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ON THE INTERPRETATION OF COMBINED TORSION AND

TENSION TESTS OF THIN-WALL TUBES

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SUMMARY

General ways of testing thin-wall tubes under combined tension and torsion as a means of checking the various theories of plasticity are discussed. Suggestions also are given for the interpretation of the tests.

INTRODUCTION

Combined torsion and tension (or compression) of thin-wall tubes constitutes one of the few testing arrangements in which a fairly general state of uniform stress can be realized without too great experimental difficulties. It is not surprising, therefore, that this arrangement has been frequently used to check the various theories of plasticity. (See, for instance, references 1 to 5.) Unfortunately, such combined tension and torsion tests are often conducted so as to keep the ratio of axial load and torque constant during any one test. In this case, the directions of the principal stresses as well as the ratios of their intensities are preserved during the plastic deformation, and various theories of plasticity furnish identical predictions. Tests of this particular type therefore do not provide a check of these theories.

In the present paper, more general ways of testing thin-wall tubes under combined tension and torsion are discussed, and suggestions are given for the interpretation of such tests.

THEORIES OF PLASTIC DEFORMATION AND PLASTIC FLOW

In the mathematical theory of plasticity two kinds of stress-strain relations are currently used to describe the mechanical behavior of quasi-isotropic metals in the strain-hardening range. In this paper, the theories of these two groups are called theories of plastic deformation and theories of plastic flow, following an apt proposal of A. A. Ilyushin (reference 6). A recent paper by G. H. Handelman, C. C. Lin, and W. Prager

(reference 7) contains a detailed discussion of the general stress-strain relations of these theories; the present discussion can therefore be restricted to the special case of combined tension and torsion of thin-wall tubes.

The stress-strain relation of a theory of plastic deformation establishes a one-to-one correspondence between the instantaneous states of stress and strain. The stress-strain relation of a theory of plastic flow, on the other hand, establishes a one-to-one correspondence between the infinitesimal increments of stress and strain when the instantaneous state of stress is known. A stress-strain relation of this kind will therefore contain the instantaneous stress components in addition to the differentials of the components of stress and strain; it must, of course, be homogeneous in these differentials, and if there is to be a one-to-one correspondence between the differentials of stress and strain, the stress-strain relation must be linear in these differentials. It is often convenient to avoid differentials by replacing them by the first derivatives of the components of stress and strain with respect to time. When written in this form, the stress-strain relations of the theories of plastic flow appear as linear forms in the time rates of change of stress and strain with coefficients which depend on the instantaneous stress. It is important to remark, however, that these stress-strain relations do not represent any viscosity effects in spite of the appearance of the rates of stress and strain. In fact, since these rates appear in a homogeneous form, the relation between stresses and strains is not affected by an arbitrary distortion of the time scale; accordingly, time enters only as a parameter which is convenient for the detailed description of the loading process. If the loads applied to a plastic body vary in such a manner that there is at least one load which always varies when the other loads vary, the intensity of this load may be used as a measure of time.

If the stress-strain relations of a theory of plastic deformation are differentiated with respect to time, the resulting relations will also be linear and homogeneous in the time derivatives of stress and strain, and hence resemble the stress-strain relations of the theory of plastic flow. However, the stress-strain relations obtained in this manner, while involving the time derivatives of stress and strain, can be integrated with respect to time. The stress-strain relations of the theories of plastic flow, on the other hand, are not supposed to be integrable in this manner.

DEVIATIONS OF STRESS AND STRAIN - POWERS

AND INVARIANTS OF THE STRESS DEVIATION

In the following discussion, the symbols are defined as

R   mean radius of tubular test specimen

w   wall thickness of test specimen

P   axial force

T   torque

σ axial stress $(P/2\pi Rw)$

τ shearing stress $\left(T/2\pi R^2 w\right)$

ϵ unit extension in axial direction

η, η^* unit contractions in circumferential and
               radial directions, respectively

θ angle of twist per unit length

γ shearing strain $(R\theta/2)$

For convenience, the axial force P and the axial stress σ are assumed
to represent tension in the present paper. In the case of compression,
the signs of P and σ must be changed in all formulas. Also, contrary
to present engineering practice, the shearing strain is defined herein
as $R\theta/2$ rather than $R\theta$. This is necessary if unduly complicated
formulas are to be avoided.

With the symbols as defined, the tensors of stress and strain are
given by

$$\begin{bmatrix} \sigma & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \epsilon & \gamma & 0 \\ \gamma & -\eta & 0 \\ 0 & 0 & -\eta^* \end{bmatrix} \tag{1}$$

The mean normal stress equals $\sigma/3$, and the mean normal strain,
$\frac{1}{3}(\epsilon - \eta - \eta^*)$.

The deviations of stress and strain are obtained by subtracting $\sigma \underline{I}/3$
and $(\epsilon - \eta - \eta^*)\underline{I}/3$. from the tensors of stress and strain, respectively,
where

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Thus,

$$\underline{S} = \begin{bmatrix} 2\sigma/3 & \tau & 0 \\ \tau & -\sigma/3 & 0 \\ 0 & 0 & -\sigma/3 \end{bmatrix} \tag{3}$$

and

$$\underline{E} = \begin{bmatrix} (2\epsilon + \eta + \eta*)/3 & \gamma & 0 \\ \gamma & -(\epsilon + 2\eta - \eta*)/3 & 0 \\ 0 & 0 & -(\epsilon - \eta + 2\eta*)/3 \end{bmatrix} \tag{4}$$

are the deviations of stress and strain. In the present paper, underscoring is used to indicate tensors.

The square of the stress deviation is obtained by squaring equation (3):

$$\underline{S}^2 = \begin{bmatrix} 4\sigma^2/9 + \tau^2 & \sigma\tau/3 & 0 \\ \sigma\tau/3 & \sigma^2/9 + \tau^2 & 0 \\ 0 & 0 & \sigma^2/9 \end{bmatrix} \tag{5}$$

Similarly,

$$\underline{S}^3 = \begin{bmatrix} 8\sigma^3/27 + \sigma\tau^2 & \sigma\tau^2/3 + \tau^3 & 0 \\ \sigma\tau^2/3 + \tau^3 & -\sigma3/27 & 0 \\ 0 & 0 & -\sigma^3/27 \end{bmatrix} \tag{6}$$

As is well known, the traces of the tensors $\underline{S}^2$ and $\underline{S}^3$ (i.e., the sums of the terms in the principal diagonals) are independent of the particular Cartesian coordinate system to which these tensors may be referred. It will be convenient to define as the second- and third-order invariants of $\underline{S}$ one-half the trace of $\underline{S}^2$ and one-third the trace of $\underline{S}^3$, respectively:

$$J = \frac{1}{3}\sigma^2 + \tau^2 \tag{7}$$

$$K = \frac{2}{27}\sigma^3 + \frac{1}{3}\sigma\tau^2 \tag{8}$$

Inspection of equations (3), (5), (6), (7), and (8) shows that

$$\underline{S}^3 = J\underline{S} + K\underline{I} \tag{9}$$

This is the special form which the Hamilton-Cayley theorem (reference 8) assumes for the stress deviation. Because of this relation, the third and all higher powers of $\underline{S}$ can be expressed in terms of $\underline{S}$, $\underline{S}^2$, and powers of the invariants J and K, as is seen by multiplying equation (9) repeatedly by $\underline{S}$ and using equation (9) to reduce the right-hand sides of the resulting equations.

REVERSIBLE AND PERMANENT STRAINS - LOADING AND UNLOADING

The theories of plastic deformation as well as those of plastic flow assume the total strain to be the sum of a reversible and a permanent component, often called elastic and plastic strains. Within the elastic range, the permanent strains vanish and all strains are reversible. Once the elastic limit has been exceeded, complete unloading of a test specimen, which has been under a state of homogeneous stress, will reveal the permanent or plastic strain associated with this state of stress. That component of strain which disappears during the unloading process is the reversible or elastic strain.

It is generally assumed that each of these two components of strain is related to the stress in a manner which does not involve the other component. In particular, the reversible strain is assumed to be related to the stress by means of the generalized law of Hooke. Moreover, it is generally assumed that all changes of volume are completely reversible, so that the permanent strain involves a change of shape but no change of volume.

In the following discussion, reversible and permanent strains will be indicated by $'$ and $''$, respectively. The assumptions just stated are then expressed by the following equations:

$$\left. \begin{aligned} \epsilon' &= \sigma/E_o \\[6pt] \eta' &= \eta^{*'} \\[6pt] &= \upsilon_o \epsilon' \\[6pt] &= \upsilon_o \sigma/E_o \\[6pt] \gamma' &= \tau/2G_o \end{aligned} \right\} \tag{10}$$

$$\eta'' + \eta^{*'} = \epsilon'' \tag{11}$$

where E_o, G_o, and υ_o denote the values of Young's modulus, shear modulus, and Poisson's ratio in the elastic range. Since

$$E_o = 2G_o(1 + \upsilon_o) \tag{12}$$

substitution of equation (10) into equation (4) and comparison with equation (3) yield

$$\underline{E}' = \underline{S}/2G_0 \tag{13}$$

The permanent strain is therefore given by

$$\underline{E}'' = \underline{E} - \underline{E}'$$

$$= \underline{E} - \underline{S}/2G_0 \tag{14}$$

Reference to equations (3) and (4) shows equation (14) to yield only three independent scalar equations. Together with equation (11), these three equations permit the permanent strains ϵ'', η'', $\eta^{*''}$, and γ'' to be computed from the measured total strains ϵ, η, η^*, γ and the stresses σ and τ; they are

$$\epsilon'' = \frac{1}{3}(2\epsilon + \eta + \eta^*) - \frac{2\sigma}{3G_0} \tag{15}$$

$$\eta'' = -\frac{1}{3}(\epsilon + 2\eta - \eta^*) + \frac{\sigma}{3G_0} \tag{16}$$

and

$$\gamma'' = \gamma - \frac{\tau}{2G_0} \tag{17}$$

Before specific stress-strain relations can be formulated, a criterion for loading and unloading must be adopted. In the case of simple tension or compression, loading corresponds to an increase in σ^2 and unloading, to a decrease. For the more general states of stress considered herein, it will be assumed that loading corresponds to an increase of the invariant J (equation (7)) and unloading, to a decrease. Thus, for loading,

$$\frac{1}{3}\sigma\dot{\sigma} + \tau\dot{\tau} > 0 \tag{18}$$

and for unloading,

$$\frac{1}{3}\sigma\dot{\sigma} + \tau\dot{\tau} < 0 \tag{19}$$

where the dots denote differentiation with respect to time.

STRESS-STRAIN RELATIONS

Discussing theories of plastic deformation, W. Prager (reference 9) established the most general stress-strain relation which is compatible with certain simple postulates. A useful transformation of this stress-strain relation is found in section 5 of reference 7. With

$$\alpha = K^2/J^3 \tag{20}$$

and

$$\underline{T} = \frac{K}{J^2}\left(\underline{S}^2 - \frac{2}{3}\,J\underline{I}\right) \tag{21}$$

this stress-strain relation may be written in the form

$$2G_o\underline{E}'' = f(J,\alpha)\left[\underline{S} + \beta(\alpha)\underline{T}\right] \tag{22}$$

For combined tension and torsion of thin-wall tubes, equation (22) may be further transformed as follows. According to equation (20),

$$\alpha = \frac{\sigma^2(2\sigma^2 + 9\tau^2)^2}{27(\sigma^2 + 2\tau^2)^3}$$

$$= \frac{\rho^2(2\rho^2 + 9)^2}{27(\rho^2 + 3)^3} \tag{23}$$

where

$$\rho = \sigma/\tau \tag{24}$$

Moreover,

$$J = \frac{1}{3}\,(\sigma^2 + 3\tau^2)$$

$$= \frac{\tau^2}{3}\,(\rho^2 + 3) \tag{25}$$

The function $f(J,\alpha)$ can therefore be considered as a function of τ and ρ:

$$f(J,\alpha) = \varphi(\tau,\rho) \tag{26}$$

Similarly, the function $\beta(\alpha)$ can be considered as a function of ρ:

$$\beta(\alpha) = \psi(\rho) \tag{27}$$

Equation (22) may thus be written in the form:

$$2G_0\underline{E}'' = \varphi(\tau,\rho) \left[\underline{S} + \psi(\rho)\underline{T}\right] \tag{28}$$

where

$$T = \tau \frac{\rho(2\rho^2 + 9)}{3(\rho^2 + 3)^2} \begin{bmatrix} 2\rho^2/9 + 1/3 & \rho/3 & 0 \\ \rho/3 & -\rho^2/9 + 1/3 & 0 \\ 0 & 0 & -\rho^2/9 - 2/3 \end{bmatrix} \tag{29}$$

The functions φ and ψ can be determined by a series of tests during each of which the value of ρ is kept constant. According to equation (28), the ratio ϵ''/γ'' will remain constant during each of these tests:

$$\frac{\epsilon''}{\gamma''} = \frac{6\rho(\rho^2 + 3)^2 + 3\rho(2\rho^2 + 9)(2\rho^2/9 + 1/3)\,\psi(\rho)}{9(\rho^2 + 3)^2 + \rho^2(2\rho^2 + 9)\,\psi(\rho)} \tag{30}$$

Each observed value of ϵ''/γ'' corresponding to a certain value of ρ therefore furnishes the value of ψ for this value of ρ.

After $\psi(\rho)$ has been obtained, the function $\varphi(\tau,\rho)$ can be determined from the shear component of equation (22):

$$\frac{2G_0\gamma''}{\tau} = \varphi(\tau,\rho)\left[1 + \psi(\rho)\frac{\rho^2(2\rho^2 + 9)}{9(\rho^2 + 3)^2}\right] \tag{31}$$

The bracketed term on the right-hand side of this equation depends only on ρ and thus remains constant during each of the aforementioned tests. A graph of the function $\varphi(\tau,\rho)$ for the particular value of ρ which is maintained constant during the test is therefore obtained by plotting

$$\varphi(\tau,\rho) = \frac{18G_0\gamma''(\rho^2 + 3)^2}{\tau\left[9(\rho^2 + 3)^2 + \rho^2(2\rho^2 + 9)\,\psi(\rho)\right]} \tag{32}$$

against τ. Here, γ'' must be computed from the observed shearing strain γ in accordance with equation (17).

As regards the theories of plastic flow, the differential stress-strain relation which is analogous to equation (23) has the form

$$2G_0 d\underline{E}'' = \Phi(\tau,\rho)\left[\underline{S} + \psi(\rho)\underline{T}\right] dJ \tag{33}$$

If $\varphi(\tau,\rho)$ and $\psi(\rho)$ are given, the function $\Phi(\tau,\rho)$ can be determined so that equations (28) and (33) predict the same mechanical behavior for

any test during which ρ is kept constant. During such a test, $\sigma = \rho\tau$, and hence

$$
\left.
\begin{aligned}
d\sigma &= \rho\ d\tau \\
&= \sigma\ d\tau/\tau \\
d\underline{S} &= \underline{S}\ d\tau/\tau \\
d\underline{T} &= \underline{T}\ d\tau/\tau \\
dJ &= 2J\ d\tau/\tau
\end{aligned}
\right\} \tag{34}
$$

Equation (28) thus yields

$$
2G_0\ d\underline{E}'' = \left(\frac{\partial\varphi}{\partial\tau} + \frac{\varphi}{\tau}\right)\left(\underline{S} + \psi\underline{T}\right) d\tau \tag{35}
$$

while equation (33) furnishes

$$
2G_0\ d\underline{E}'' = 2\Phi\ \frac{J}{\tau}\left(\underline{S} + \psi\underline{T}\right) d\tau \tag{36}
$$

On account of equation (25), comparison of equations (35) and (36) yields

$$
\Phi = \frac{3}{2\tau(\rho^2 + 3)}\left(\frac{\partial\varphi}{\partial\tau} + \frac{\varphi}{\tau}\right) \tag{37}
$$

SUGGESTED TEST

If the functions Φ and φ are related to each other by means of equation (37), the stress-strain relations, equations (28) and (33), furnish the same prediction for any test during which ρ is kept constant. A decision between the theories of plastic deformation and plastic flow therefore requires more general tests during which ρ is allowed to vary. For mild steel with negligible strain hardening such tests have been made by K. Hohenemser and W. Prager (reference 3); for materials with pronounced strain hardening, however, systematic tests of this kind do not seem to have been carried out as yet. A possible method by which these tests could be carried out is given in the following discussion.

A decisive difference between the two kinds of theory of plastic action is revealed in a test in which simple torsion is followed by tension during which the torque is maintained constant. At the instant of beginning tension $\sigma = 0$ and thereafter $d\tau = 0$. Accordingly,

$$
dJ = \frac{2}{3}\left(\sigma\ d\sigma + 3\tau\ d\tau\right) = 0 \tag{38}
$$

at this instant, and equation (33) yields

$$d\underline{E}'' = 0 \tag{39}$$

According to the theories of plastic flow, the initial value of the ratio $d\sigma/d\epsilon$ should therefore equal Young's modulus E_0 quite independently of the precise form of the functions Φ and Ψ. According to the theories of plastic deformation, on the other hand,

$$2G_0 d\underline{E}'' = (\underline{S} + \Psi\underline{T})\, d\varphi + \varphi(d\underline{S} + \Psi\, d\underline{T} + \underline{T}\, d\Psi) \tag{40}$$

Now, at the instant of beginning tension $\sigma = 0$, that is, $\rho = 0$, and thereafter $d\tau = 0$, that is, $d\rho = d\sigma/\tau$. Thus, the following normal components in the direction of the axis of the tube are obtained:

$$\left.\begin{aligned} \underline{S} \;)\; & 0 \\ d\underline{S} \;)\; & \tfrac{2}{3}d\sigma \\ \underline{T} \;)\; & 0 \\ d\underline{T} \;)\; & \tfrac{1}{9}d\sigma \\ d\underline{E}'' \;)\; & d\epsilon'' \end{aligned}\right\} \tag{41}$$

Equation (40) yields then

$$9E_0\, d\epsilon'' = \varphi(\tau,0)\,(1 + \upsilon_0)\,\left[6 + \Psi(0)\right] d\sigma \tag{42}$$

where G_0 has been replaced by its value from equation (12). Since $d\epsilon' = d\sigma/E_0$ and $d\epsilon = d\epsilon' + d\epsilon''$,

$$\frac{d\sigma}{d\epsilon} = \frac{9E_0}{9 + \varphi(\tau,0)\,(1 + \upsilon_0)\,\left[6 + \Psi(0)\right]} \tag{43}$$

Similarly,

$$\frac{d\sigma}{d\gamma} = \frac{2G_0}{\partial\varphi/\partial\rho} \tag{44}$$

Since the functions φ and Ψ can be determined from the tests described in the preceding section, the right-hand sides of equations (43) and (44) can be computed as functions of τ and equation (28) can be checked by experiment.

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