The total number of coefficients in the polynomial (eq. (1)) is $m + 1$; therefore, j unknown coefficients remain.

For the least-squares method to be applicable, the number of unknown coefficients, j, must be less than N. When $j = N$, the polynomial passes through each point and has the specified derivatives at point (x_0,y_0). In this case, the j unknown coefficients can be found more simply than by the least-squares method. In the present analysis, $j < N$.

The j unknown coefficients are to have values that make the polynomial (eq. (1)) a least-squares polynomial for the N points (x_1,y_1), (x_2,y_2), . . . (x_N,y_N). That is, the unknown coefficients are to have the values that minimize the sum of the squares of the differences between the y values calculated by equation (1) and the given y values for x_1, x_2, . . . x_N; thus,

$$S = \sum_{n=0}^{N}\left(C_0 + C_1 z_n + C_2 z_n^2 + C_3 z_n^3 + \ . \ . \ . + C_m z_n^m - y_n\right)^2 \tag{4}$$

is to be a minimum, where

$$z_n = x_n - x_0$$

The substitution of 0 for 1 can be made in the lower limit of the summation because $z_0 = 0$ and $C_0 = y_0$. The expression for S can be written as

$$S = \sum_{n=0}^{N}\Big[\left(C_{u_1} z_n^{u_1} + C_{u_2} z_n^{u_2} + C_{u_3} z_n^{u_3} + \ . \ . \ . + C_{u_j} z_n^{u_j}\right) - \Big(y_n - C_0 -$$

$$C_{k_1} z_n^{k_1} - C_{k_2} z_n^{k_2} - C_{k_3} z_n^{k_3} - \ . \ . \ . \ . - C_{k_{m-j}} z_n^{k_{m-j}}\Big)\Big]^2 \tag{5}$$

where $C_{u_1}, C_{u_2}, C_{u_3}, \ldots C_{u_j}$ are the j unknown coefficients and C_0 and $C_{k_1}, C_{k_2}, C_{k_3}, \ldots C_{k_{m-j}}$ are the $(m - j + 1)$ known coefficients. The integers $u_1, u_2, u_3, \ldots u_j$ used as subscripts and superscripts are not necessarily consecutive. Now let

$$w_n = y_n - C_0 - C_{k_1} z_n^{k_1} - C_{k_2} z_n^{k_2} - C_{k_3} z_n^{k_3} - \ldots - C_{k_{m-j}} z_n^{k_{m-j}} \tag{6}$$

Equation (5) then becomes

$$S = \sum_{n=0}^{N} \left(C_{u_1} z_n^{u_1} + C_{u_2} z_n^{u_2} + C_{u_3} z_n^{u_3} + \ldots + C_{u_j} z_n^{u_j} - w_n \right)^2 \tag{7}$$

The sum S is a function of the j unknown coefficients $C_{u_1}, C_{u_2}, C_{u_3}, \ldots C_{u_j}$. For S to be a minimum it is necessary that

$$\frac{\partial S}{\partial C_{u_1}} = 0, \; \frac{\partial S}{\partial C_{u_2}} = 0, \; \frac{\partial S}{\partial C_{u_3}} = 0, \; \ldots \; \frac{\partial S}{\partial C_{u_j}} = 0 \tag{8}$$

The application of the conditions given by equations (8) to equation (7) results in the following equations:

$$\left.\begin{aligned}
&\sum_{n=0}^{N} \left(C_{u_1} z_n^{u_1} + C_{u_2} z_n^{u_2} + C_{u_3} z_n^{u_3} + \ldots + C_{u_j} z_n^{u_j} - w_n \right) z_n^{u_1} = 0 \\
&\sum_{n=0}^{N} \left(C_{u_1} z_n^{u_1} + C_{u_2} z_n^{u_2} + C_{u_3} z_n^{u_3} + \ldots + C_{u_j} z_n^{u_j} - w_n \right) z_n^{u_2} = 0 \\
&\sum_{n=0}^{N} \left(C_{u_1} z_n^{u_1} + C_{u_2} z_n^{u_2} + C_{u_3} z_n^{u_3} + \ldots + C_{u_j} z_n^{u_j} - w_n \right) z_n^{u_3} = 0 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{n=0}^{N} \left(C_{u_1} z_n^{u_1} + C_{u_2} z_n^{u_2} + C_{u_3} z_n^{u_3} + \ldots + C_{u_j} z_n^{u_j} - w_n \right) z_n^{u_j} = 0
\end{aligned}\right\} \tag{9}$$

or

$$\left.\begin{aligned}
C_{u_1}\sum_{n=0}^{N} z_n^{2u_1} + C_{u_2}\sum_{n=0}^{N} z_n^{u_2+u_1} + C_{u_3}\sum_{n=0}^{N} z_n^{u_3+u_1} + \ldots + C_{u_j}\sum_{n=0}^{N} z_n^{u_j+u_1} &= \sum_{n=0}^{N} w_n z_n^{u_1} \\
C_{u_1}\sum_{n=0}^{N} z_n^{u_1+u_2} + C_{u_2}\sum_{n=0}^{N} z_n^{2u_2} + C_{u_3}\sum_{n=0}^{N} z_n^{u_3+u_2} + \ldots + C_{u_j}\sum_{n=0}^{N} z_n^{u_j+u_2} &= \sum_{n=0}^{N} w_n z_n^{u_2} \\
C_{u_1}\sum_{n=0}^{N} z_n^{u_1+u_3} + C_{u_2}\sum_{n=0}^{N} z_n^{u_2+u_3} + C_{u_3}\sum_{n=0}^{N} z_n^{2u_3} + \ldots + C_{u_j}\sum_{n=0}^{N} z_n^{u_j+u_3} &= \sum_{n=0}^{N} w_n z_n^{u_3} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
C_{u_1}\sum_{n=0}^{N} z_n^{u_1+u_j} + C_{u_2}\sum_{n=0}^{N} z_n^{u_2+u_j} + C_{u_3}\sum_{n=0}^{N} z_n^{u_3+u_j} + \ldots + C_{u_j}\sum_{n=0}^{N} z_n^{2u_j} &= \sum_{n=0}^{N} w_n z_n^{u_j}
\end{aligned}\right\} \qquad (10)$$

Now, for simplicity in writing, introduce the notation

$$\sum_{n=0}^{N} z_n^{(\ )} = \sigma_{(\ )}$$

and

$$\sum_{n=0}^{N} w_n z_n^{(\ )} = \phi_{(\ )}$$

Equations (10) for $C_{u_1}, C_{u_2}, C_{u_3}, \ldots C_{u_j}$ then become

$$\left.\begin{aligned}
C_{u_1}\sigma_{2u_1} + C_{u_2}\sigma_{u_2+u_1} + C_{u_3}\sigma_{u_3+u_1} + \ldots + C_{u_j}\sigma_{u_j+u_1} &= \phi_{u_1} \\
C_{u_1}\sigma_{u_1+u_2} + C_{u_2}\sigma_{2u_2} + C_{u_3}\sigma_{u_3+u_2} + \ldots + C_{u_j}\sigma_{u_j+u_2} &= \phi_{u_2} \\
C_{u_1}\sigma_{u_1+u_3} + C_{u_2}\sigma_{u_2+u_3} + C_{u_3}\sigma_{2u_3} + \ldots + C_{u_j}\sigma_{u_j+u_3} &= \phi_{u_3} \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
C_{u_1}\sigma_{u_1+u_j} + C_{u_2}\sigma_{u_2+u_j} + C_{u_3}\sigma_{u_3+u_j} + \ldots + C_{u_j}\sigma_{2u_j} &= \phi_{u_j}
\end{aligned}\right\} \qquad (11)$$

The values of $C_{u_1}, C_{u_2}, C_{u_3}, \ldots C_{u_j}$ are determined by solving the j simultaneous linear equations (eqs. (11)). A convenient method of solution is given in chapter I of reference 3.

NUMERICAL EXAMPLE

A simple example is given to illustrate the use of the method for a specific case.

A third-degree polynomial is to pass through the point $\left(x_0 = 5, y_0 = 125\right)$ with a slope of 75 and is to be a least-square polynomial for the points:

x	y
1	1
2	8
4	64
7	343
8	512
9	729
10	1000

The polynomial is

$$y = C_0 + C_1 z + C_2 z^2 + C_3 z^3$$

where

$$z = x - 5$$

The requirement that $y = 125$ for $x = 5$ $(z = 0)$ leads to

$$C_0 = 125$$

From equation (3) with $p = 1$ and $\left(\frac{dy}{dz}\right)_{z=0} = 75$, the resulting value obtained for C_1 is

$$C_1 = 75$$

In this example there is one known derivative and the polynomial is of the third degree; therefore,

$$m - j = 1$$

$$m = 3$$

Consequently, $j = 2$ and

$$C_{u_1} = C_2$$

$$C_{u_2} = C_3$$

With $C_0 = 125$ and $C_{k_1} = C_1 = 75$, equation (6) becomes

$$w_n = y_n - 125 - 75z_n$$

In order to simplify the computation of C_2 and C_3, the following table was prepared:

n	x_n	z_n	z_n^2	z_n^3	z_n^4	z_n^5	z_n^6	y_n	$y_n z_n^2$	$y_n z_n^3$
0	5	0	0	0	0	0	0	125	0	0
1	1	-4	16	-64	256	-1,024	4,096	1	16	-64
2	2	-3	9	-27	81	-243	729	8	72	-216
3	4	-1	1	-1	1	-1	1	64	64	-64
4	7	2	4	8	16	32	64	343	1,372	2,744
5	8	3	9	27	81	243	729	512	4,608	13,824
6	9	4	16	64	256	1,024	4,096	729	11,664	46,656
7	10	5	25	125	625	3,125	15,625	1,000	25,000	125,000
$\sum$			80	132	1,316	3,156	25,340		42,796	187,880

With $u_1 = 2$, $u_2 = 3$, and $u_j = 3$ equations (11) become

$$C_2\sigma_4 + C_3\sigma_5 = \phi_2$$

$$C_2\sigma_5 + C_3\sigma_6 = \phi_3$$

where

$$\sigma_4 = \sum_{n=0}^{7} z_n^{4} = 1,316$$

$$\sigma_5 = \sum_{n=0}^{7} z_n^{5} = 3,156$$

$$\sigma_6 = \sum_{n=0}^{7} z_n^{6} = 25,340$$

and

$$\phi_2 = \sum_{n=0}^{7} w_n z_n^{2}$$

$$= \sum_{n=0}^{7}\left(y_n - 125 - 75z_n\right)z_n^{2}$$

$$= \sum_{n=0}^{7} y_n z_n^{2} - 125 \sum_{n=0}^{7} z_n^{2} - 75 \sum_{n=0}^{7} z_n^{3}$$

$$= 42,796 - 125(80) - 75(132)$$

$$= 22,896$$

and

$$\phi_3 = \sum_{n=0}^{7} w_n z_n^3$$

$$= \sum_{n=0}^{7} (y_n - 125 - 75z_n)z_n^3$$

$$= \sum_{n=0}^{7} y_n z_n^3 - 125 \sum_{n=0}^{7} z_n^3 - 75 \sum_{n=0}^{7} z_n^4$$

$$= 187,880 - 125(132) - 75(1316)$$

$$= 72,680$$

The equations for C_2 and C_3 then are

$$1316C_2 + 3156C_3 = 22,896$$

$$3156C_2 + 25,340C_3 = 72,680$$

Solving for C_2 and C_3 results in

$$C_2 = 15$$

$$C_3 = 1$$

The required cubic equation then is

$$y = 125 + 75z + 15z^2 + z^3$$

or

$$y = 125 + 75(x - 5) + 15(x - 5)^2 + (x - 5)^3$$

which reduces to

$$y = x^3$$

the equation from which the set of values of y against x was obtained.

CONCLUSIONS

A method is presented for finding an mth-degree polynomial that passes through a specified point with $(m - j)$ specified derivatives $(1 \leqq j \leqq m)$ and is a least-squares polynomial for other points which are spaced at unequal intervals of the independent variable.

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NACA-Langley - 9-1-54 - 125