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CRITICAL STRESS OF THIN-WALLED CYLINDERS IN AXIAL COMPRESSION

By S. B. Batdorf, Murry Schildcrout, and Manuel Stein

SUMMARY

Empirical design curves are presented for the critical stress of thin-walled cylinders loaded in axial compression. These curves are plotted in terms of the nondimensional parameters of small-deflection theory and are compared with theoretical curves derived for the buckling of cylinders with simply supported and clamped edges. An empirical equation is given for the buckling of cylinders having a length-radius ratio greater than about 0.75.

The test data obtained from various sources follow the general trend of the theoretical curve for cylinders with clamped edges, agreeing closely with the theory in the case of short cylinders, but falling considerably below the theoretical results for long cylinders. The discrepancy in the case of long cylinders increases with increasing values of the ratio of radius to wall thickness. Plotting curves for different values of this ratio reduces the scatter in the test data and a certain degree of correlation with theory is achieved. Advantage is taken of this correlation to obtain estimated design curves for cylinders with simply supported edges, for which little experimental information is available.

REVIEW OF PREVIOUS WORK ON PROBLEM

Solutions to the problem of the determination of the critical stress of thin-walled cylinders subjected to axial compression have been presented by a large number of authors. Southwell, Timoshenko, Flügge, and numerous other authors have obtained theoretical solutions by the use of the small-deflection theory. (See, for example, references 1 to 4.) The value given by the small-deflection theory for the buckling stress of a thin-walled cylinder of moderate length having simply supported edges is

$$\sigma_x = \frac{Et}{r\sqrt{3(1-\mu^2)}} = 0.608\frac{Et}{r} \qquad (1)$$

where
σ_x critical compressive stress
E Young's modulus
t wall thickness of cylinder
r radius of cylinder
μ Poisson's ratio (in the present paper μ is taken to be 0.316 whenever a value is assigned to it)

Experiments (references 5 to 10) have shown that the actual critical stress is much lower than that predicted by equation (1). Except in the case of short cylinders, the experiments usually give values only 15 to 50 percent of that predicted theoretically; moreover, the observed buckle pattern is different from that predicted on the basis of theory. A number of attempts have been made to explain these discrepancies theoretically. Flügge (reference 3) considered the deviation of the actual edge supports from the support conditions assumed in the theoretical treatment. Donnell (reference 5) and also Flügge considered the initial deviation from the perfect cylindrical shape. Neither of the two attempted explanations satisfactorily accounts for the discrepancy existing between the theoretical and experimental values of the buckling stresses of cylinders.

Von Kármán and Tsien (reference 11) introduced a large-deflection theory to account for the buckling behavior of long cylinders. They showed that a long cylinder can be in equilibrium in a buckled state at a stress that is much smaller than the critical stress of small-deflection theory and also succeeded in accounting for the buckle pattern observed in the early stages of buckling. Reference 11 suggested that when a cylinder has an initial imperfection or is subjected to a shock, it might pass into one of these buckled states without ever having reached the critical load given by equation (1). Based on the same approach, a theory for the buckling stresses of perfect cylinders was proposed by Tsien (reference 12), which gave for loading by rigid screw-power testing machine

$$\sigma_x = 0.370\frac{Et}{r} \qquad (2)$$

and for loading by ideal hydraulic testing machine or dead weight

$$\sigma_x = 0.238\frac{Et}{r} \qquad (3)$$

The large-deflection theories fail in two respects to describe adequately the buckling behavior of actual cylinders. First, the theories are formulated only for long cylinders; equations (2) and (3) seriously underestimate the critical stress of very short cylinders. Second, even for long cylinders, attempts to determine experimentally the numerical coefficient C in the buckling formula

$$\sigma_x = CE\frac{t}{r} \qquad (4)$$

have resulted in appreciable experimental scatter. The experimental scatter is due at least in part to the initial imperfections of construction always present in real cylinders. (See fig. 1.)
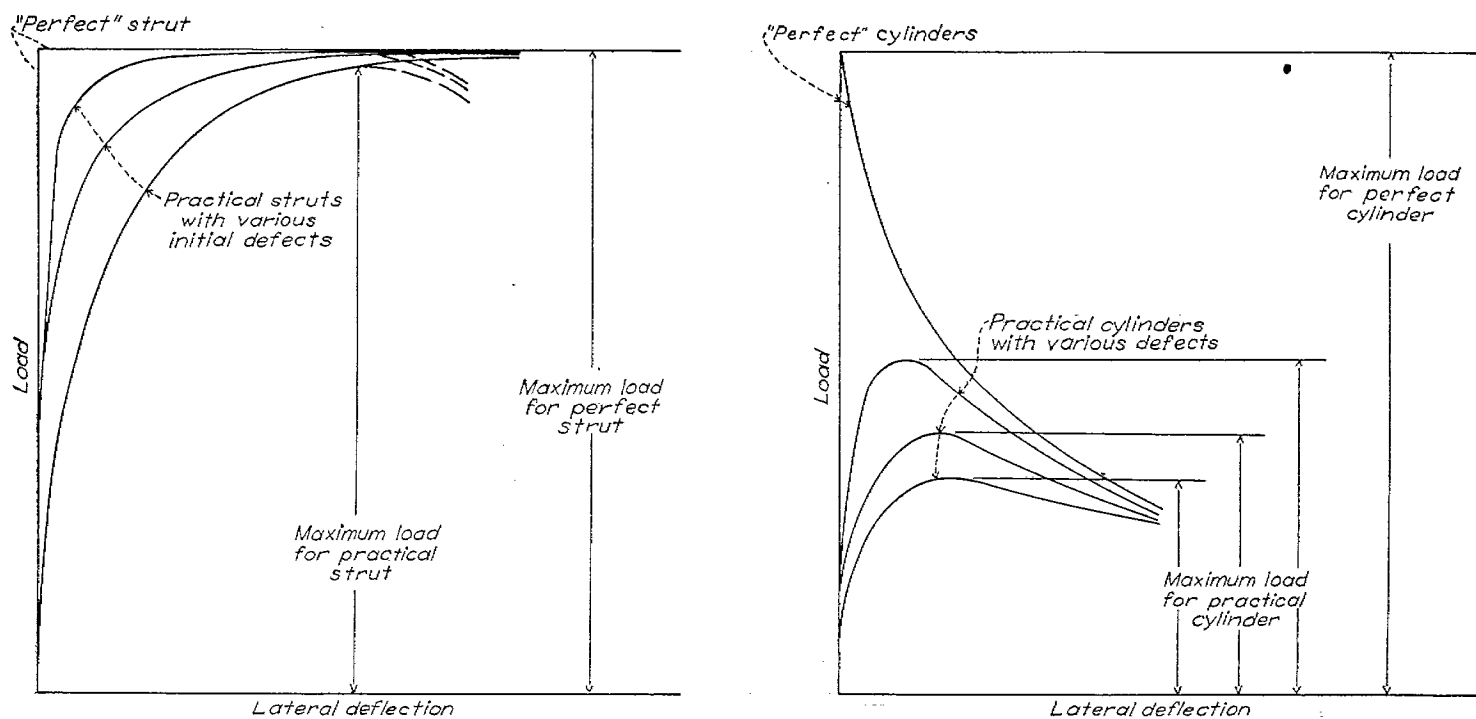
FIGURE 1.—Effect of initial defects upon the maximum compressive loads of struts and cylinders. (From reference 13, fig. 18.)

In the absence of a complete and satisfactory theoretical solution for the critical stress of cylinders, a number of authors have proposed empirical formulas derived from test data (references 6 to 8). One such formula, which takes into account the length of the cylinder, is due to Ballerstedt and Wagner (reference 8):

$$\frac{\sigma_x}{E} = 3.3 \left(\frac{t}{L}\right)^2 + 0.2 \left(\frac{t}{r}\right) \tag{5}$$

The first parameter in this equation $\left(\frac{t}{L}\right)^2$ is appropriate for flat sheet and the second parameter $\frac{t}{r}$ is included to take into account the effect of curvature. More recently Kanemitsu and Nojima (reference 9) compiled all available previous experimental results and conducted a number of tests of their own. The formula of Wagner and Ballerstedt was modified in reference 9 to bring it into better agreement with experiment as follows:

$$\frac{\sigma_x}{E} = 0.16 \left(\frac{t}{L}\right)^{1.3} + 9 \left(\frac{t}{r}\right)^{1.6} \tag{6}$$

Within its range of application $\left(0.1 < \frac{L}{r} < 1.5; \right.$ $500 \leq \frac{r}{t} \leq 3000 \left.\right)$ equation (6) is in considerably better agreement with experiment than equation (5) but, because of the change in the exponents of the parameters, equation (6) does not have any rational basis and must be regarded as

purely empirical. A complete divorce of theory and experiment, however, cannot be regarded as a satisfactory permanent settlement of the problem, and the present report attempts to bring theory and experiment into reasonable accord.

CONTRIBUTION OF PRESENT PAPER

In the present paper the available test data for critical stresses of cylinders are reexamined and theoretical results are used as a guide in fairing the curves, in extending the range of validity of the existing empirical results, and in achieving a more rational interpretation of the test data. For this purpose the test data are plotted in terms of the parameters of cylinder theory and are compared with theoretical results derived in the appendix on the basis of small-deflection theory.

The cylinder-theory parameters used are

$$k_x = \frac{\sigma_x t L^2}{D \pi^2}$$

and

$$Z = \frac{L^2}{rt} \sqrt{1 - \mu^2}$$

where

D flexural stiffness of plate per unit length $\left(\frac{Et^3}{12(1-\mu^2)}\right)$

L length of cylinder

Z curvature parameter

k_x critical stress coefficient appearing in the equation

$$\sigma_x = \frac{k_x \pi^2 D}{L^2 t}$$

The experimental data are used as the principal guide in determining the critical compressive stresses of long cylinders (large values of Z) and the theoretical results are used mainly to supplement the test data in determining the critical stress of very short cylinders (small values of Z). The experimental scatter is reduced by presenting different curves for cylinders with different values of the ratio of radius to wall thickness on the assumption that for long cylinders this ratio furnishes some indication of the initial imperfections of the cylinder. Although these curves were determined partly on the basis of theoretical considerations, they are for convenience referred to herein as empirical curves.

RESULTS AND CONCLUSIONS

The critical compressive stress for cylinders is given by the equation

$$\sigma_x = \frac{k_x \pi^2 D}{L^2 t} \tag{7}$$

where the values of k_x may be obtained from figure 2 for cylinders with either clamped or simply supported edges. The design curves for cylinders with clamped edges are established by the test results reported in references 5 to 9. (See fig. 3.) Each curve was faired through a series of test points which were plotted for cylinders with nearly the same ratio of radius to wall thickness r/t. The estimated (dashed) parts of the design curves for simple support were obtained by fairing between the known experimental curves for long cylinders (large values of Z), which according to theory should be the same whether the cylinders have simply supported or clamped edges, and the theoretical curves for very short cylinders (small values of Z).
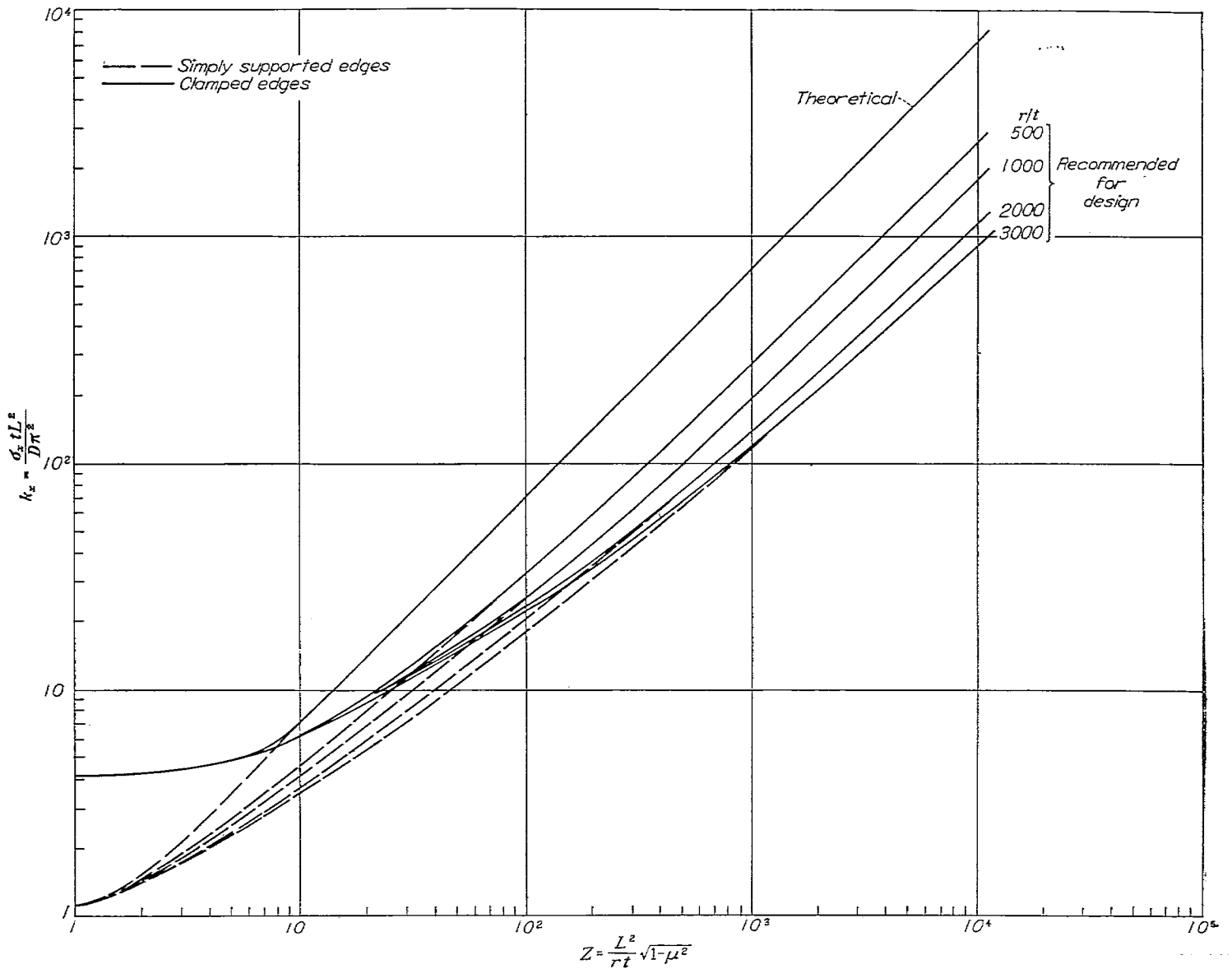

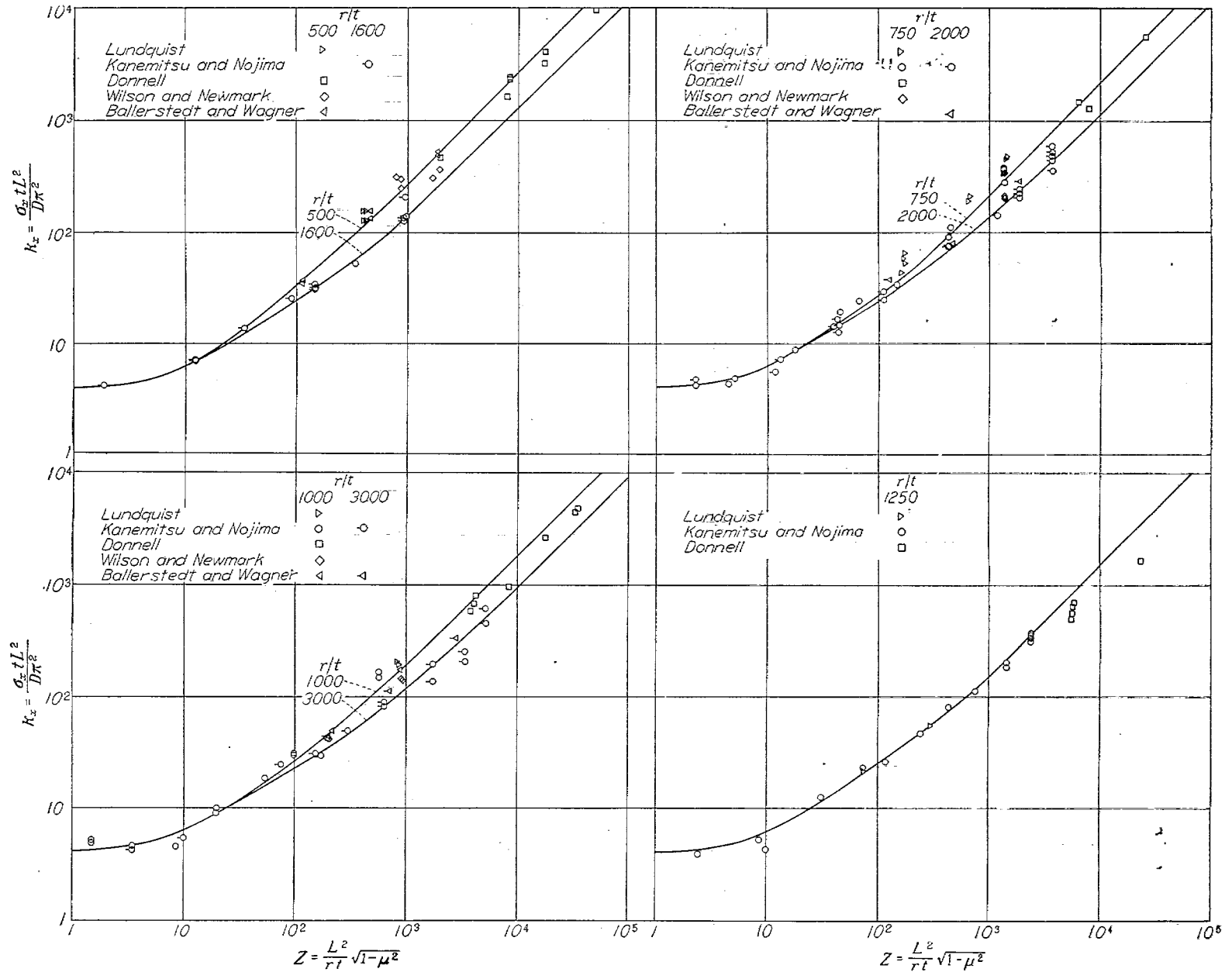FIGURE 2.—Critical stress coefficients for thin-walled circular cylinders subjected to axial compression.

FIGURE 3.—Comparison of test results with design curves recommended for cylinders with clamped edges. (Data from references 5 to 9.)

For long cylinders the buckling stress is considerably below the theoretical buckling stress, the amount of the discrepancy depending on the ratio of radius to wall thickness. For very short cylinders the values of the critical stresses approach those for flat plates (simply supported ends, $k_x = 1$; clamped ends, $k_x = 4$), for which the agreement between theoretical and experimental results is known to be good. The general trend of each empirical curve is similar to that of the theoretical curve, indicating the existence of a certain degree of correlation between theory and test data.

At large values of Z, the curves for k_x become straight lines given by the formula

$$k_x = 1.15 CZ \qquad (8)$$

where C depends on the ratio of radius to wall thickness of the cylinders in the manner shown in figure 4. From equations (7) and (8) the following expression for the

critical stress is obtained

$$\sigma_x = CE\frac{t}{r} \qquad (9)$$

Equations (8) and (9) may be used when the length of the cylinder is more than about ¾ of the radius. The empirical curves of reference 10 indicate that the critical stress is substantially independent of length when the length is greater than about ¾ of the radius. (This result may be checked by noting that for $Z > 0.5\frac{r}{t}$ the experimental curves of figure 2 are substantially straight lines of unit slope.)

In figure 5, the empirical formula of Kanemitsu and Nojima (equation (6) of the present paper, the best previously published formula for the buckling of cylinders) is plotted in terms of the parameters k_x and Z. The curves are cut
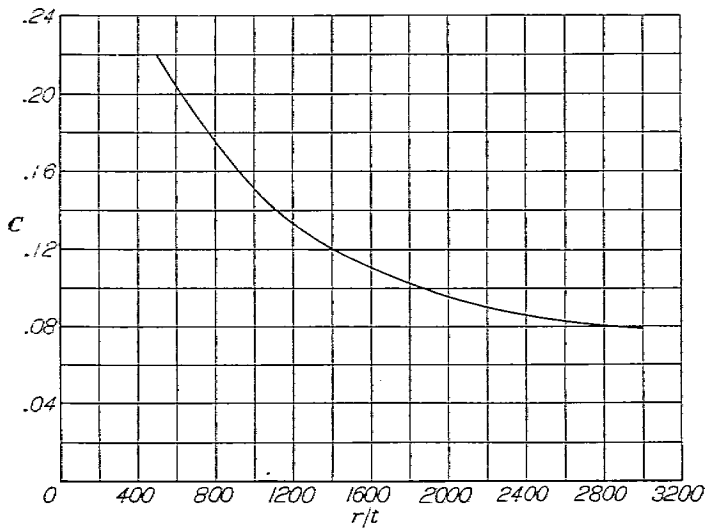
FIGURE 4.—Coefficient for computing critical axial compressive stresses of intermediate length and long cylinders $\left(\begin{array}{c}k_x=1.15CZ \\ \sigma_x=CE\frac{t}{r}\end{array}\right\}$ for $\frac{L}{r}\geqq 0.75$ $\Big)$.

off at those values of Z corresponding to the lower limits of the range of dimensions within which the formula was intended to apply. In general, for the range covered, the curves are in reasonable agreement with the test data and with the curves of the present paper for cylinders with clamped edges. The practical importance of the present approach lies in the fact that the use of the theoretical parameters and the theoretical solutions as a guide in fairing the curves permits the removal of the lower limits on these curves and also permits estimated curves to be drawn for the buckling stresses for simply supported cylinders, although experimental data are available only for cylinders with clamped edges.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
    NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
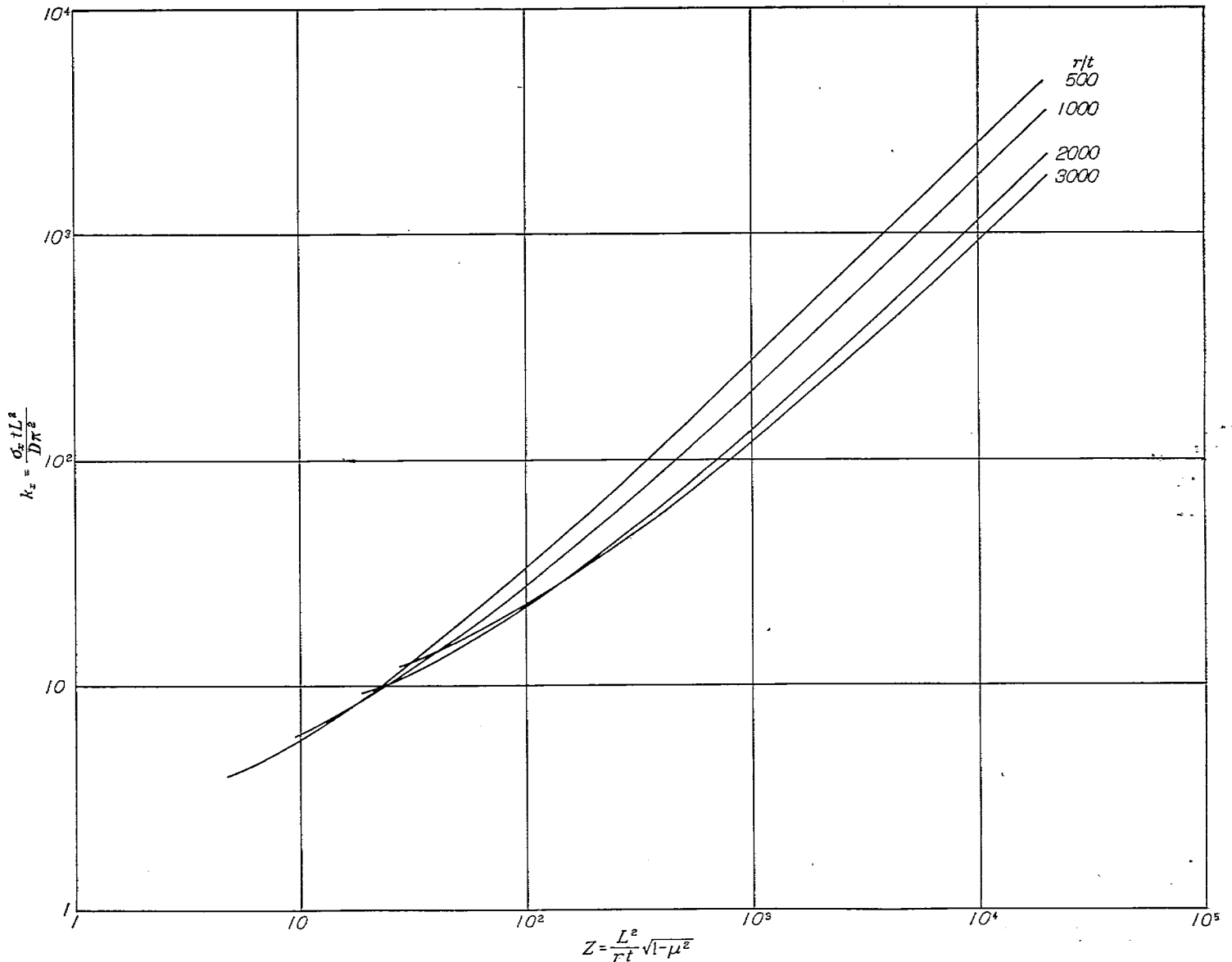   LANGLEY FIELD, VA., March 20, 1947.

FIGURE 5.—Critical stress coefficients for thin-walled cylinders implied by the formula $\frac{\sigma_x}{E}$ =0.16 $\left(\frac{t}{L}\right)^{1.3}$+9 $\left(\frac{t}{r}\right)^{1.6}$ given by Kanemitsu and Nojima in reference 9.

APPENDIX

THEORY FOR CYLINDERS BUCKLING UNDER AXIAL COMPRESSION

SYMBOLS

m positive integer

r radius of cylinder

t wall thickness of cylinder

w radial component of displacement, positive outward

x axial coordinate of cylinder

y circumferential coordinate of cylinder

C coefficient appearing in $\sigma_x = CE\,\dfrac{t}{r}$

D flexural stiffness of plate per unit length $\left(\dfrac{Et^3}{12(1-\mu^2)}\right)$

E Young's modulus

L length of cylinder

Q operator on w defined in appendix

Z curvature parameter $\left(\dfrac{L^2}{rt}\sqrt{1-\mu^2}\ \text{or}\ \left(\dfrac{L}{r}\right)^2\dfrac{r}{t}\sqrt{1-\mu^2}\right)$

a_m coefficient of deflection function

k_x critical-compressive-stress coefficient appearing in the

     formula $\sigma_x = \dfrac{k_x \pi^2 D}{L^2 t}$

$M_m = [(m-1)^2 + \beta^2]^2 + \dfrac{12Z^2(m-1)^4}{\pi^4[(m-1)^2+\beta^2]^2} - (m-1)^2 k_x$

V_m deflection function defined in the appendix

$\beta = \dfrac{L}{\lambda}$

λ half wave length of buckles in circumferential direction

μ Poisson's ratio

σ_x critical axial compressive stress

$\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

$\nabla^4 = \dfrac{\partial^4}{\partial x^4} + 2\dfrac{\partial^4}{\partial x^2 \partial y^2} + \dfrac{\partial^4}{\partial y^4}$

∇^{-4} the inverse of ∇^4 defined by $\nabla^{-4}\nabla^4 w = w$

THEORETICAL SOLUTION

The critical compressive stress at which buckling occurs in a cylindrical shell may be obtained by solving the equation of equilibrium.

Equation of equilibrium.—The equation of equilibrium for a slightly buckled cylindrical shell under axial compression is (reference 14)

$$D\nabla^4 w + \frac{Et}{r^2}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + \sigma_x t\frac{\partial^2 w}{\partial x^2} = 0 \qquad (A1)$$

where x is the coordinate in the axial direction and y is the coordinate in the circumferential direction. Dividing through equation (A1) by D gives

$$\nabla^4 w + \frac{12Z^2}{L^4}\nabla^{-4}\frac{\partial^4 w}{\partial x^4} + k_x\frac{\pi^2}{L^2}\frac{\partial^2 w}{\partial x^2} = 0 \qquad (A2)$$

where the dimensionless parameters Z and k_x are defined by

$$Z = \frac{L^2}{rt}\sqrt{1-\mu^2}$$

$$k_x = \frac{\sigma_x t L^2}{D\pi^2}$$

The equation of equilibrium may be represented by

$$Qw = 0 \qquad (A3)$$

where Q is defined by

$$Q = \nabla^4 + \frac{12Z^2}{L^4}\nabla^{-4}\frac{\partial^4}{\partial x^4} + k_x\frac{\pi^2}{L^2}\frac{\partial^2}{\partial x^2}$$

Method of solution.—Equation (A2) may be solved by use of the Galerkin method as outlined in reference 15. When this method is applied, the deflection w is expressed in series form as follows

$$w = \sum_{m=1}^{j} a_m V_m \qquad (A4)$$

The set of functions V_m are chosen to satisfy the boundary conditions but need not satisfy the equation of equilibrium. The coefficients a_m are determined by the equations

$$\int_0^{2\lambda}\int_0^L V_m Qw\,dx\,dy = 0 \qquad (m = 1, 2, 3, \ldots j) \quad (A5)$$

In the present paper the deflection functions were chosen to satisfy the following conditions on w at the ends of the cylinder: For simply supported edges

$$w = \frac{\partial^2 w}{\partial x^2} = 0$$

For clamped edges

$$w = \frac{\partial w}{\partial x} = 0$$

Simply supported edges.—An expansion for w that is sinusoidal in the circumferential direction and perfectly general (subject to the boundary conditions for simple support) in the axial direction is

$$w = \sin\frac{\pi y}{\lambda}\sum a_m\sin\frac{m\pi x}{L} \qquad (A6)$$

where λ is the half wave length of the buckles in the circumferential direction. (Equation (A6) is equivalent to equation (A4) if

$$V_m = \sin \frac{\pi y}{\lambda} \sin \frac{m \pi x}{L} \qquad (A7)$$

Substitution of expressions (A6) and (A7) into equation (A5) and integration over the limits indicated give

$$m^2 k_x = (\beta^2 + m^2)^2 + \frac{12 Z^2 m^4}{\pi^4 (m^2 + \beta^2)^2} \qquad (m = 1, 2, 3, \ldots) \qquad (A8)$$

where

$$\beta = \frac{L}{\lambda}$$

The minimum value of k_x for a given Z is found by assuming a value for m and minimizing k_x with respect to β. This procedure is followed for various values of m until a minimum k_x is reached. Figure 2 presents the theoretical critical stress coefficients for cylinders with simply supported edges subjected to axial compression.

Clamped edges.—A procedure similar to that used for cylinders with simply supported edges may be followed for cylinders with clamped edges. The deflection function used is the following series

$$w = \sin \frac{\pi y}{\lambda} \sum a_m \left(\cos \frac{(m-1)\pi x}{L} - \cos \frac{(m+1)\pi x}{L} \right) \qquad (A9)$$

Each term of this series satisfies the conditions on w at the edges. The function V_m is now defined as

$$V_m = \sin \frac{\pi y}{\lambda} \left(\cos \frac{(m-1)\pi x}{L} - \cos \frac{(m+1)\pi x}{L} \right) \qquad (A10)$$
$$(m = 1, 2, 3, \ldots)$$

After the same operations are carried out for clamped edges as those carried out for the case of simply supported edges, the following equations result:

For $m = 1$,
$$a_1(2M_1 + M_3) - a_3 M_3 = 0$$
For $m = 2$,
$$a_2(M_2 + M_4) - a_4 M_4 = 0 \qquad (A11)$$
For $m = 3, 4, 5, \ldots ,$
$$a_m(M_m + M_{m+2}) - a_{m-2} M_m - a_{m+2} M_{m+2} = 0$$

where

$$M_m = [(m-1)^2 + \beta^2]^2 + \frac{12 Z^2 (m-1)^4}{\pi^4 [(m-1)^2 + \beta^2]^2} - (m-1)^2 k_x$$
$$(m = 1, 2, 3, \ldots)$$

These equations have a solution if the following infinite determinant vanishes:

$m=1$	$2M_1 + M_3$	0	$-M_3$	0	0	0	0	0	$\ldots$
$m=2$	0	$M_2 + M_4$	0	$-M_4$	0	0	0	0	$\ldots$
$m=3$	$-M_3$	0	$M_3 + M_5$	0	$-M_5$	0	0	0	$\ldots$
$m=4$	0	$-M_4$	0	$M_4 + M_6$	0	$-M_6$	0	0	$\ldots$
$m=5$	0	0	$-M_5$	0	$M_5 + M_7$	0	$-M_7$	0	$\ldots$
$m=6$	0	0	0	$-M_6$	0	$M_6 + M_8$	0	$-M_8$	$\ldots$
$m=7$	0	0	0	0	$-M_7$	0	$M_7 + M_9$	0	$\ldots$
$m=8$	0	0	0	0	0	$-M_8$	0	$M_8 + M_{10}$	$\ldots$
$\ldots$									$\ldots$

(A12)

If the rows and columns are rearranged the infinite determinant can be factored into the product of two infinite subdeterminants. The critical stresses may then be obtained from the following equation:

$m=1$	$2M_1 + M_3$	$-M_3$	0	0	$\ldots$	0	0	0	0	$\ldots$	
$m=3$	$-M_3$	$M_3 + M_5$	$-M_5$	0	$\ldots$	0	0	0	0	$\ldots$	
$m=5$	0	$-M_5$	$M_5 + M_7$	$-M_7$	$\ldots$	0	0	0	0	$\ldots$	
$m=7$	0	0	$-M_7$	$M_7 + M_9$	$\ldots$	0	0	0	0	$\ldots$	
$\ldots$					$\ldots$					$\ldots$	$=0$ (A13)
$m=2$	0	0	0	0	$\ldots$	$M_2 + M_4$	$-M_4$	0	0	$\ldots$	
$m=4$	0	0	0	0	$\ldots$	$-M_4$	$M_4 + M_6$	$-M_6$	0	$\ldots$	
$m=6$	0	0	0	0	$\ldots$	0	$-M_6$	$M_6 + M_8$	$-M_8$	$\ldots$	
$m=8$	0	0	0	0	$\ldots$	0	0	$-M_8$	$M_8 + M_{10}$	$\ldots$	
$\ldots$					$\ldots$					$\ldots$	

The infinite subdeterminant involving terms with odd subscripts corresponds to a symmetrical buckling pattern (a buckling pattern symmetrical about the plane perpendicular to and bisecting the axis of the cylinder). The infinite subdeterminant involving terms with even subscripts corresponds to an antisymmetrical buckling pattern. For brevity these subdeterminants will be referred to as the odd determinant and the even determinant, respectively.

The first approximation is
Odd determinant:
$$2M_1 + M_3 = 0 \qquad (A14)$$
Even determinant:
$$M_2 + M_4 = 0 \qquad (A15)$$

The second approximation is
Odd determinant:
$$2M_1(M_3 + M_5) + M_3M_5 = 0 \qquad (A16)$$
Even determinant:
$$M_2(M_4 + M_6) + M_4M_6 = 0 \qquad (A17)$$

These equations show that for a selected value of the curvature parameter Z the critical buckling stress of a cylinder depends upon the circumferential wave length. Since a structure buckles at the lowest stress at which instability can occur, k_x is minimized with respect to the wave length by substituting values of β into the equations until the minimum k_x can be obtained from a plot of k_x against β. For a given Z the lower of the two values obtained from equations (A14) and (A15) is the first approximation of the critical buckling stress and, similarly, the lower of the two values of k_x obtained from equations (A16) and (A17) is the second approximation of the critical buckling stress.

Figure 2 presents the theoretical critical stress coefficients for cylinders with clamped edges in axial compression as obtained from the second approximation, together with the exact curve for the case of simply supported edges. Although this solution is an upper-limit solution, the second approximation for the critical stress coefficient of a cylinder with clamped edges must be very close to being exact for intermediate and large values of Z because it is almost identical with the exact solution for a cylinder with simply supported

edges, and the critical stress of a cylinder with clamped edges cannot be less than the critical stress for a cylinder with simply supported edges. For values of Z approaching zero, the accuracy of the second approximation is indicated by the fact that it coincides with the known exact solution ($k_x = 4$) for a long flat plate with clamped edges.

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