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ON SUBSONIC COMPRESSIBLE FLOWS BY A METHOD OF CORRESPONDENCE

I - METHODS FOR OBTAINING SUBSONIC CIRCULATORY COMPRESSIBLE FLOWS ABOUT TWO-DIMENSIONAL BODIES

By Abe Gelbart
Syracuse University

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SUMMARY

By means of the general solutions of the hodograph equations for compressible fluids, certain solutions corresponding to solutions of the hodograph equations of an incompressible fluid are used to find flow patterns of compressible fluids. When the adiabatic equation of state is used, only a general method is outlined.

The method appears to lead to the solution of the problem of subsonic flows with circulation around arbitrary bodies, as the method of Theodorsen does for incompressible fluids. A second paper, part II, illustrates the method for some given bodies. For the linearized equation of state, the results obtained include some of the results of Von Kármán and Tsien, as well as some of the recent results of Bers. The method can be used for flows with circulation as well as without circulation.

INTRODUCTION

It is well known that the nonlinear compressible-flow equations in the physical plane can be reduced to linear equations which have the form of generalized Cauchy-Riemann equations by the change of the independent variables in the physical plane to the independent variables of the hodograph plane. The successful technique applied to the solution of the incompressible-flow problems around given bodies was achieved only because the theory of analytic functions had been developed previously. It would, therefore, be suspected that similar results for the theory of compressible fluids could be obtained if there were developed a theory corresponding to the theory of analytic functions for the generalized Cauchy-Riemann equations representing the flow of a compressible fluid in the hodograph plane.

Development of such a theory has been partly achieved by the theory of Σ-monogenic functions (references 1 and 2). This theory has not yet attained the perfection that the theory of analytic functions has. However, the Σ-monogenic Taylor series and the Σ-monogenic Laplace transforms may be partly able to overcome the weaknesses of this method. Some efforts in this direction are being made in this paper.

Even with the complete theory of Σ-monogenic functions one of the chief drawbacks of the approach lies in the difficulty of transforming the solutions from the hodograph plane to the physical plane.

The method of the theory of correspondence has been very briefly outlined in reference 3. The procedure is to obtain the flow of an incompressible fluid around some given closed body, to transform the complex potential of the flow to the hodograph, and then to obtain the particular Σ-monogenic function of the infinite set of solutions which possesses the desired properties.

It may seem, at first, that this technique is too general to be of practical use, but there already exist some interesting results of the application of this method. Furthermore, this technique is certainly not new. It has been used with much success by Chaplygin, Von Kármán, Tsien, Bergman, Bers, and others. (See references 4 to 12.) However, much is believed to be new in the specific use of this method when employing Σ-monogenic functions. When the method is applied directly to that of a source and a sink of an incompressible fluid, the method yields a source and a sink of a compressible fluid.

It is the aim of this paper to elaborate on the correspondence method in a general way and to apply it in particular to the flow of a compressible fluid under the linearized equation of state. It should be mentioned that this method has already been applied, under the assumption of the linearized equation of state, by others, notably by Chaplygin, von Kármán, Tsien, and Bers.

By dealing with the method in all its generality, a formula is obtained involving an arbitrary analytic function. By choosing particular values of this arbitrary function, of which the derivative is regular in the exterior of a region including the origin, the particular formulas of Tsien (reference 6) and Bers (reference 8) are obtained. Once the arbitrary analytic function is determined for a given flow, velocities, stagnation points, and so forth, can be readily computed. In a second paper by Bartnoff and Gelbart (reference 13), some flows around given bodies are calculated.

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SYMBOLS

k, α, β	constants
p	pressure
ρ	density
q	velocity
p_1, q_1	particular values of the quantities
o	subscript referring to the state of the fluid at rest
u	horizontal velocity
v	vertical velocity
θ	angle that velocity vector makes with horizontal
q_∞	velocity of undisturbed stream
$\tilde{q}$	distorted velocity
a	velocity of sound
w	complex variable in distorted hodograph plane
σ	complex variable in hodograph plane
z	complex variable in physical plane
M	Mach number (q/a)
φ, Φ	potential functions
ψ, Ψ	stream functions
Ω	complex potential $(\varphi + i\psi)$
ζ	complex variable in auxiliary plane
γ	ratio of specific heats

K	constant depending on velocity of undisturbed stream $K = \left(\frac{1 + \sqrt{1 + q^2_\infty}}{q_\infty}\right)$
$G(\zeta)$	complex potential of an incompressible fluid in auxiliary plane
Γ	circulation of an incompressible fluid
τ	an analytic function of the velocity q
$W^{(n)}, i\tilde{W}^{(n)}$	generalized complex powers
$Q^{(n)}, Q^{*(n)}$	generalized real powers
$\grave{}$	generalized differentiation
$\cdot$	generalized multiplication
$C_{n,\nu}$	binomial coefficients
E	generalized complex exponential function
c, c^*	generalized real cosine function
s, s^*	generalized real sine function
L_Σ	generalized Laplace transform
$\overline{(\quad)}$	complex conjugate of $(\quad)$
$\lvert\quad\rvert$	absolute value symbol

COMPRESSIBLE FLOWS UNDER THE ASSUMPTION OF THE ADIABATIC EQUATION OF STATE

The four basic relations for the potential flow of a steady two-dimensional fluid that will be assumed are:

Equation of state, $$p = k\rho^\gamma \qquad (1)$$

where

p pressure

ρ density

γ ratio of the specific heats

k constant

Bernoulli's equation,

$$\frac{q^2}{2} + \int \frac{dp}{\rho} = \text{constant} \tag{2}$$

The continuity equation,

$$\operatorname{div}\left(\frac{\rho}{\rho_0}\vec{q}\right) = 0 \tag{3}$$

and the circulation equation (for irrotational fluids),

$$\operatorname{curl}(\vec{q}) = 0 \tag{4}$$

The velocity of sound is given by

$$a^2 = \frac{dp}{d\rho} \tag{5}$$

From equation (5) the Bernoulli equation can be written in the form

$$\frac{q^2}{2} + \int a^2 \frac{d\rho}{\rho} = \text{constant} \tag{6}$$

which often is more convenient. From these assumptions on the fluid, the first two equations give rise to the relations for density, pressure, the velocity of sound, and so forth, in terms of the velocity only. The third and fourth equations give rise to the equations of motion of the flow.

The subscript zero on the variables p, ρ, and a will indicate the particular value of the variable at a stagnation point. It will be assumed throughout that

$$\rho_o = a_o = 1 \qquad (7)$$

This is equivalent to introducing the dimensionless variables ρ/ρ_o and q/a_o.

From equations (1) and (2), with the aid of equation (5), the following well-known relations are established:

$$a^2 = a_o^2 - [(\gamma - 1)/2]q^2 \qquad (8)$$

$$p = p_o\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_o^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$= p_o\left(1 + \frac{\gamma - 1}{2}\frac{q^2}{a^2}\right)^{-\frac{\gamma}{\gamma-1}} \qquad (9)$$

$$\rho = \rho_o\left(1 - \frac{\gamma - 1}{2}\frac{q^2}{a_o^2}\right)^{\frac{1}{\gamma-1}}$$

$$= \rho_o\left(1 + \frac{\gamma - 1}{2}\frac{q^2}{a^2}\right)^{-\frac{1}{\gamma-1}} \qquad (10)$$

$$M^2 = -\frac{q}{\rho}\frac{d\rho}{dq}$$

$$= q^2/\left(a_o^2 - \frac{\gamma - 1}{2}q^2\right) \qquad (11)$$

These quantities are all given in terms of the single variable q.

From the basic relations (3) and (4), and the fact that the flow is potential, there exist two functions φ, the potential function, and ψ, the stream function, that satisfy the equations

$$\left.\begin{aligned} \varphi_x &= \frac{1}{\rho}\psi_y \\ \varphi_y &= \frac{1}{\rho}\psi_x \end{aligned}\right\} \qquad (12)$$

where the subscripts indicate partial differentiation with respect to the variables indicated. The independent variables x and y are the coordinates in the physical plane of the position of the particle of the fluid. System (12) represents the flow of a compressible fluid.

Let u and v be the horizontal and the vertical velocities, respectively, of a particle of the fluid at a point (x,y). Then

$$\left.\begin{aligned} u &= q \cos \theta \\ v &= q \sin \theta \end{aligned}\right\} \tag{13}$$

where q is the magnitude of the velocity $\vec{q}$, and θ the angle that $\vec{q}$ makes with the x-axis.

From the physical definitions of the potential function φ and the stream function ψ, it follows that

$$d\varphi = u\,dx + v\,dy \tag{14}$$

$$d\psi = -\rho v\,dx + \rho u\,dy \tag{15}$$

Since $d\varphi = \varphi_x\,dx + \varphi_y\,dy$ and $d\psi = \psi_x\,dx + \psi_y\,dy$, system (12) can be obtained directly by comparing these equations with equations (14) and (15).

From equations (14) and (15) it follows that

$$\begin{aligned} d\varphi + i\,\frac{1}{\rho}\,d\psi &= u\,dx + v\,dy + i(-v\,dx + u\,dy) \\ &= (u - iv)(dx + i\,dy) \\ &= qe^{-i\theta}\,dz \end{aligned}$$

where $z = x + iy$. Thus it follows that

$$dz = \frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho}\,d\psi\right) \tag{16}$$

In the hodograph, the coordinates are the polar coordinates (q,θ). Equation (16) is in the nature of a transformation from the hodograph coordinates of the flow to the physical coordinates (x,y). It is this fact that makes equation (16) of fundamental importance for the approach used in this paper.

Upon eliminating first one and then the other of the two unknown functions, φ and ψ, from equations (12), the second-order partial differential equations

$$\left(\rho\varphi_x\right)_x + \left(\rho\varphi_y\right)_y = 0 \tag{17}$$

$$\left(\frac{1}{\rho}\psi_x\right)_x - \left(\frac{1}{\rho}\psi_y\right)_y = 0 \tag{18}$$

are obtained

From equation (14) it is deduced that

$$q = \sqrt{{\varphi_x}^2 + {\varphi_y}^2} \tag{19}$$

so that ρ is a function of the unknown φ. Equations (17) and (18) are therefore nonlinear. It is precisely this condition that has led previous investigators to transform the flow equations from the independent variables (x,y) in the physical plane to the independent variables (q,θ) in the more geometrically complicated hodograph plane. As will be shown, these geometric complications can be circumvented after the linearized equations in the hodograph variables are used to advantage.

The flow equations in the hodograph variables have been derived from equations (12) by many previous writers. For the sake of completeness, however, an outline of a derivation will be presented here. (See reference 10.)

Equation (16) is first differentiated with respect to θ:

$$\frac{\partial z}{\partial \theta} = \frac{1}{q} e^{i\theta}\left(\frac{\partial \varphi}{\partial \theta} + i \frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\right) \tag{20}$$

and then with respect to q:

$$\frac{\partial z}{\partial q} = \frac{1}{q} e^{i\theta}\left(\frac{\partial \varphi}{\partial q} + i \frac{1}{\rho}\frac{\partial \psi}{\partial q}\right) \tag{21}$$

Equation (20) is differentiated with respect to q:

$$\frac{\partial^2 z}{\partial q \partial \theta} = e^{i\theta}\left[-\frac{1}{q^2}\frac{\partial \varphi}{\partial \theta} + i \frac{d}{dq}\left(\frac{1}{\rho q}\frac{\partial \psi}{\partial \theta}\right)\right] + \frac{1}{q} e^{i\theta}\left(\frac{\partial^2 \varphi}{\partial q\, \partial \theta} + i \frac{1}{\rho}\frac{\partial^2 \psi}{\partial q\, \partial \theta}\right) \tag{22}$$

and equation (21) is differentiated with respect to θ:

$$\frac{\partial^2 z}{\partial\theta\,\partial q} = \frac{i}{q} e^{i\theta}\left(\frac{\partial\varphi}{\partial q} + i\,\frac{1}{\rho}\,\frac{\partial\psi}{\partial q}\right) + \frac{1}{q} e^{i\theta}\left(\frac{\partial^2\varphi}{\partial\theta\,\partial q} + i\,\frac{1}{\rho}\,\frac{\partial^2\psi}{\partial\theta\,\partial q}\right) \tag{23}$$

Since the left-hand sides of equations (22) and (23) are equal, it follows that

$$\frac{1}{q} e^{i\theta}\left(\frac{\partial\varphi}{\partial q} + i\,\frac{1}{\rho}\,\frac{\partial\psi}{\partial q}\right) = e^{i\theta}\left[-\frac{1}{q^2}\,\frac{\partial\varphi}{\partial\theta} + i\,\frac{d}{dq}\left(\frac{1}{\rho q}\right)\frac{\partial\psi}{\partial\theta}\right] \tag{24}$$

The flow equations in the hodograph variables are obtained by equating the real and the imaginary parts of equation (24):

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= \frac{q}{\rho}\,\frac{\partial\psi}{\partial q}\\ \frac{\partial\varphi}{\partial q} &= q\,\frac{d}{dq}\left(\frac{1}{\rho q}\right)\frac{\partial\psi}{\partial\theta}\end{aligned}\right\} \tag{25}$$

From equation (11)

$$\frac{d\rho}{dq} = -\frac{\rho}{q} M^2$$

It follows that

$$\begin{aligned}\frac{d}{dq}\left(\frac{1}{\rho q}\right) &= -\frac{1}{\rho^2 q}\,\frac{d\rho}{dq} - \frac{1}{\rho q^2}\\ &= \frac{1}{\rho q^2} M^2 - \frac{1}{\rho q^2} = \frac{1}{\rho q^2}\left(M^2 - 1\right)\end{aligned}$$

Equations (25) can now be written in a more suitable form:

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial\theta} &= \frac{q}{\rho}\,\frac{\partial\psi}{\partial q}\\ \frac{\partial\varphi}{\partial q} &= -\frac{1 - M^2}{\rho q}\,\frac{\partial\psi}{\partial\theta}\end{aligned}\right\} \tag{26}$$

The fact that M and ρ are functions of q only makes equations (26) linear. These equations can be handled mathematically with much greater ease than can the nonlinear equations (12) in the physical plane. The theory of Σ-monogenic functions has been set up for just this purpose. Though the solutions of equations (26) that are obtained give the complex potential of the flow in terms of the velocity of a particle of the fluid rather than in terms of the position of the particle, the fundamental transformation, equation (16), enables these solutions to be transformed back to the physical plane. The fact that the right-hand side of equation (16) is an exact differential facilitates greatly the process of carrying out this transformation.

SOLUTIONS OF THE COMPRESSIBLE FLOW EQUATIONS AS FUNCTIONS OF THE HODOGRAPH VARIABLES

Equations (26) can be written in a more general form for the purpose of mathematical treatment:

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial \theta} &= \tau_1(q) \frac{\partial \psi}{\partial q} \\ \frac{\partial \varphi}{\partial q} &= -\tau_2(q) \frac{\partial \psi}{\partial \theta} \end{aligned}\right\} \tag{27}$$

where τ_1 and τ_2 are positive analytic functions of q. The condition that τ_2 be positive is equivalent to assuming that the flow remains subsonic. Many of the mathematical results that will be presented here hold also when τ_2 becomes negative, so that flows involving supersonic velocities can be treated. This has been carried out elsewhere for very special flows. (See reference 3.)

Consider the function represented by the line integral

$$\begin{aligned} F(\sigma) &= \Phi(\theta,q) + i\Psi(\theta,q) \\ &= \int_{\theta_0,q_0}^{\theta,q} \left(\varphi d\theta - \tau_2(q)\psi\, dq\right) + i\int_{\theta_0,q_0}^{\theta,q} \left(\psi\, d\theta + \frac{1}{\tau_1(q)}\varphi\, dq\right) \end{aligned} \tag{28}$$

where θ_0, q_0 are fixed values of θ and q and $\sigma = \theta + iq$. By virtue of equations (27) a simple computation verifies that each of the two integrands in the right-hand side of equation (28) is an exact differential.

This establishes that F is a function of the point (θ,q) and has the same value when the integration is taken along any path, provided proper regard is given to singularities.

The second important fact about $F(\theta,q)$ is that its real part Φ and its imaginary part Ψ again satisfy the system of equations (27). This affords a method of generating particular solutions of system (27).

It is trivial that when $\varphi = 1$ and $\psi = 0$, this pair is a solution of equations (27). When this solution is substituted into equation (28)

$$\begin{aligned}\Phi + i\Psi &= W(\sigma) \\ &= \theta + i\int_0^q \frac{1}{\tau_1}\,dq\end{aligned} \tag{29}$$

If the real and the imaginary parts of equation (29) are again substituted into equation (28), the next solution gives

$$W^{(2)} = \theta^2 + 2i\,\theta\int_0^q \frac{1}{\tau_1}\,dq - 2!\int_0^q \tau_2 \int_0 \frac{1}{\tau_1}\,dq^2 \tag{30}$$

The superscript 2 in parentheses indicates that the function has the nature of a power, as is observed when $\tau_1 = \tau_2 = 1$. Here equation (30) reduces to $(\theta + iq)^2 = \sigma^2$.

The notation $F(\sigma_0;\sigma)$, $\sigma_0 = \theta_0 + iq_0$, indicates that θ_0 and q_0 are the lower limits of integration in equation (28). For example,

$$W^{(2)}(\sigma_0;\sigma) = (\theta - \theta_0)^2 + 2i\,(\theta - \theta_0)\int_{q_0}^q \frac{1}{\tau_1}\,dq - 2!\int_{q_0}^q \tau_2 \int_{q_0} \frac{1}{\tau_1}\,dq^2$$

and corresponds to the function $(\sigma - \sigma_0)^2$. When $\sigma_0 = 0$, $F(\sigma_0;\sigma)$ is usually written $F(\sigma)$.

Let

$$Q^{(0)} \equiv 1$$

and

$$Q^{(n)}(q) = \begin{cases} n! \overbrace{\int \frac{1}{\tau_1} \int \tau_2 \cdots \int \frac{1}{\tau_1}\, dq^n}^{n \text{ times}}, & n \text{ odd} \\ n! \overbrace{\int \tau_2 \int \frac{1}{\tau_1} \cdots \int \frac{1}{\tau_1}\, dq^n}^{n \text{ times}}, & n \text{ even} \end{cases} \tag{31}$$

If this operation is repeated n times the resulting solution is

$$W^{(n)} = (\theta + iQ)^{(n)}$$

$$= \sum_{\nu=0}^{n} C_{n,\nu}\, \theta^{\nu}\, i^{n-\nu}\, Q^{(n-\nu)}(q) \tag{32}$$

where $C_{n,\nu}$ is the νth binomial coefficient.

If another trivial solution is taken as a first solution, say, $\varphi = 0, \psi = 1$, and substituted in equation (28), it follows that

$$F(\sigma) = i \cdot W(\sigma)$$

$$= i\tilde{W}(\sigma)$$

$$= i\left(\theta + i\int \tau_2\, dq\right) \tag{33}$$

The symbolic notation $i.W$ is used to represent a generalization of the concept of multiplication. It will be seen that this notation is rather useful. When this process of substitution is repeated

$$i \cdot W^{(2)}_{(\sigma)} = i\tilde{W}^{(2)}_{(\sigma)}$$

$$= i\left(\theta^2 + 2i\,\theta \int \tau_2\, dq - 2! \int \frac{1}{\tau_1} \int \tau_2\, dq^2\right) \tag{34}$$

Let

$$Q^{*(o)} \equiv 1$$

and

$$Q^{*(n)} = \begin{cases} n! \overbrace{\int \tau_2 \int \frac{1}{\tau_1} \cdots \int \tau_2 \, dq^n,}^{n \text{ times}} & n \text{ odd} \\ n! \int \frac{1}{\tau_1} \int \tau_2 \cdots \int \tau_2 \, dq^n & n \text{ even} \end{cases} \tag{35}$$

Again, if this operation is repeated n times the resulting solution is

$$\begin{aligned} i \cdot W^{(n)} &= i\tilde{W}^{(n)} \\ &= i \sum_{\nu=0}^{n} C_{n,\nu} \, \theta^{\nu} \, i^{n-\nu} \, Q^{*(n-\nu)}(q) \end{aligned} \tag{36}$$

Finally, if the trivial solution $\varphi = \alpha$, $\psi = \beta$ is taken, α and β being constants, n-repetitions of the operation (28) yield the solution

$$a \cdot W^{(n)} = \alpha W^{(n)}(\sigma) + \beta \left[i \cdot W^{(n)} \right] \tag{37}$$

where $a = \alpha + i\beta$.

Complex-valued functions of which the real and imaginary parts satisfy the system (26) are said to be Σ-monogenic functions. Thus, $W^{(n)}$ and $i \cdot W^{(n)} = i\tilde{W}^{(n)}$ are Σ-monogenic functions. These are more specifically referred to as formal powers, since when $\tau_1 = \tau_2 = 1$, they reduce to $\sigma^n = (\theta + iq)^n$ and $i\sigma^n = i(\theta + iq)^n$. The formal product of the two solutions a and $W^{(n)}$ is $a \cdot W^{(n)}$, which itself is a solution. It should be noted that the ordinary product $aW^{(n)}$ is not a solution of the system (26) when a is complex.

As Σ-multiplication of a constant and a formal power was introduced, so may Σ-integration be introduced. The Σ-integration of the Σ-monogenic function, $f(\sigma) = \varphi + i\psi$, is defined as

$$\begin{aligned} F(\sigma) &= \Phi + i\Psi \\ &= \int^{\theta q} (\varphi \, d\theta - \tau_2 \psi \, dq) + i \int^{\theta q} \left(\psi \, d\theta + \frac{1}{\tau_1} \varphi \, dq \right) \\ &= \int f(\sigma) d_{\Sigma} \sigma \end{aligned} \tag{38}$$

The $\sum$ refers to the coefficient matrix of system (27).

$$\sum = \begin{Vmatrix} 1 & \tau_1 \\ 1 & \tau_2 \end{Vmatrix} \tag{39}$$

And Σ-differentiation of the Σ-monogenic function, $f(\sigma) = \varphi + i\psi$, is defined as

$$\dot{f}(\sigma) = \frac{d_\Sigma f(\sigma)}{d_\Sigma \sigma}$$

$$= \varphi_x + i\psi_x \tag{40}$$

From the relations of equations (27)

$$\frac{d_\Sigma f}{d_\Sigma \sigma} = \varphi_x + i\psi_x$$

$$= \tau_1 \psi_y - \frac{1}{\tau_2} \varphi_y \tag{41}$$

Again, when $\tau_1 = \tau_2 = 1$, Σ-integration and Σ-differentiation reduce to ordinary complex integration and differentiation.

A more elaborate definition of Σ-integration and differentiation has been given in reference 2. It has been shown in reference 2 that

$$\frac{d_\Sigma F(\sigma)}{d_\Sigma \sigma} = f(\sigma) \tag{42}$$

and

$$\int \dot{F}(\sigma) d_\Sigma \sigma = F(\sigma) \tag{43}$$

Thus, Σ-integration and Σ-differentiation are inverse processes. Also, the Σ-derivative of a Σ-monogenic function is a Σ-monogenic function.

From the preceding definitions it can be verified by direct computation that the Σ-integration and the Σ-differentiation of the formal powers follow similar rules to those of ordinary integration and differentation of the powers of a complex variable; more precisely,

$$\int a \cdot W^{(n)} d_\Sigma q = \frac{a \cdot W^{(n+1)}}{n + 1} \tag{44}$$

and

$$\frac{d_\Sigma}{d_\Sigma \sigma}\left[a\cdot W^{(n)}\right] = na\cdot W^{(n-1)} \tag{45}$$

Since system (27) is linear, the sum of two solutions is a solution. Hence, a formal polynomial of the nth degree,

$$f(\sigma) = a_0 + a_1\cdot W + \ . \ . \ . + a_n\cdot W^{(n)}, \quad a_n \neq 0 \tag{46}$$

is a Σ-monogenic function. The formal power series

$$f(\sigma) = \sum_{n=0}^{\infty} a_n\cdot W^{(n)} \tag{47}$$

represents a Σ-monogenic function, provided the series converges uniformly and absolutely in some neighborhood of σ_0.

It has been shown (reference 2) that every complex solution of system (27) can be represented uniquely by a formal power series of the form of equation (47). (This statement is true only when $\tau_1 \tau_2 > 0$, that is, in the elliptic case or subsonic flow.) The coefficients are given by the formula

$$a_n = \frac{1}{n!} \left. \frac{d_\Sigma^{\,n} f(\sigma)}{d_\Sigma \sigma^n} \right|_{\sigma = \sigma_0} \tag{48}$$

Since every solution of system (27) can be represented, at least in some neighborhood, as a formal power series and since every complex potential in the hodograph variables of a compressible fluid is a solution of system (27), it would appear that it remains only to choose that particular solution which is the desired flow around a given body with prescribed conditions. This approach, however, still has some very serious difficulties.

The formal powers presented previously are solutions of system (27) in closed form and are valid throughout their regions of regularity. The first powers $W(\sigma)$ and $i\cdot W(\sigma)$ are known to represent a compressible vortex, respectively. The higher powers appear not to represent any flows of interest. Linear combinations of the powers, that is, formal polynomials, have not yet been studied sufficiently to determine whether they represent flows of interest. The solutions represented by formal power series are first, not in closed form and, second, not valid throughout their regions of regularity. Furthermore, since in aerodynamics the majority of solutions of interest are those that are regular throughout the exterior of a closed region, solutions corresponding to the analytic functions σ^{-n} must be obtained in either closed form or in a series that can be extended throughout its region of regularity. What is most desired is a theorem analogous to that of Laurent in analytic functions, for Σ-monogenic functions. Such a theorem, if it can be found, might quickly lead to the solutions of the major problems in compressibility.

It is hardly hoped that the complicated problem of compressibility can be solved in any simple way. Any results in this direction, therefore, are of interest.

Some progress along these lines has been made by Bers and Gelbart (reference 2) by extending to Σ-monogenic functions some of the results on the Laplace transform. These give, in closed form, solutions of system (27) that are different from the formal powers $a \cdot W^{(n)}$, and in half planes correspond to the analytic functions σ^{-n}.

Consider the function

$$a \cdot E(\sigma_o;\alpha,\sigma) = \sum_{n=o}^{\infty} \frac{\alpha^n a}{n!} W^{(n)}(\sigma_o;\sigma) \tag{49}$$

where as before $a = \alpha + i\beta$ (see references 1 and 2) and α is real. For the sake of brevity

$$\left.\begin{aligned} 1 \cdot E(\sigma_o;\alpha,\sigma) &= E(\sigma_o;\alpha,\sigma) \\ E(0;\alpha,\sigma) &= E(\alpha,\sigma) \end{aligned}\right\} \tag{50}$$

This function is termed the "Σ-exponential function" for obvious reasons.

A simple computation shows that

$$E(\alpha,\sigma) = e^{\alpha\theta}\left[c(\alpha,q) + is(\alpha,q)\right] \tag{51}$$

and

$$i \cdot E(\alpha,\sigma) = ie^{\alpha\theta}\left[c^*(\alpha,q) + is^*(\alpha,q)\right] \tag{52}$$

where

$$\left.\begin{aligned} c(\alpha,q) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} Q^{(2n)}(q) \\ c^*(\alpha,q) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} Q^{*(2n)}(q) \end{aligned}\right\} \tag{53a}$$

$$\left.\begin{aligned} s(\alpha,q) &= \sum_{n=0}^{\infty} \frac{(-1)\alpha^{2n+1}}{(2n+1)!} Q^{(2n+1)}(q) \\ s^*(\alpha,q) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} Q^{*(2n+1)}(q) \end{aligned}\right\} \tag{53b}$$

From the definition of $E(\alpha,\sigma)$ and that of Σ-differentiation, it is clear that

$$\frac{d_\Sigma}{d_\Sigma \sigma}[E(\alpha,\sigma)] = \alpha E(\alpha,\sigma) \tag{54}$$

Further properties of functions c, s, c^*, and s^* are

$$\left.\begin{aligned} s(\alpha,0) &= s^*(\alpha,0) = 0 \\ c(\alpha,0) &= c^*(\alpha,0) = 1 \end{aligned}\right\} \tag{55}$$

and

$$\left.\begin{aligned} s'(\alpha,q) &= \alpha \frac{c(\alpha,q)}{\tau_1(q)} \\ s^{*\prime}(\alpha,q) &= \alpha\, \tau_2(q) c^*(\alpha,y) \\ c'(\alpha,q) &= -\alpha\, \tau_2(q) s(\alpha,q) \\ c^{*\prime}(\alpha,q) &= -\alpha \frac{s^*(\alpha,q)}{\tau_1(q)} \end{aligned}\right\} \tag{56}$$

the differentiation being with respect to q. These functions may be regarded as generalizations of the trigonometric functions and have many properties analogous to those of the trigonometric functions. For example, they satisfy Sturm–Liouville differential equations and possess such properties as

$$c(\alpha,q)c^*(\alpha,q) + s(\alpha,q)s^*(\alpha,q) \equiv 1 \tag{57}$$

Now consider the function

$$\begin{aligned} f(\sigma) &= L_{\Sigma}[F(\alpha)] \\ &= \int_0^\infty F(\alpha) \cdot E(-\alpha,\sigma)d\alpha \end{aligned} \tag{58}$$

where $F(\alpha) = F_1(\alpha) + iF_2(\alpha)$ is a complex-valued function of the real variable α, $F_1(\alpha)$ and $F_2(\alpha)$ being real functions. This is called the Σ–Laplace transform of $F(\alpha)$, for when $\tau_1 = \tau_2 = 1$ the integral in equation (58) reduces to the ordinary Laplace transform of the function $F(\alpha)$.

Equation (58) may be rewritten in the more convenient form

$$\begin{aligned} f(\sigma) &= \varphi(\theta,q) + i\psi(\theta,q) \\ &= \int_0^\infty F_1(\alpha)E(-\alpha,\sigma)d\alpha + \int_0^\infty F_2(\alpha)\ [i\cdot E(-\alpha,\sigma)]\ d\alpha \\ &= \int_0^\infty e^{-\alpha\theta}\ c(\alpha,q)F_1(\alpha)d\alpha - i\int_0^\infty e^{-\alpha\theta}\ s(\alpha,q)F_1(\alpha)d\alpha \\ &\quad + i\int_0^\infty e^{-\alpha\theta}\ c^*(\alpha,q)F_2(\alpha)d\alpha + \int_0^\infty e^{-\alpha\theta}\ s^*(\alpha,q)F_2(\alpha)d\alpha \end{aligned} \tag{59}$$

Then

$$\phi\ (\alpha,q) = \int_0^\infty e^{-\alpha\theta}\ c(\alpha,q)F_1(\alpha)d\alpha + \int_0^\infty e^{-\alpha\theta}\ s^*(\alpha,q)F_2(\alpha)d\alpha \tag{60}$$

and

$$\psi\ (\theta,q) = -\int_0^\infty e^{-\alpha\theta}\ s(\alpha,q)F_1(\alpha)d\alpha + \int_0^\infty e^{-\alpha\theta}\ c^*(\alpha,q)F_2(\alpha)d\alpha \tag{61}$$

Two facts may be demonstrated about the function in equation (58):

1. If β is the abscissa of convergence of the Laplace integral

$$\int_0^\infty e^{-\alpha\sigma} F(\alpha)d\alpha \tag{62}$$

then the generalized Laplace integral

$$\int_0^\infty F(\alpha) \cdot E(-\alpha,\sigma)d\alpha \tag{63}$$

converges uniformly in the semi-infinite strip

$$\theta \geq \theta_0 > \beta$$

$$q_1 \leq q \leq q_2$$

2. Integral (63) represents a Σ-monogenic function in the open half plane of convergence of integral (62).

Because of theorem 2, expression (63) represents a new class of solutions of system (27) in closed form. Hence, for every function $F(\alpha)$ for which integral (63) converges (from theorem 1 this class is known to be wide), expression (63) is the complex potential of a compressible flow in the hodograph coordinates.

For certain functions $F(\alpha)$ it is easy to show that the formal series expansion of function

$$f(\alpha) = \int_0^\infty F(\alpha)\cdot E(-\alpha,\sigma)d\alpha$$

converges in a smaller region than does function $f(\alpha)$. The Σ-Laplace representation of a function may thus be regarded as a means of continuing analytically a function that is given in the form of a formal power series.

Because

$$\frac{1}{\sigma} = \int_0^\infty e^{-\alpha\sigma}\, d\alpha \tag{64}$$

in the right half plane, it seems natural to define the negative Σ-power function by

$$W^{(-1)} = \int_0^\infty E(-\alpha,\sigma)d\alpha \tag{65}$$

Since Σ-differentiation under the integral sign is permitted, representations for the higher negative Σ-powers can be obtained. By Σ-differentiating each side of equation (65) n times and using equation (54), it follows that

$$\frac{d^n_\Sigma}{d_\Sigma \sigma^n}\left[W^{(-1)}\right] = \int_0^\infty E(-\alpha,\sigma)\alpha^n \, d\alpha \tag{66}$$

In view of equation (66) it seems convenient to define the negative Σ-powered functions by

$$W^{(-n)} = \int_0^\infty E(-\alpha,\sigma) \frac{\alpha^{n-1}}{(n-1)!} \, d\alpha \tag{67}$$

Generalizations of other special functions of a complex variable may be obtained in a similar way.

In order that functions $W^{(-n)}$, as defined by equation (67), play a similar role to that of the inverse powers it would be necessary to show that they possess a pole of the nth order at the origin and be regular everywhere else. This has not yet been shown. In fact, $W^{(-n)}$ is defined only for the right half plane, $Rl\sigma > 0$. It can be defined in the left half plane, $Rl\sigma < 0$, by

$$W^{(-n)} = \int_0^\infty E(\alpha,\sigma) \frac{\alpha^{n-1}}{(n-1)!} \, d\alpha \tag{68}$$

On the imaginary axis other than at the origin it can be defined as the limit of the function as σ approaches a point on the imaginary axis. It has not yet been proved that $W^{(-n)}$, defined in the right and the left half plane by equations (67) and (68), respectively, is continuous across the imaginary axis. If its truth is assumed, some progress might be made in the study of compressible flows.

In order to study the flows that arise from these solutions, it is necessary to transform them from the hodograph back to the physical plane.

Consider the solution

$$\begin{aligned}\varphi + i\psi &= W^{(n)} \\ &= \sum_{n=0}^{\infty} i^r c_{n,r}\, \theta^{n-r}\, Q^{(r)}\end{aligned} \tag{69}$$

The real part is

$$\varphi = \sum_{r=0}^{\left[\frac{n}{2}\right]} (-1)^r C_{n,2r}\, \theta^{n-2r}\, Q^{(2r)} \tag{70}$$

and the imaginary part is

$$\psi = \sum_{r=0}^{\left[\frac{n-1}{2}\right]} (-1)^r C_{n,2r+1}\, \theta^{n-2r-1}\, Q^{(2r+1)} \tag{71}$$

where the brackets in the upper limit of the summation indicate that the integer chosen is the smaller one nearer to the number within the brackets.

After substituting (70) and (71) into equation

$$dz = \frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho}\,d\psi\right)$$

$$= \frac{e^{i\theta}}{q}\left[\varphi_\theta\, d\theta + \varphi_q\, dq + \frac{i}{\rho}\left(\psi_\theta\, d\theta + \psi_q\, dq\right)\right] \tag{72}$$

and integrating, it follows that

$$z = e^{i\theta}\left\{\frac{1}{q}\sum_{r=0}^{\left[\frac{2n-3+(-1)^{n+1}}{4}\right]}\left[(-1)^r(n-2r)C_{n,2r}\, Q^{(2r)}\,\cdot \sum_{s=0}^{n-2r-1} i^{s-1}\,\frac{(n-2r-1)!\,\theta^{n-2r-1-s}}{n-2r-1-s}\right]\right.$$

$$\left. + \frac{1}{\rho q}\sum_{r=0}^{\left[\frac{2n-3+(-1)^{n+1}}{4}\right]}\left[(-1)^r(n-2r-1)C_{n,2r+1}\, Q^{(2r+1)}\,\cdot \sum_{s=0}^{n-2r-2} i^{s}\,\frac{(n-2r-2)!\,\theta^{n-2r-s-2}}{(n-2r-s-2)!}\right]\right\} \tag{73}$$

Equation (73) together with equation (69) can be regarded as the parametric equations of the compressible flow of $\varphi + i\psi$ in the physical plane.

Similarly, for the solution

$$\varphi + i\psi = i\tilde{W}^{(n)}$$
$$= i \sum_{r=0}^{\infty} i^r C_{n,r}\, \theta^{n-r}\, Q^{*(r)} \tag{74}$$

$$\left.\begin{aligned} z = e^{i\theta} \Bigg\{ & \frac{1}{q} \sum_{r=0}^{\left[\frac{2n-3+(-1)^{n+1}}{4}\right]} \Bigg[(-1)^r (n-2r) C_{n,2r}\, Q^{*(2r)} \cdot \sum_{s=0}^{n-2r-1} i^{s-1} \frac{(n-2r-1)!\, \theta^{n-2r-s-1}}{(n-2r-s-1)!} \Bigg] \\ & + \frac{1}{\rho q} \sum_{r=0}^{\left[\frac{2n-3+(-1)^{n+1}}{4}\right]} \Bigg[(-1)^r (n-2r-1) Q^{*(2r+1)} \cdot \sum_{s=0}^{n-2r-2} i^{s} \frac{(n-2r-2)!\, \theta^{n-2r-s-2}}{(n-2r-s-2)!} \Bigg] \Bigg\} \end{aligned}\right\} \tag{75}$$

By taking linear combinations of the formal power solutions, other flows can be obtained in the physical plane in parametric form.

If the generalized Laplace integral solutions are taken, then

$$\varphi + i\psi = \int_0^\infty F(\alpha)\cdot E(-\alpha,\sigma)d\sigma$$
$$= \int_0^\infty e^{-\alpha\theta}\left[c(\alpha,q)F_1(\alpha) + c^*(\alpha,q)F_2(\alpha)\right]d\alpha$$
$$+ i\int_0^\infty e^{-\alpha\theta}\left[c^*(\alpha,q)F_2(\alpha)d\alpha - e^{-\alpha\theta}\, s(\alpha,q)F_1(\alpha)\right]d\alpha$$

and

$$z = -\frac{1}{q}\int_0^\infty \frac{\alpha e^{(i-\alpha)\theta}}{i-\alpha}\left[c(\alpha,q)F_1(\alpha) + s^*(\alpha,q)F_2(\alpha)\right]d\alpha$$
$$-\frac{1}{q\rho}\int_0^\infty \frac{\alpha e^{(i-\alpha)\theta}}{i-\alpha}\left[c^*(\alpha,q)F_2(\alpha) - s(\alpha,q)F_1(\alpha)\right]d\alpha \qquad (76)$$

Further investigation is required to determine precisely under what conditions the flow is l around a closed body and uniform at infinity. These investigations seem within reach by the method here indicated. This could yield mixed subsonic-supersonic flows with subsonic free-stream velocities under the assumption of the adiabatic equation of state.

As an application of the method outlined in this paper, fluids of which the flows are everywhere subsonic (more precisely, flows satisfying the linearized equation of state) are chosen. Precise conditions are obtained for flows around closed bodies with prescribed velocities at infinity. Flows under these hypotheses have been investigated with much success by Chaplygin, von Kármán, Tsien, Bers, and others. Of particular interest in this connection for flows around closed bodies is the work of Bers (reference 8).

In the sequel to the present report (reference 13) a detailed investigation of the flow around a circle is made (under the assumption of the linearized equation of state). This is done primarily because the flow around a circle appears to be basic for the study of flows around given bodies, as in the case of incompressible fluids.

COMPRESSIBLE FLUIDS UNDER THE ASSUMPTION OF THE LINEARIZED EQUATION OF STATE

Of the four basic relations of the flow, equations (1) to (4), the first, the equation of state, is replaced by

$$p - p_1 = k\left(\frac{1}{\rho} - \frac{1}{\rho_1}\right) \qquad (77)$$

where ρ_1 is the density for a given velocity q_1, and p_1 is the corresponding pressure.

If pressure-volume curves are drawn of equations (1) and (77), it is observed that when k is suitably chosen equation (77) represents a line tangent to the curve represented by equation (1) at the point p_1, $1/\rho_1$. Von Kármán and Tsien used this fact partly to justify the use of the linearized equation of state for the study of subsonic flows.

From equations (5) and (77)

$$\frac{dp}{d\rho} = a^2 = -\frac{k}{\rho^2} \tag{78}$$

Hence

$$a^2 \rho^2 = -k \tag{79}$$

From equation (6)

$$\frac{q^2}{2} + \int \frac{a^2 \rho^2}{\rho^3} d\rho = \text{constant}$$

and from equation (79)

$$\frac{q^2}{2} + a^2 \rho^2 \int \frac{d\rho}{\rho^3} = \text{constant}$$

Thus,

$$q^2 - a^2 = \text{constant} \tag{80}$$

It follows that

$$q^2 - a^2 = \frac{q^2/a^2 - 1}{1/a^2}$$

$$= \frac{M - 1}{-\rho^2/k}$$

$$= \text{constant}$$

or

$$\frac{1 - M^2}{\rho^2} = \text{constant} \tag{81}$$

By the normalization, $a_o = \rho_o = 1$, the constants in the right-hand sides of equations (80) and (81) are readily determined to be minus unity and unity, respectively. Thus,

$$a^2 = 1 + q^2 \tag{82}$$

and

$$\begin{aligned} \rho^2 &= 1 - M^2 \\ &= 1 - \frac{q^2}{1 + q^2} \\ &= \frac{1}{1 + q^2} \end{aligned} \tag{83}$$

From equation (77)

$$p - p_1 = k\left(\frac{1}{\sqrt{1 + q^2}} - \frac{1}{\rho_1}\right) \tag{84}$$

Again, from the definitions of stream function φ and potential function ψ of an incompressible fluid

$$\left.\begin{aligned} d\varphi &= u\ dx + v\ dy \\ d\psi &= -v\ dx + u\ dy \end{aligned}\right\} \tag{85}$$

also

$$\left.\begin{aligned} d\varphi &= \varphi_x\ dx + \varphi_y\ dy \\ d\psi &= \psi_x\ dx + \psi_y\ dy \end{aligned}\right\} \tag{86}$$

Comparing systems (85) and (86), it follows that

$$\left.\begin{aligned} \varphi_x &= \psi_y \\ \varphi_y &= -\psi_x \end{aligned}\right\} \tag{87}$$

These are the Cauchy-Riemann equations. This establishes the well-known fact that the complex potential $\Omega = \varphi + i\psi$ of an incompressible flow is an analytic function of the complex variable $z = x + iy$, the complex coordinates in the physical plane.

When the second of equations (85) is multiplied by i and the two equations added

$$d\Omega = (u - iv)\ (dx + i\ dy)$$
$$= qe^{-i\theta}\ dz \tag{88}$$

or

$$dz = \frac{1}{q}\ e^{i\theta}\ d\Omega \tag{89}$$

By proceeding as was done in the derivation of equations (25), the change of the variables (x,y) to the independent variables (θ,q) in equations (87) leads to equations

$$\left.\begin{aligned}\psi_\theta &= q\ \psi_q \\ \varphi_q &= -\frac{1}{q}\psi_\theta\end{aligned}\right\} \tag{90}$$

The transformation that will symmetrize system

$$\left.\begin{aligned}\varphi_\theta &= \tau_1\ (q)\ \psi_q \\ \varphi_q &= \tau_2\ (q)\ \psi_\theta\end{aligned}\right\} \tag{91}$$

is given by

$$\tilde{q} = \int^q \sqrt{\frac{\tau_2(q)}{\tau_1(q)}}\ dq, \qquad \theta = \theta \tag{92}$$

System (90) is therefore symmetrized by the transformation

$$\tilde{q} = \int_{q_\infty}^{q} \frac{dq}{q} = \log \frac{q}{q_\infty} \tag{93}$$

and reduces to the Cauchy-Riemann equations

$$\left.\begin{aligned}\varphi_\theta &= \psi_{\tilde{q}} \\ \varphi_{\tilde{q}} &= -\psi_\theta\end{aligned}\right\} \tag{94}$$

Thus, the complex potential of an incompressible flow, given in terms of the independent variables of the θ, $\tilde{q}$-plane is an analytic function of the complex variable $w = \theta + i\tilde{q}$.

Equations

$$\left.\begin{aligned} \varphi_\theta &= \frac{q}{\rho}\psi_q \\ \varphi_q &= \frac{1 - M^2}{\rho q}\psi_\theta \end{aligned}\right\} \tag{95}$$

can similarly be symmetrized by the transformation

$$\tilde{q} = \int_\infty^q \frac{\sqrt{1 - M^2}}{q}\, dq \tag{96}$$

Equations (95) then become

$$\left.\begin{aligned} \varphi_\theta &= \frac{\sqrt{1 - M^2}}{\rho}\psi_{\tilde{q}} \\ \varphi_{\tilde{q}} &= -\frac{\sqrt{1 - M^2}}{\rho}\psi_\theta \end{aligned}\right\} \tag{97}$$

However, under the assumption of the linearized equation of state

$$\frac{\sqrt{1 - M^2}}{\rho} = 1$$

so that equations (97) become the Cauchy-Riemann equations

$$\left.\begin{aligned} \varphi_\theta &= \psi_{\tilde{q}} \\ \varphi_{\tilde{q}} &= -\psi_\theta \end{aligned}\right\} \tag{98}$$

Again, the complex potential of a compressible flow under the assumption of the linearized equation of state is an analytic function of the complex variable $w = \theta + i\tilde{q}$. The w-plane shall be referred to as the distorted hodograph plane.

Any analytic function of w may be regarded as the complex potential of either an incompressible flow in the physical plane or in the hodograph plane, or a compressible flow in the distorted hodograph under the assumption of this section, and vice versa; then, in general, to every incompressible flow around a given body with a given free-stream velocity there corresponds a compressible flow (everywhere subsonic) around the same body with the same free-stream velocity.

This correspondence can be expressed as follows: Given the complex potential $\Omega(w)$ of an incompressible flow, or of a compressible flow in the distorted hodograph, then there exists an analytic function $w = g(\zeta)$ such that $\Omega[g(\zeta)]$ is the complex potential in the distorted hodograph ζ-plane of a given compressible flow.

Since

$$\sqrt{1 - M^2} = \rho, \quad \text{and} \quad \rho = -\frac{1}{\sqrt{1 + q^2}}$$

$$\begin{aligned}\tilde{q} &= \int_{q_\infty}^{q} \frac{\sqrt{1 - M^2}}{q}\,dq \\ &= \int_{q_\infty}^{q} -\frac{dq}{q\sqrt{1 + q^2}}\end{aligned} \tag{99}$$

and

$$\tilde{q} = \log \frac{Kq}{1 + \sqrt{1 + q^2}} \tag{100}$$

where

$$K = \frac{1 + \sqrt{1 + q^2}}{q_\infty} \tag{101}$$

Hence

$$e^{\tilde{q}} = \frac{Kq}{1 + \sqrt{1 + q^2}} \tag{102}$$

and

$$\frac{1}{q} = \frac{K}{2e^{\tilde{q}}} - \frac{e^{\tilde{q}}}{2K} \tag{103}$$

Also

$$\frac{1}{\rho q} = \frac{\sqrt{1 + q^2}}{q}$$

$$= \frac{K}{2e^{\tilde{q}}} + \frac{e^{\tilde{q}}}{2} \tag{104}$$

Thus, equation (16) may be put into the more convenient form,

$$dz = \frac{e^{i\theta}}{q}\left[d\left(\frac{\Omega + \bar{\Omega}}{2}\right) + \frac{i}{\rho}\left(\frac{\Omega - \bar{\Omega}}{2i}\right)\right]$$

$$= \frac{e^{i\theta}}{2}\left(\frac{1}{q} + \frac{1}{\rho q}\right) d\Omega + \frac{e^{i\theta}}{2}\left(\frac{1}{q} - \frac{1}{\rho q}\right) d\bar{\Omega} \tag{105}$$

From equations (103) and (104), equation (105) becomes

$$dz = \frac{e^{i\theta}}{q} \cdot \frac{K}{e^{\tilde{q}}} d\Omega - \frac{e^{i\theta}}{2} \frac{e^{\tilde{q}}}{K} d\bar{\Omega}$$

$$= \frac{K}{2} e^{iw} d\Omega - \frac{1}{2K} e^{i\bar{w}} d\bar{\Omega} \tag{106}$$

where $w = \theta + i\tilde{q}$. Thus,

$$z = \frac{K}{2}\int e^{iw} d\Omega - \frac{1}{2K}\overline{\int e^{-iw} d\Omega} \tag{107}$$

where $\Omega(w)$, the complex potential of a compressible fluid in the distorted hodograph plane, is an analytic function of w.

Since the complex potential of a compressible fluid remains the complex potential of a compressible fluid under an analytic transformation, set

$$w(\zeta) = -i \log \frac{2f'(\zeta)}{KG'(\zeta)} \tag{108}$$

or

$$e^{iw} = \frac{2f'(\zeta)}{KG'(\zeta)} \tag{109}$$

and

$$e^{-iw} = \frac{KG'(\zeta)}{2f'(\zeta)} \tag{110}$$

where $f(\zeta)$ and $G(\zeta) = \Omega[w(\zeta)]$ are analytic, $f(\zeta)$ being arbitrary and $G(\zeta)$ the complex potential of an incompressible flow around a given body in the physical ζ-plane.

When w is considered a function of ζ, equation (107) may be written as

$$z = \frac{K}{2}\int e^{iw(\zeta)}G'(\zeta)d\zeta - \frac{1}{4}\int \overline{e^{-iw(\zeta)}G'(\zeta)d\zeta} \tag{111}$$

When equation (108) is substituted into equation (111), equation (111) becomes

$$z = f(\zeta) - \frac{1}{4}\int \overline{\frac{[G'(\zeta)]^2}{f'(\zeta)}\, d\zeta} \tag{112}$$

It should be emphasized that $f(\zeta)$ is an arbitrary analytic function of ζ, while $G(\zeta)$ is an incompressible flow in the ζ-plane, ζ being the physical coordinates of the flow, and $G(\zeta) = \Omega[w(\zeta)]$ is the corresponding compressible flow in the w-plane. The relation,

$$\begin{aligned} \operatorname{Im} G(\zeta) &= \operatorname{Im} \Omega[w(\zeta)] \\ &= \text{constant} \end{aligned}$$

represents streamlines of the incompressible flow as well as streamlines of the compressible flow. If ζ moves along a streamline of the incompressible flow, then w, given by equation (108), traverses a streamline of the compressible flow in the hodograph plane and z, by the transformation (108), traces out a streamline in the physical plane of the compressible flow.

It is convenient to consider $G(\zeta)$ the incompressible flow around a circle of radius R, with circulation Γ, and having a free-stream velocity q_∞; namely

$$G(\zeta) = q_\infty\left(\zeta + \frac{R^2}{\zeta}\right) - \frac{i\Gamma}{2\pi}\log\frac{\zeta}{R} \tag{113}$$

Since $e^{iw} = e^{i(\theta+i\tilde{q})}$, it follows from equation (110) that

$$e^{-i\theta}\, e^{\tilde{q}} = \frac{K}{2}\,\frac{G'(\zeta)}{f'(\zeta)} \tag{114}$$

Thus,

$$e^{\hat{q}} = \frac{K}{2}\left|\frac{G'(\zeta)}{f'(\zeta)}\right|$$

and recalling the relation (102),

$$q = \frac{2K\,e^{\tilde{q}}}{K^2 - e^{2\tilde{q}}}$$

$$= \frac{\left|\frac{G'}{f'}\right|}{1 - \frac{1}{4}\left|\frac{G'}{f'}\right|^2} \tag{115}$$

If, then, $f(\zeta)$ is so determined that a prescribed flow around a given body is obtained, it follows that for a given value ζ equation (112) determines a point of the flow and equation (115) determines the velocity at that point.

At a stagnation point $q = 0$. From equation (115), $q = 0$ when $G'(\zeta) = 0$. Thus, stagnation points of the flow occur wherever stagnation points occur in the incompressible flow $G(\zeta)$ in the ζ-plane.

If as $\zeta \longrightarrow \infty$, $z \longrightarrow \infty$, and $G'(\zeta)$ is bounded away from zero at infinity, then, from equation (115), $f'(\zeta)$ must be regular and unequal to zero at infinity, if the flow is to have a velocity $q_\infty \neq 0$ at infinity. Thus, the most general form that $f(\zeta)$ can have is

$$f(\zeta) = b_{-1} + b_{o1}\zeta + b_1\log\zeta + \sum_{n=2}^{\infty} b_n\frac{1}{\zeta^{n-1}},\quad b_o \neq 0 \tag{116}$$

and

$$f'(\zeta) = b_0 + b_1 \frac{1}{\zeta} - \sum_{n=2}^{\infty} (n-1) b_n \frac{1}{\zeta} n \tag{117}$$

where the values of b may be complex constants.

It is known from incompressible-flow theory that uniform flows past closed bodies must also have this form. Therefore, the second term in the right-hand side of equation (112) takes the form

$$\overline{g(\zeta)} = -\frac{1}{4}\int \overline{\frac{[G'(\zeta)]^2}{f'(\zeta)}}\, d\zeta$$

$$= \bar{C}_{-1} + \bar{C}_0 \bar{\zeta} + \bar{C}_1 \log \bar{\zeta} + \sum_{n=2}^{\infty} \bar{C}_n \frac{1}{\bar{\zeta}^{n-1}} \tag{118}$$

If the flow at infinity is to be horizontal and of magnitude q_∞, then from equation (114) and the fact that $\tilde{q}_\infty = 0$

$$\lim_{\zeta \to \infty} \frac{K}{2} \frac{G'(\zeta)}{f'(\zeta)} = 1 \tag{119}$$

When $G(z)$ is given by equation (113),

$$\lim_{\zeta \to \infty} f'(\zeta) = b_0$$

$$= \lim_{z \to \infty} \left[\frac{K}{2} G'(\zeta)\right]$$

$$= \frac{K}{2} q_\infty \tag{120}$$

where $\lim_{\zeta \to \infty} f'(\zeta) = b_0$ is the condition that the flow at infinity be horizontal and $q = q_\infty$.

In order to examine the shape of the body in the stream of the flow, the circle $\zeta = Re^{i\varphi}$ is mapped through equation (112) into the z-plane. In order for the circle to map into a closed body, it follows from equations (116) and (118) that

$$b_1 \, i\varphi - \bar{c}_1 \, i\varphi = 0$$

or

$$b_1 - \bar{c}_1 = 0 \tag{121}$$

It is of interest to note that, when $f(\zeta) = \zeta$, the formula of Tsien (reference 6) is obtained; namely

$$z = \zeta - \frac{1}{4} \int \overline{[G'(\zeta)]^2 \, d\zeta} \tag{122}$$

The coefficient in the integral term is different only because of a different normalization.

Again, when

$$f(\zeta) = \int \left[G'(\zeta)\right]^{1-1/n} d\zeta$$

Bers' formula is obtained; namely

$$z = \int \left[G'(\zeta)\right]^{1-1/n} d\zeta - \frac{1}{4} \int \overline{\left[G'(\zeta)\right]^{1+1/n} d\zeta} \tag{123}$$

Here, too, the formulas differ only by a normalizing factor.

In Bers' formula n is arbitrary within limits. This freedom enables him to determine the conditions for the flow around a closed body. Since $f(\zeta)$ in formula (112) is arbitrary and analytic, there is an infinite number of arbitrary coefficients. The first two coefficients determine the flow at infinity and that the flow be around a closed body. The other arbitrary coefficients can be fixed to determine the flow around a given body.

A similar technique to that developed by Theodorsen and Garrick (references 14 and 15) for determining the coefficients of $f(\zeta)$ might be developed in order to obtain a prescribed body. Bers has employed this approach by another method with some success. It should be noted that when $M = 0$ the transformation (112) reduces to the initial transformation used by Theodorsen (reference 14). This implies that if an integral equation were set up from equation (112) it would reduce to Theodorsen's when $M = 0$, so that the integral equation would be a generalization of Theodorsen's transformation.

In another paper Bartnoff determined $f(\zeta)$ such that the circle $\zeta = Re^{i\varphi}$ goes into a unit circle in the z-plane (accuracy to within a few percent), thus giving the compressible flow around a circle. This has been done by others, notably by Bers.

The right-hand side of the transformation (112) is invariant under an analytic transformation. For, let

$$\zeta = \zeta(\xi) \tag{124}$$

be an analytic function of ξ; then by direct substitution of equation (124) into equation (112), the transformation becomes

$$z = f\left[\zeta(\xi)\right] - \frac{1}{4}\int \frac{\overline{\left\{d/d\xi\ G\left[\zeta(\xi)\right]\right\}^2}}{d/d\xi\ f\left[\zeta(\xi)\right]}\,d\xi \tag{125}$$

Because the transformation (112) is invariant under a conformal transformation the subsonic flow of a compressible fluid around a circle is of basic importance.

Syracuse University,
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