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TECHNICAL NOTE 2132

THE CALCULATION OF MODES AND FREQUENCIES OF A MODIFIED  
STRUCTURE FROM THOSE OF THE UNMODIFIED STRUCTURE

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Washington

July 1950

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## SUMMARY

A method is developed for the calculation of the natural coupled or uncoupled frequencies and modes of a structure with modifications, such as the addition of concentrated masses or springs, directly from the known modes and frequencies of the unmodified structure. The modes of the modified structure are expanded in terms of the modes of the unmodified structure. A characteristic equation and a frequency determinant, the order of which is twice the number of modifications, are derived by the use of the Galerkin method. Numerical examples are presented to show the accuracy of the method and the number of modes and frequencies of the unmodified structure necessary for agreement with exact solutions.

## INTRODUCTION

The calculation of the natural modes and frequencies of an airplane structure is usually required for various loading conditions. The variation of conditions may be brought about by changes in pay load, changes in the amount of fuel carried, the addition of tip tanks, and so forth. These changes may be regarded as modifications to a primary structure. In addition to weight changes, the addition of elastic restraints, such as spring supports which may be used in ground vibration tests, may also be considered as modifications. Thus, in this paper the basic or primary structure is known as unmodified, and the structure after masses and springs are added is known as modified.

Present methods of calculating modes and frequencies require a separate and independent calculation for each modification to the primary structure. In order to simplify the calculations of these modes and frequencies, a method is developed in this paper that allows, with very little extra work, the calculation of the modes and frequencies of a modified structure directly from the modes and frequencies of the unmodified structure.

The paper presents both a theoretical analysis and numerical examples. In the theoretical analysis a frequency determinant and the modal functions are derived. The order of the frequency determinant is, in general, twice the number of modifications to the primary structure. Each element of the determinant is a series having terms that are functions of the modal shapes and frequencies of the primary structure. The numerical examples illustrate the convergence of the series and the accuracy with which the frequencies and modes of the modified structure can be calculated.

#### SYMBOLS

$EI$	flexural stiffness
$GJ$	torsional stiffness
$y$	deflection of elastic axis of modified beam
$\phi$	angle of twist of cross section of modified beam
$y_n$	deflection of elastic axis of unmodified beam in $n$ th coupled mode normalized to give unit tip deflection
$\phi_n$	angle of twist of cross section of unmodified beam in $n$ th coupled mode
$\omega$	a natural frequency of modified beam
$\omega_n$	natural frequency of unmodified beam in $n$ th coupled mode
$M_i$	magnitude of concentrated mass at station $x_i$
$\alpha_i$	spring constant of spring at station $x_i$
$a_n$	proportion of $n$ th mode of unmodified beam present in a modified-beam mode
$e$	distance between center of gravity of cross section of beam and elastic axis; positive when center of gravity lies forward of elastic axis
$e_i$	distance between elastic axis and center of gravity of concentrated mass or distance between elastic axis and spring (see fig. 2); positive when center of gravity of mass or spring lies forward of elastic axis
$k$	radius of gyration of cross section of beam about elastic axis

$k_i$  for a mass, radius of gyration of mass about elastic axis; for a spring,  $e_i$

$m$  mass of unmodified beam per unit length

$x$  spanwise coordinate measured from center line

$[P]_i$  lateral load caused by mass or spring at station  $x_i$

$[T]_i$  torque about elastic axis caused by mass or spring at station  $x_i$

$$[P_n]_i = K_i \left( [\bar{y}_n]_i + e_i [\bar{\phi}_n]_i \right)$$

$$[T_n]_i = K_i \left( e_i [\bar{y}_n]_i + k_i^2 [\bar{\phi}_n]_i \right)$$

$$N_n = \int_{-L}^L m \left( \bar{y}_n'^2 + 2e_i \bar{y}_n \bar{\phi}_n' + k_i^2 \bar{\phi}_n'^2 \right) dx$$

$L$  semispan of beam

$$K_i \begin{cases} = M_i \omega^2 & \text{for concentrated mass at station } x_i \\ = -\alpha_i & \text{for spring at station } x_i \end{cases}$$

$f$  frequency coefficient

$\delta(x - x_i)$  function such that its value is zero at every point except argument  $x_i$  and its value at this point is infinity in

$$\text{such a way that } \int_{-L}^L \delta(x - x_i) dx = 1$$

Subscripts:

$i, j$  integers referring to locations of concentrated masses or springs

$m, n$  integers referring to modes of unmodified beam

$A, B, C$  locations of masses in example 2

#### THEORETICAL ANALYSIS

Development of characteristic equation.- Figure 1 illustrates a typical structure that is to be considered in the development of the method presented in this paper. It consists of a primary beam of any

spanwise variation of flexural stiffness  $EI$ , torsional stiffness  $GJ$ , and mass  $m$ . The modifications to the primary structure are the addition of any number of concentrated masses of magnitude  $M_1, M_2, \dots$  at stations  $x_1, x_2, \dots$ , respectively, and any number of springs with spring constants  $\alpha_5, \alpha_6, \dots$  at stations  $x_5, x_6, \dots$ , respectively.

The method developed in this paper is based on the assumption that the coupled modes  $y_n$  and  $\varphi_n$ , normalized to give a unit tip bending deflection, are known for the primary structure. If the uncoupled modes of the modified structure are desired, uncoupled modes of the primary structure must of course be used.

The procedure presented in this paper is based upon the Galerkin method (reference 1). The differential equations for the modified beam in free harmonic coupled vibration, vibrating at a natural frequency  $\omega$ , are

$$\frac{d^2}{dx^2} \left( EI \frac{d^2 y}{dx^2} \right) - m\omega^2(y + e\varphi) - \sum_i [P]_i \delta(x - x_i) = 0 \quad (1a)$$

and

$$\frac{d}{dx} \left( GJ \frac{d\varphi}{dx} \right) + m\omega^2(e y + k^2 \varphi) + \sum_i [T]_i \delta(x - x_i) = 0 \quad (1b)$$

where  $y$  and  $\varphi$  are the maximum coupled bending and torsional deflections of the modified beam, respectively, and  $[P]_i$  and  $[T]_i$  are the concentrated loads and torques, respectively, caused by the modifications at station  $x_i$  and are defined as

$$\left. \begin{aligned} [P]_i &= K_i ([y]_i + e_i [\varphi]_i) \\ [T]_i &= K_i (e_i [y]_i + k^2 [\varphi]_i) \end{aligned} \right\} \quad (i = 1, 2, \dots) \quad (2)$$

where

$$K_i \begin{cases} = M_i \omega^2 & \text{for concentrated masses} \\ = -\alpha_i & \text{for springs} \end{cases}$$

$e_i$  distance from the elastic axis to the center of gravity of concentrated mass or, in the case of a spring, to the center line of the spring (see fig. 2)

$k_i$  radius of gyration of mass about elastic axis for concentrated masses, or, in the case of a spring,  $e_i$

$\delta(x - x_i)$  a function such that its value is zero at every point except the argument  $x_i$  and its value at this point is infinity in such a way that  $\int_{-L}^L \delta(x - x_i) dx = 1$

The coupled bending and torsional deformations  $y$  and  $\varphi$  can be expressed in terms of the coupled modes of the unmodified beam by infinite series

$$\left. \begin{aligned} y &= a_0 y_0 + a_1 y_1 + \dots = \sum_{m=0}^{\infty} a_m y_m \\ \varphi &= a_0 \varphi_0 + a_1 \varphi_1 + \dots = \sum_{m=0}^{\infty} a_m \varphi_m \end{aligned} \right\} \quad (3)$$

where the coefficients  $a_0, a_1, \dots$  can be considered as generalized coordinates that give the proportions of the primary-beam modes that are present in a natural mode of the modified structure.

Since  $y_m$  and  $\varphi_m$  are coupled natural modes of the primary structure, they must satisfy the differential equations

$$\left. \begin{aligned} \frac{d^2}{dx^2} \left( EI \frac{d^2 y_m}{dx^2} \right) &= m \omega_m^2 (y_m + e \varphi_m) \\ \frac{d}{dx} \left( GJ \frac{d \varphi_m}{dx} \right) &= -m \omega_m^2 (e y_m + k^2 \varphi_m) \quad (m = 0, 1, \dots) \end{aligned} \right\} \quad (4)$$

and the orthogonality conditions, if  $m \neq n$ ,

$$\int_{-L}^L m y_m (y_n + e \varphi_n) dx + \int_{-L}^L m \varphi_m (k^2 \varphi_n + e y_n) dx = 0 \quad (5a)$$

and, if  $m = n$ ,

$$\int_{-L}^L m(y_m^2 + 2ey_m\phi_m + k^2\phi_m^2)dx = N_m \quad (5b)$$

In addition, the primary-structure modes  $y_n$  and  $\phi_n$  satisfy the same boundary conditions as the modified-structure modes  $y$  and  $\phi$ .

The procedure which is used to obtain the characteristic equation is based on the Galerkin method and is as follows: The infinite series for  $y$  and  $\phi$  (equations (3)) are substituted into the differential equations (1a) and (1b); then equations (1a) and (1b) are multiplied by  $y_n$  and  $\phi_n$ , respectively, and are integrated over the length of the beam. This procedure yields

$$\int_{-L}^L y_n \sum_{m=0}^{\infty} a_m \frac{d^2}{dx^2} \left( EI \frac{d^2 y_m}{dx^2} \right) dx - \int_{-L}^L y_n m \omega^2 \sum_{m=0}^{\infty} a_m (y_m + e\phi_m) dx - \sum_i \int_{-L}^L y_n [P]_i \delta(x - x_i) dx = 0 \quad (n = 0, 1, 2, \dots) \quad (6a)$$

and

$$\int_{-L}^L \phi_n \sum_{m=0}^{\infty} a_m \frac{d}{dx} \left( GJ \frac{d\phi_m}{dx} \right) dx + \int_{-L}^L \phi_n m \omega^2 \sum_{m=0}^{\infty} a_m (ey_m + k^2\phi_m) dx + \sum_i \int_{-L}^L \phi_n [T]_i \delta(x - x_i) dx = 0 \quad (n = 0, 1, 2, \dots) \quad (6b)$$

With the use of the definition of  $\delta(x - x_i)$ , the differential equations shown in equations (4), and the orthogonality conditions in equations (5), the subtraction of equation (6b) from (6a) can be simplified to

$$a_n N_n (\omega_n^2 - \omega^2) - \sum_i \left( [P]_i [y_n]_i + [T]_i [\phi_n]_i \right) \quad (7)$$

( $n = 0, 1, 2, \dots$ )

If  $[P]_i$  and  $[T]_i$  are expanded in terms of  $a_n y_n$  and  $a_n \phi_n$  (equations (2) and (3)), equations (7) would lead to a set of simultaneous homogeneous equations in the coefficients  $a_n$ . The order of the frequency determinant that would result from the direct consideration of this set of equations would be equal to the number of terms used in the deflection functions (equations (3)). A simplified frequency determinant can be developed, however, in the following manner.

Simplified frequency determinant.- The solution of equation (7) for  $a_n$  gives

$$a_n = \frac{1}{N_n(\omega_n^2 - \omega^2)} \sum_i \left( [P]_i [y_n]_i + [T]_i [\phi_n]_i \right) \quad (8)$$

The loads  $[P]$  and  $[T]$  due to any modification at a particular station  $x_j$  can be written

$$[P]_j = \sum_{n=0}^{\infty} a_n [P_n]_j \quad (j = 1, 2, \dots, i) \quad (9a)$$

and

$$[T]_j = \sum_{n=0}^{\infty} a_n [T_n]_j \quad (j = 1, 2, \dots, i) \quad (9b)$$

where

$$\left. \begin{aligned} [P_n]_j &= K_j \left( [y_n]_j + e_j [\phi_n]_j \right) \\ [T_n]_j &= K_j \left( e_j [y_n]_j + k_j^2 [\phi_n]_j \right) \end{aligned} \right\} \quad (10)$$

The substitution of equation (8) into equations (9) gives

$$\left. \begin{aligned} [P]_j &= \sum_i [P]_i \sum_{n=0}^{\infty} \frac{[P_n]_j [y_n]_i}{N_n(\omega_n^2 - \omega^2)} + \sum_i [T]_i \sum_{n=0}^{\infty} \frac{[P_n]_j [\phi_n]_i}{N_n(\omega_n^2 - \omega^2)} \\ [T]_j &= \sum_i [P]_i \sum_{n=0}^{\infty} \frac{[T_n]_j [y_n]_i}{N_n(\omega_n^2 - \omega^2)} + \sum_i [T]_i \sum_{n=0}^{\infty} \frac{[T_n]_j [\phi_n]_i}{N_n(\omega_n^2 - \omega^2)} \end{aligned} \right\} \quad (11)$$

(j = 1, 2, \dots, i)

where  $[P]_i$  and  $[T]_i$  are unknowns.



Equations (11) can be written for every station at which a modification is made; therefore two equations are obtained for every concentrated mass or spring on the structure. These equations are shown in table 1.

Since these equations are homogeneous in  $[P]_i$  and  $[T]_i$ , the determinant of these equations, which is the bracketed part of table 1, must vanish in order to have real values of  $[P]_i$  and  $[T]_i$ . The order of this simplified frequency determinant is now proportional to the number of modifications instead of the number of terms in the expansion. Although each term of the determinant is an infinite series, for practical problems it is necessary to include only the first few terms of each series, each term of which corresponds to a known mode of the unmodified structure. As the examples show subsequently, at least as many frequencies and modes of the unmodified beam must be known as the number of frequencies and modes of the modified beam desired. The rapid convergence of the calculated frequencies to the exact frequencies as the number of terms of the series is increased is shown in the examples.

Calculation of the modes and frequencies.- For most cases perhaps the simplest method of obtaining the natural frequency  $\omega$  of the modified beam from the frequency determinant is to evaluate the determinant for several trial values of  $\omega$  in the expected vicinity of the natural frequency and plot the value of the determinant against the trial frequency  $\omega$ . In most cases only three or four points are needed to obtain the natural frequency.

This method of evaluating the natural frequencies is especially advantageous when the effect of several loading conditions in which the modifications do not change position but only magnitude is being studied. Under this condition the value of each summation in all the elements of the determinant change only in proportion to the changes in the values of  $K_i$ .

To calculate the natural mode after its frequency has been established, the relative values of  $[P]_i$  and  $[T]_i$  have to be obtained. In order to find these relative values, one of the loads is given an arbitrary value of unity and the resulting set of nonhomogeneous equations is solved simultaneously for the relative values of the other loads. In the process one of the equations can be discarded, but it has been found best simply to add two of the equations together before the solution is made.

With the values of the frequency and the loads  $[P]_i$  and  $[T]_i$ , equation (8) is used to evaluate the coefficients  $a_n$ , and then the modal functions can be found from equations (3). At least the same number of primary-beam modes should be included in the calculation of the modified-beam mode as terms used in the evaluation of the series.

The convergence of the calculated mode to the exact mode as the number of primary-beam modes included in the calculation is increased is shown in the examples.

### EXAMPLES

In order to illustrate the method and to show the accuracy with which the frequencies and modes of the modified beams can be calculated, the uncoupled symmetrical bending frequencies and modes of a uniform free-free beam are calculated with the following modifications made at the elastic axis of the beam (see fig. 3):

Example 1 - Equal masses placed at the tips of the beam

Example 2 - Masses placed at the center line of the span and third points of the beam

Example 3 - A spring placed at the center line of the beam

Because the masses and springs were placed at the elastic axis,  $e_i$  and  $k_i$  are zero or

$$[P]_i = K_i [y]_i \quad (12a)$$

and

$$[T]_i = 0 \quad (12b)$$

Also, because only the symmetrical uncoupled modes are required, only symmetrical uncoupled modes and frequencies of the primary beam are used.

Modes and frequencies of the unmodified beam. - The frequencies of the unmodified uniform free-free beam are

$$\omega_n^2 = f_n^4 \frac{EI}{mL^4}$$

where

$$\begin{array}{ll} f_0^4 = 0 & f_3^4 = 5,571 \\ f_1^4 = 31.28 & f_4^4 = 19,262 \\ f_2^4 = 913.6 & f_5^4 = 49,600 \end{array}$$

The constants  $N$  are

$$N_0 = 2mL$$

and, for  $n \neq 0$ ,

$$N_n = \frac{1}{2} mL$$

The modes of the unmodified uniform beam, normalized to give a unit tip deflection, are given by the following equations:

$$y_n = \frac{1}{2} \left( \frac{\cosh f_n \frac{x}{L}}{\cosh f_n} + \frac{\cos f_n \frac{x}{L}}{\cos f_n} \right) \quad (n = 0, 1, 2, \dots)$$

Example 1: Calculation of the frequencies and modes of a uniform free-free beam with concentrated masses at the tips (fig. 3(b)).— Example 1 illustrates the use of the method for the case of symmetrical modifications. The placement of the masses at the tips gives a severe test of the accuracy of the method. The results obtained for this example are compared with an exact solution of the differential equation.

Since the modifications are symmetrically placed, that is,  $x_1 = x_2 = L$ ,

$$[y_n]_1 = [y_n]_2$$

With the use of this condition and equation (12), the frequency determinant for equal tip masses can be written as

$$-\frac{mL}{M} + 2\omega^2 \sum_{n=0}^{\infty} \frac{[y_n]_1^2}{\frac{N_n}{mL} (\omega_n^2 - \omega^2)} = 0$$

The results obtained from the solution of this frequency equation when  $\frac{mL}{M} = 4$  are given in the following table for

$$\omega^2 = r^4 \frac{EI}{mL^4}$$

Mode	Number of terms in series	Calculated $f^2$	Exact $f^2$	Percent error
First	3	4.15	4.15	0
First	2	4.17	4.15	.5
Second	4	25.0	24.9	.4
Second	3	25.2	24.9	1.2
Third	6	65.0	64.8	.3
Third	5	65.2	64.8	.6
Third	4	65.8	64.8	1.5

Examination of this table shows that the use of the series with a minimum number of terms evaluates the frequency of first three modes with errors of less than 2 percent, which is sufficient for most uses.

Once the frequencies of the modified beam are found, the proportion of each of the primary-beam modes present in a modified-beam mode is calculated from equation (8). Because the values of  $[P]_i$  are equal, equation (8) becomes

$$a_n = - \frac{2[y_n]_1^2}{N_n(\omega_n^2 - \omega^2)}$$

after  $[P]_i$  is assumed to be unity.

The comparison of the solution for the first three modified-beam modes with the exact modes is shown in figure 4.

Example 2: Calculation of the frequencies and modes of a uniform free-free beam with concentrated masses at the center line and third points of the beam (fig. 3(c)). - Example 2 illustrates the use of the method when more than one modification is made to the primary structure. The results for this example are compared with calculations obtained from a matrix iteration process which is considered as exact.

The magnitudes of the masses and the assumed geometrical properties of the beam are shown in figure 3(c). Since, as in example 1, two of the masses are symmetrically placed, the frequency determinant can be reduced to

$$\left[ \begin{array}{c} \begin{array}{c} \text{[P]}_C \\ -1 + 2 \sum_{n=0}^{\infty} \frac{K_C [\bar{y}_n]_C^2}{N_n (\omega_n^2 - \omega^2)} \\ \\ 2 \sum_{n=0}^{\infty} \frac{K_A [\bar{y}_n]_C [\bar{y}_n]_A}{N_n (\omega_n^2 - \omega^2)} \end{array} \\ \begin{array}{c} \text{[P]}_A \\ \sum_{n=0}^{\infty} \frac{K_C [\bar{y}_n]_A [\bar{y}_n]_C}{N_n (\omega_n^2 - \omega^2)} \\ \\ -1 + \sum_{n=0}^{\infty} \frac{K_A [\bar{y}_n]_A^2}{N_n (\omega_n^2 - \omega^2)} \end{array} \end{array} \right] = 0$$

The results obtained from the solutions of this frequency determinant are as follows:

Mode	Number of terms in series	Calculated $\omega$	Exact $\omega$	Percent error
First	3	21.8	21.8	0
First	2	22.0	21.8	.9
Second	5	91.4	91.4	0
Second	4	93.0	91.4	1.7
Second	3	100.7	91.4	10.2
Third	5	261.8	261.3	.2
Third	4	269.5	261.3	3.1

For this example equation (8) becomes

$$a_n = -\frac{1}{N_n (\omega_n^2 - \omega^2)} \left( 2 [\bar{y}_n]_C K_C [\bar{y}]_C + [\bar{y}_n]_A K_A [\bar{y}]_A \right)$$

when  $[\bar{P}]_C$  is assumed to be unity and the relative value of  $[\bar{P}]_A$  is calculated from the determinant just shown, in the manner previously described. The comparison of the calculated and the exact solutions for the first three modes is shown in figure 5.

The large error in the calculation of the frequency of the second mode with the minimum number of terms is caused by the large peak deflection that occurs in the second mode near the point where the masses are placed.

Example 3: Calculation of the frequencies and modes of a uniform free-free beam with a spring at the center line.- Example 3 is presented to show how the method is used when a beam is modified by the addition of a spring instead of a mass. The results obtained are compared with an exact solution of the differential equation.

Because only one modification is used, the frequency determinant consists of only one term

$$1 + \sum_{n=0}^{\infty} \frac{\alpha [y_n]_0^2}{N_n (\omega_n^2 - \omega^2)} = 0$$

The results obtained from the solution of this frequency equation when a spring is used with a spring constant

$$\alpha = 20 \frac{EI}{L^3}$$

are tabulated as follows for

$$\omega^2 = f^4 \frac{EI}{mL^4}$$

Mode	Number of terms in series	Calculated f <sup>2</sup>	Exact f <sup>2</sup>	Percent error
First	3	2.49	2.49	0
First	2	2.51	2.49	.8
Second	4	7.01	7.00	.1
Second	3	7.02	7.00	.3
Third	5	30.6	30.6	0
Third	4	30.6	30.6	0

The proportion of each of the primary-beam modes present in the modified-beam modes, as in other examples, is obtained from equation (8), which can be written

$$a_n = - \frac{[y_n]_0}{N_n (\omega_n^2 - \omega^2)}$$

The comparison of the solutions for the first three modified-beam modes with the exact solutions is shown in figure 6.

#### CONCLUDING REMARKS

The method presented herein can be used once the frequencies and modes of the unmodified structure are known. Thus the method is advantageous when the modes and frequencies of the unmodified structure either are known or are required along with the modes and frequencies of the structure with various modifications.

The amount of time necessary to calculate a natural frequency and mode of a modified structure once the primary-structure modes and frequencies are known by the method presented herein is generally shorter than any of the standard methods. Also, in most cases that can be encountered, all the calculations necessary can be done with a slide rule. Most of the calculations done for one frequency and mode can be used in the calculations of other frequencies and modes.

Langley Aeronautical Laboratory  
National Advisory Committee for Aeronautics  
Langley Air Force Base, Va., April 24, 1950

#### REFERENCE

1. Duncan, W. J.: Galerkin's Method in Mechanics and Differential Equations. R. & M. No. 1798, British A.R.C., 1937.

TABLE 1  
EQUATIONS FOR STRUCTURE WITH CONCENTRATED MASSES AND SPRINGS

$[P]_1$	$[T]_1$	$[P]_2$	$[T]_2$	...	$[P]_1$	$[T]_1$
$-1+ \sum_{n=0}^{\infty} \frac{[P]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [y]_2}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [\phi]_2}{K_n (\omega_n^2 - \omega^2)}$	...	$\sum_{n=0}^{\infty} \frac{[P]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$
$\sum_{n=0}^{\infty} \frac{[T]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$-1+ \sum_{n=0}^{\infty} \frac{[T]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_1 [y]_2}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_1 [\phi]_2}{K_n (\omega_n^2 - \omega^2)}$	...	$\sum_{n=0}^{\infty} \frac{[T]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$
$\sum_{n=0}^{\infty} \frac{[P]_2 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_2 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$	$-1+ \sum_{n=0}^{\infty} \frac{[P]_2 [y]_2}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_2 [\phi]_2}{K_n (\omega_n^2 - \omega^2)}$	...	$\sum_{n=0}^{\infty} \frac{[P]_2 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_2 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$
$\sum_{n=0}^{\infty} \frac{[T]_2 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_2 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_2 [y]_2}{K_n (\omega_n^2 - \omega^2)}$	$-1+ \sum_{n=0}^{\infty} \frac{[T]_2 [\phi]_2}{K_n (\omega_n^2 - \omega^2)}$	...	$\sum_{n=0}^{\infty} \frac{[T]_2 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_2 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$
.....						
$\sum_{n=0}^{\infty} \frac{[P]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [y]_2}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [\phi]_2}{K_n (\omega_n^2 - \omega^2)}$	...	$-1+ \sum_{n=0}^{\infty} \frac{[P]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[P]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$
$\sum_{n=0}^{\infty} \frac{[T]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_1 [y]_2}{K_n (\omega_n^2 - \omega^2)}$	$\sum_{n=0}^{\infty} \frac{[T]_1 [\phi]_2}{K_n (\omega_n^2 - \omega^2)}$	...	$\sum_{n=0}^{\infty} \frac{[T]_1 [y]_1}{K_n (\omega_n^2 - \omega^2)}$	$-1+ \sum_{n=0}^{\infty} \frac{[T]_1 [\phi]_1}{K_n (\omega_n^2 - \omega^2)}$





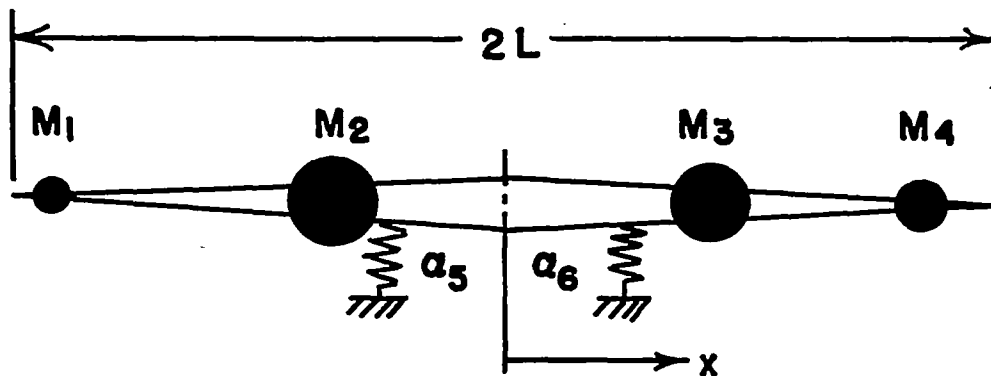


Figure 1.— Structure used in theoretical analysis.

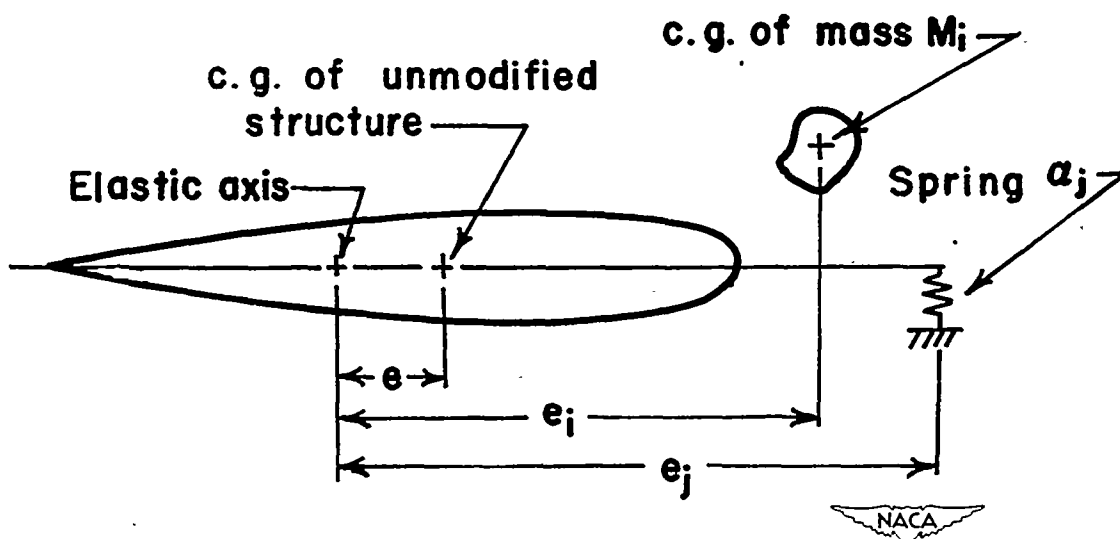
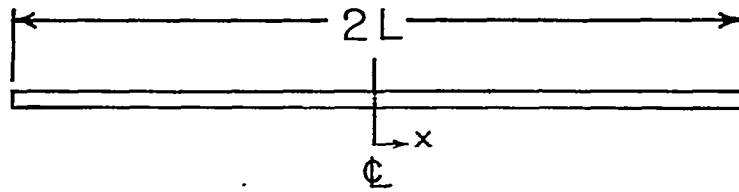
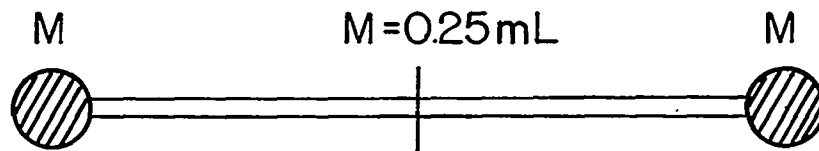


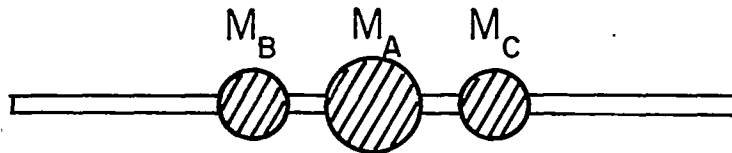
Figure 2.— Definition of the eccentricities.



(a) Unmodified uniform free-free beam.



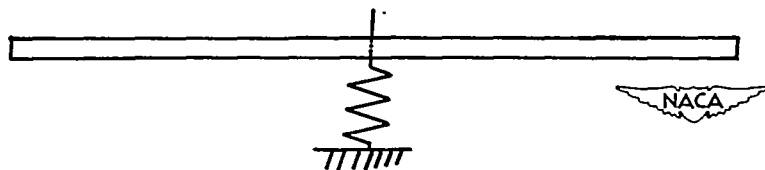
(b) Example 1—Addition of masses at tip.



$$0.25M_A = M_B = M_C = 0.942$$

$$\sqrt{\frac{EI}{mL^4}} = 5.514$$

(c) Example 2—Addition of multiple masses.



(d) Example 3—Addition of a spring.

Figure 3.—Unmodified beam with modifications used in examples.

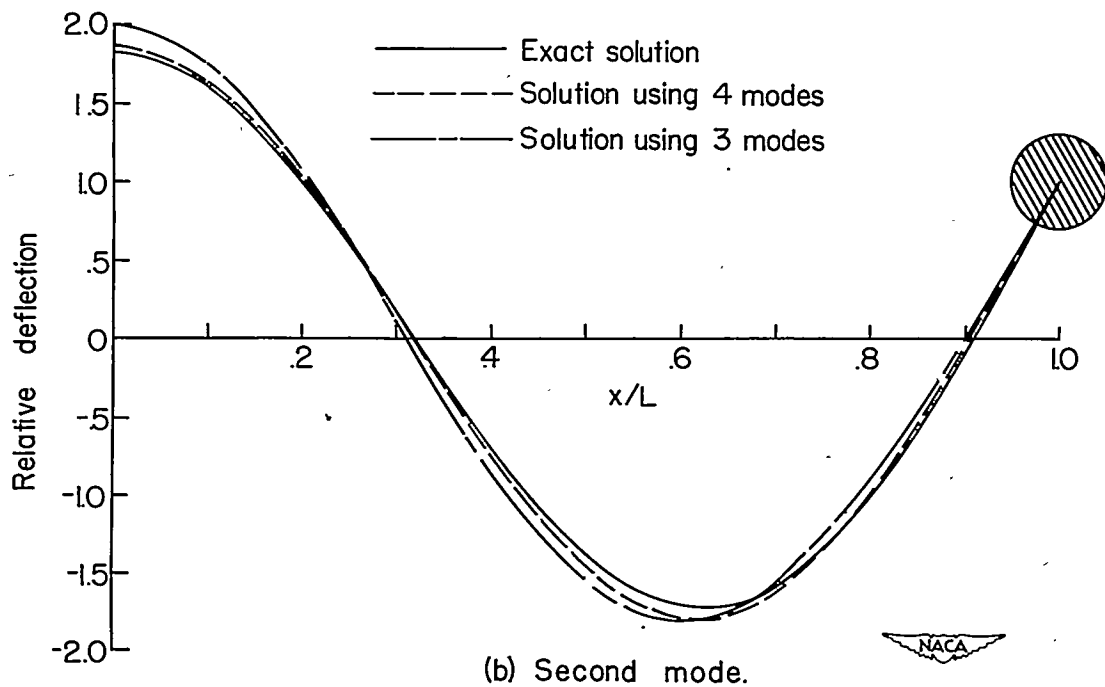
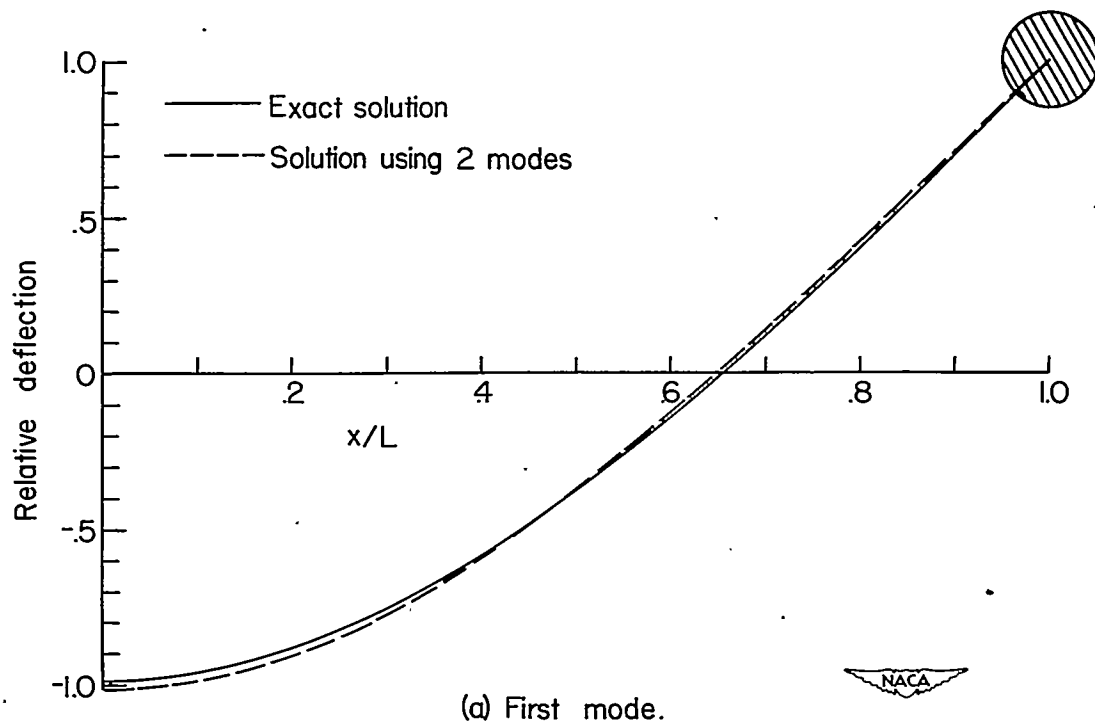
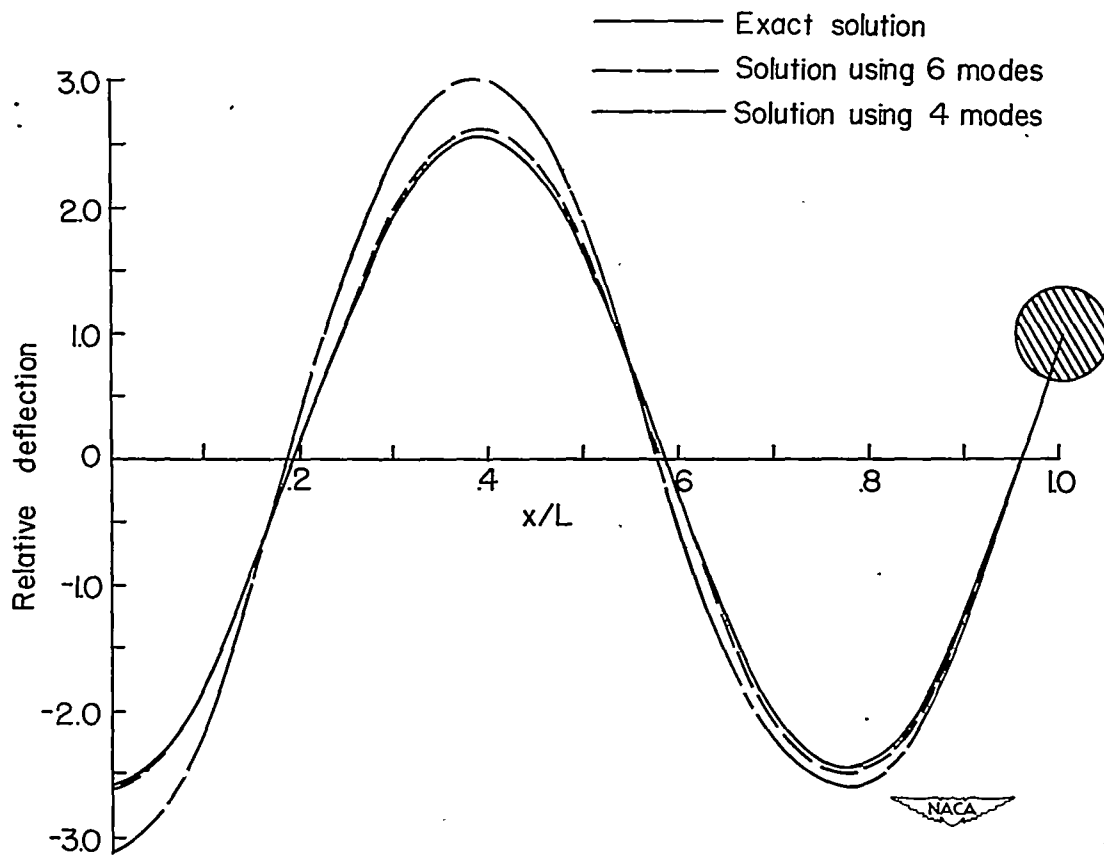


Figure 4.—Comparison between calculated and exact solutions in example I.



(c) Third mode.

Figure 4.- Concluded.

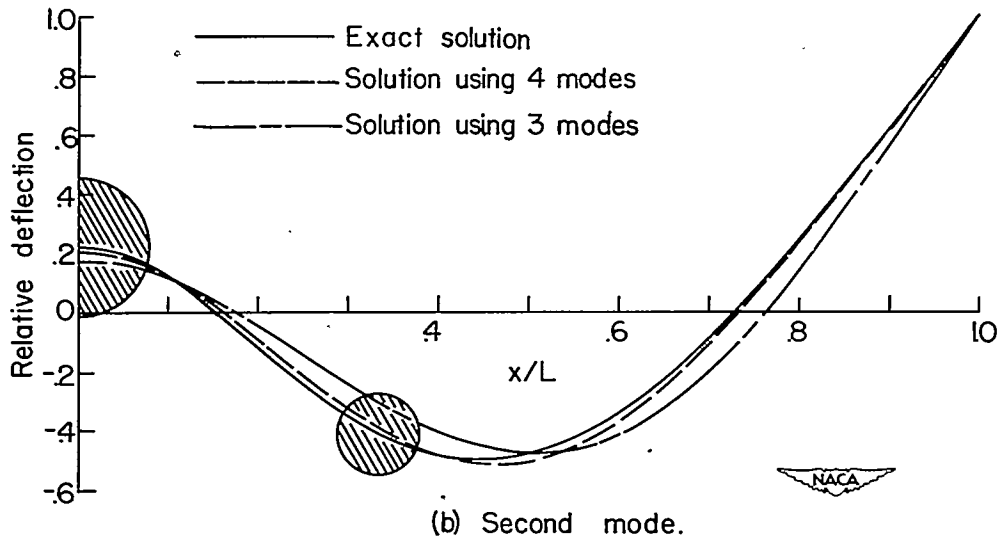
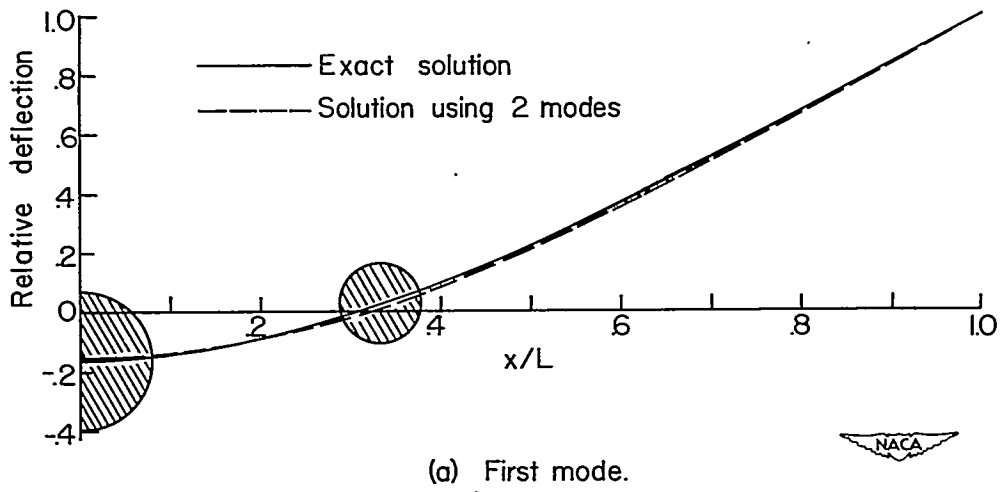


Figure 5.—Comparison between calculated and exact solutions in example 2.

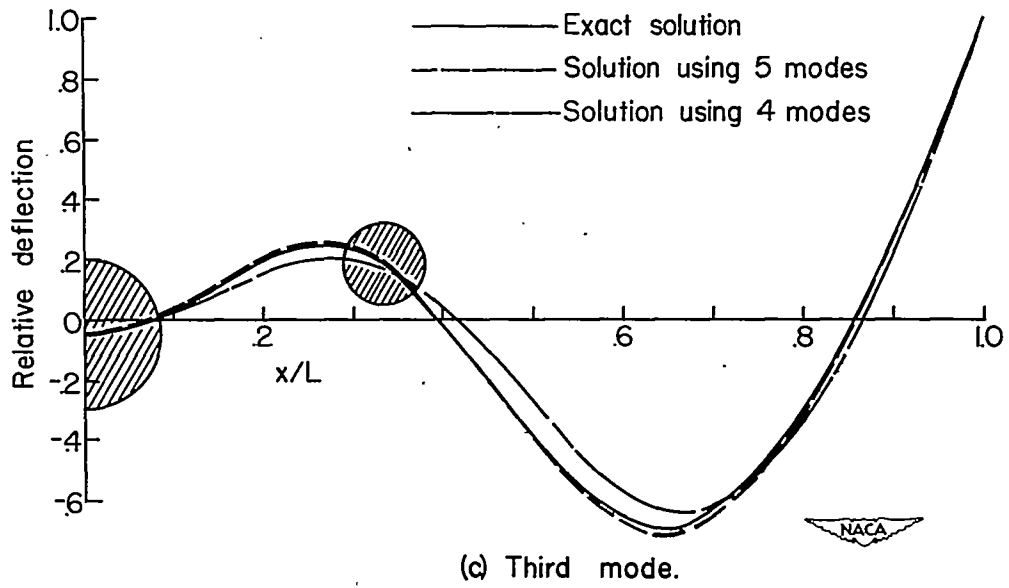
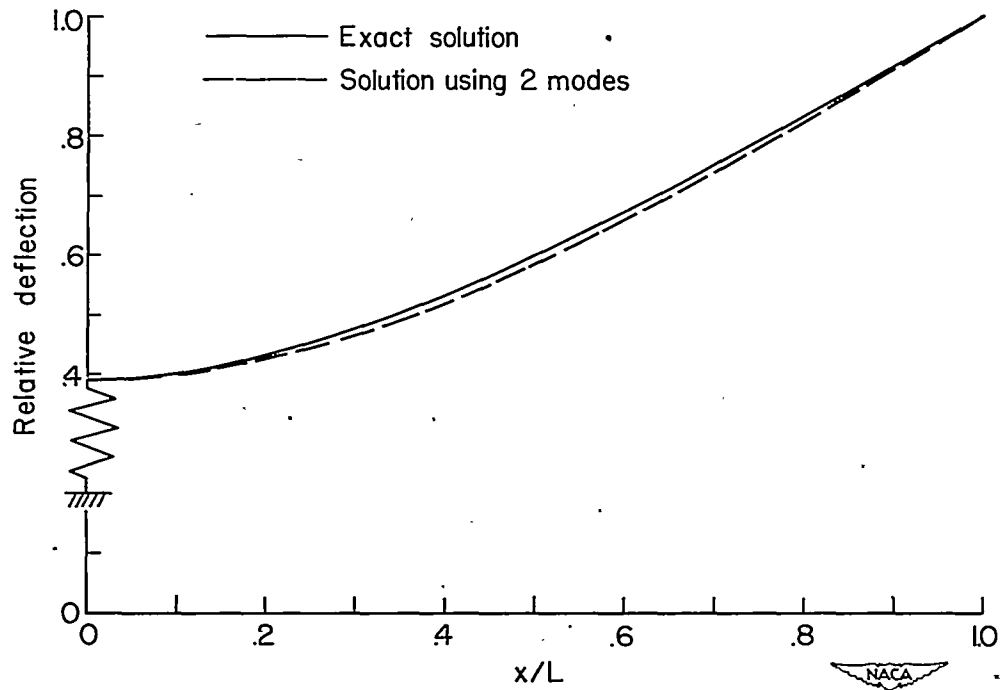
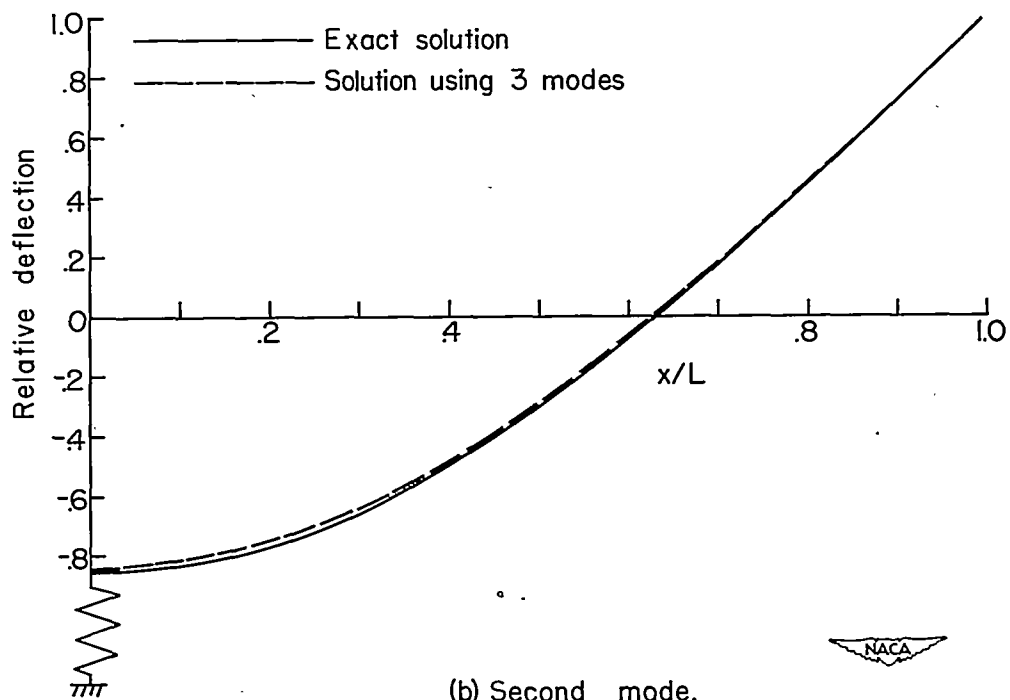


Figure 5.-Concluded

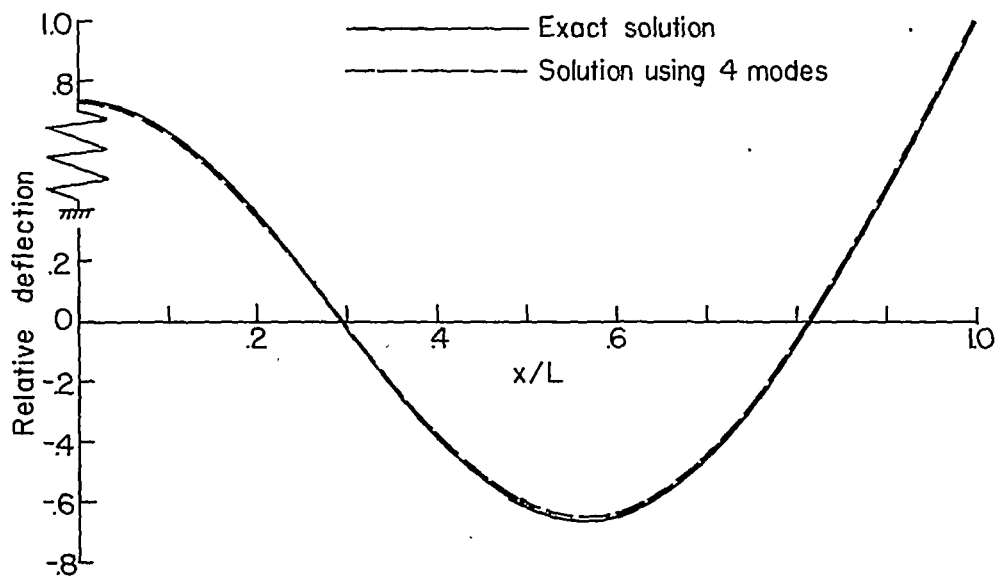


(a) First mode.



(b) Second mode.

Figure 6.—Comparison between calculated and exact solutions in example 3.



(c) Third mode.



Figure 6.- Concluded.