NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1006

VELOCITY DISTRIBUTION ON WING SECTIONS OF ARBITRARY SHAPE IN COMPRESSIBLE POTENTIAL FLOW

I - SYMMETRIC FLOWS OBEYING THE SIMPLIFIED DENSITY-SPEED RELATION

By Lipman Bers
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Washington
April 1946

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SUMMARY

As a first step toward the computation of the velocity distribution along a wing profile of arbitrary shape in a compressible fluid, the circulation-free flow around a symmetrical profile is treated under the assumption of the simplified density-speed relation due to Tchaplygin, Kármán, and Tsien. The velocity distribution problem is reduced to a non-linear integral equation which is solved by a fairly rapidly convergent iteration method. Numerical examples are given.

INTRODUCTION

The central problem in the two-dimensional theory of a potential flow of a perfect fluid around an airfoil profile is that of determining the pressure distribution on a profile of given shape if the speed and direction of the flow at infinity (undisturbed flow) are known. A solution of this problem should consist not merely of giving a mathematical existence proof but of indicating a method for obtaining numerical results of reasonable accuracy in a reasonable amount of time.

The difficulty of the problem depends essentially upon the prescribed speed at infinity. If this speed does not exceed a certain limiting value (depending upon the profile) the flow will be everywhere subsonic. For higher values of the speed at infinity the flow becomes partly supersonic (mixed or supercritical flow). Finally, it is probable that for too high values of the speed at infinity a potential flow becomes either mathematically impossible or unstable. The case of mixed flow is the more important one, both from the practical and theoretical points of view. Nevertheless, it seems that the complete solution of the problem of everywhere subsonic flows is a necessary prerequisite for a successful attack on the problem of transition through the speed of sound. (In fact, at present the very existence of mixed flows past a profile has not yet been proved.)

In view of the admitted difficulty of the problem it is advisable to develop the mathematical apparatus by considering first the simplest possible cases. The most radical simplification would be, of course, to neglect compressibility altogether. Under these assumptions the pressure distribution problem has been solved completely. (See references 1 and 2.) In the present report the following two simplifying assumptions are made:

A. Only circulation-free flows around symmetrical profiles are considered.

B. It is assumed that the velocity potential satisfies the simplified differential equation resulting from the so-called Chaplygin-Kármán-Tsien equation of state. (Cf. references 3, 4, and 5.)

Some remarks may be made concerning this second assumption. In general, the velocity potential $\varphi(x,y)$ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left(\rho\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho\frac{\partial \varphi}{\partial y}\right) = 0 \tag{1}$$

where ρ is the density of the fluid. Since the density is a given function of the speed q

$$\rho = \rho(q), \qquad q^2 = \left(\frac{\partial \varphi}{\partial x}\right)^2 + \left(\frac{\partial \varphi}{\partial y}\right)^2 \tag{2}$$

equation (1) is nonlinear. The function $\rho(q)$ is determined by the pressure-density relation (equation of state). In an isentropic flow the pressure p satisfies the relation

$$p \rho^{-\gamma} = \text{constant} \tag{3}$$

where γ is the ratio of specific heats for constant pressure and constant volume. (The standard value of γ is 1.405.) This implies the density-speed relation

$$\rho = \rho_0 \left(1 - \frac{\gamma - 1}{2} \frac{q^2}{a_0^2}\right)^{1/(\gamma - 1)} \tag{4}$$

where a_0 is the speed of sound at a stagnation point and ρ_0 the stagnation density. Chaplygin noticed that the equation satisfied by the potential becomes simpler if the density-speed relation is taken in the form

$$\rho = \rho_0 \left(1 + \frac{q^2}{a_0^2}\right)^{-\frac{1}{2}} \tag{5}$$

This relation may be obtained formally from (4) by setting $\gamma = -1$. Though this value of γ violates fundamental physical laws, it should be observed that only the density-speed relation and not the pressure-density relation enters in the equation for the potential.

As a matter of fact, the function (5) behaves qualitatively in the same way as does the function (4) within the subsonic range; that is, for $0 \leqq q^2 \leqq 2a_0^2/(1 + \gamma)$, and for small values of q/a_0 the function (5) gives a good numerical approximation to (4).

Von Kármán and Tsien justify the use of the value $\gamma = -1$ by the remark that it is possible to determine such values of the constants A and B that the pressure-density relation

$$p = A/\rho + B$$

will give a good approximation to the relation (3) for values of p and ρ close to some preassigned values, say to the values of p and ρ for the undisturbed flow. This remark is of interest as far as computations of the pressure distribution are concerned. It in no way affects the velocity distribution for, as it was already noticed, the differential equation for the potential function depends only upon the density-speed relation, and the preceding pressure-density relation leads to the same equation (5) no matter what values are assigned to A and B.

It should be emphasized, however, that the primary purpose of this report is not to facilitate the use of the application of the approximate relation (5) but rather to develop methods which could be extended to the case of the actual density-speed relation.

In the following, use will be made of certain results contained in a previous report. (See reference 6.)

This investigation, conducted at Brown University, was sponsored by and conducted with the financial assistance of the National Advisory Committee for Aeronautics.

The author largely profited from several instructive discussions he had with Professor S. E. Warschawski. He also is indebted to Mr. Charles Saltzer for competent assistance.

SYMBOLS

$A(\omega)$	auxiliary function defined by equation (35)
a	local speed of sound
a_o	speed of sound at a stagnation point
$B(\omega)$	auxiliary function defined by equation (35)
C, C_j	positive constants
dS^2	non-Euclidean length element defined by equation (22)
$E(P)$	domain exterior to the profile P
F	integral transformation defined in section 5
$f(\omega)$	function defining the mapping of the circle into the profile P
$f_n(\omega)$	nth approximation to the function $f(\omega)$

G	complex potential of a compressible flow
$g(\sigma)$	function inverse to $f(\omega)$
g_{ik}	coefficients of the metric (22)
$h(\omega)$	function defined by equation (44)
Im( )	imaginary part of ( )
P	profile surrounded by a compressible flow
p	pressure
q	local speed
q_∞	speed of the undisturbed flow
$\tilde{q}(\sigma)$	value of q at a boundary point
q^*	distorted speed
q^*_∞	distorted speed of the undisturbed flow
$\tilde{q}^*(\sigma)$	value of q^* at a boundary point
M	local Mach number
M_∞	stream Mach number
R	radius of the circle in the ζ-plane
Re( )	real part of ( )
s	arc length measured along P
S	length of the curve P
t	parameter occurring in section 8
u,v	components of the velocity
w^*	distorted velocity
x,y	Cartesian coordinate in the z-plane
z	complex variable

$Z(s)$	coordinate of the profile as a function of the arc length
Z_L	leading edge
Z_T	trailing edge
α	angle at the trailing end
γ	exponent in the adiabatic relation
ϵ	thickness parameter of a symmetrical Joukowski profile
ϵ_i	constants occurring in section 8
$\Theta(\sigma)$	slope of the profile P
θ	angle between the velocity vector and the x-axis
$\tilde{\theta}$	value of θ on the boundary
ζ	auxiliary complex variable
λ	square of the distorted speed of the undisturbed flow
Λ	function defined by equation (54)
ξ, η	Cartesian coordinates in the ζ-plane
ρ	density
ρ_0	stagnation density
σ	dimensionless length parameter along the profile P
φ	velocity potential
$\tilde{\varphi}$	value of φ at the boundary
χ	auxiliary analytic function defined by equation (34)
ψ	stream function
ω	argument of a point on the circle $\|\zeta\| = R$

ANALYSIS

1. The Boundary Value Problem

Consider a symmetrical profile P in the plane of the complex variable $z = x + iy$. It will be assumed that P is a smooth curve, except, perhaps, for a sharp angle at the trailing edge z_T, that the x-axis is parallel to the axis of symmetry of the profile and that the profile is given by an equation of the form

$$z = Z(s), \quad 0 \leq s \leq S \tag{6}$$

where s is the arc length on the curve P measured in the counterclockwise direction from the point z_T. Then S is the total length of the profile and

$$z_L = Z(S/2)$$

is the leading edge. It will be convenient to introduce the dimensionless parameter

$$\sigma = 2\pi s/S \tag{7}$$

The function

$$\Theta(\sigma) = \arg Z'(\sigma S/2\pi) \tag{8}$$

where

$$Z'(s) = dZ/ds$$

depends only upon the shape but not upon the size or position of P. Note that by virtue of the foregoing assumptions

$$\Theta(0) = \pi - \alpha/2, \quad \Theta(\pi) = 3\pi/2, \quad \Theta(2\pi) = 2\pi + \alpha/2$$

where α is the angle at the trailing edge, $0 \leq \alpha \leq \pi$, and

$$\Theta(2\pi - \sigma) = 3\pi - \Theta(\sigma), \quad 0 \leq \sigma \leq \pi \tag{9}$$

The equation of the curve P may be written in the form

$$z = Z(\sigma S/2\pi) = \frac{S}{2\pi}\int_0^{\sigma} e^{i\Theta(\sigma)}\, d\sigma + Z_T \qquad (10)$$

Now let $a_o\varphi(x,y)$ be the potential of a circulation-free flow of a compressible fluid past the profile P; that is, a function such that

$$u = a_o \frac{\partial\varphi}{\partial x}, \qquad v = a_o \frac{\partial\varphi}{\partial y}$$

are the components of the velocity in the x- and y-directions, respectively, a_o being the speed of sound at a stagnation point. The function $\varphi(x,y)$ is defined and one-valued in the domain $E(P)$ exterior to P and satisfies the boundary condition

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad P \qquad (11)$$

as well as the condition

$$\frac{\partial\varphi}{\partial x} \longrightarrow \frac{q_\infty}{a_o}, \qquad \frac{\partial\varphi}{\partial y} \longrightarrow 0 \quad \text{as } z \longrightarrow \infty \qquad (12)$$

Here $\partial/\partial n$ denotes differentiation in the direction normal to P, and q_∞ is the speed of the flow far away from the profile (undisturbed flow).

The conjugate complex velocity is given by

$$w = u - iv = qe^{-i\theta} \qquad (13)$$

where q is the speed and θ the angle between the velocity vector and the x-axis. The function θ satisfies the condition

$$\begin{aligned} \theta &= \Theta - \pi \quad \text{on the upper bank of } P \\ \theta &= \Theta - 2\pi \quad \text{on the lower bank of } P \end{aligned} \qquad (14)$$

and

$$\theta \longrightarrow 0 \quad \text{as} \quad z \longrightarrow \infty$$

Under the assumption of the approximate density-speed relation (cf. Introduction)

$$\rho = \frac{\rho_0}{\sqrt{1 + q^2/a_0^2}}$$

the equation of continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

takes the form

$$\left\{1 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right\} \frac{\partial^2\varphi}{\partial x^2} - 2\,\frac{\partial\varphi}{\partial x}\,\frac{\partial\varphi}{\partial y}\,\frac{\partial^2\varphi}{\partial x\partial y} + \left\{1 + \left(\frac{\partial\varphi}{\partial x}\right)^2\right\} \frac{\partial^2\varphi}{\partial y^2} = 0 \qquad (15)$$

This is the classical equation of a minimal surface.

The determination of the flow around a given profile P requires the integration of the differential equation (15) under the boundary conditions (11) and (12). In the case of an incompressible flow the corresponding boundary value problem can be reduced to the problem of mapping the domain $E(P)$ conformally into a domain exterior to a circle. A similar mapping will be defined presently for the flow considered here.

2. Mapping of the Profile into a Circle

The stream function of the flow $\psi(x,y)$ is defined by the equations

$$\frac{\partial\varphi}{\partial x} = \frac{\rho_0}{\rho}\,\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\varphi}{\partial y} = -\,\frac{\rho_0}{\rho}\,\frac{\partial\psi}{\partial x}$$

This function is constant along any streamline and can be normalized so that

$$\psi = 0 \quad \text{on} \quad P$$

The complex potential $G(z)$ is defined by

$$G(z) = \varphi(x,y) + i\psi(x,y) \tag{16}$$

Let the potential φ be normalized so that

$$\varphi_L = -\varphi_T$$

where φ_L denotes the value of φ at $z = z_L$ and φ_T the value of φ at $z = z_T$. This can always be achieved by adding a constant to φ. The function

$$G = G(z) \tag{17}$$

maps the domain $E(P)$ into the domain in the G-plane exterior to the slit

$$\psi = 0, \quad -\varphi_T \leq \varphi \leq \varphi_T \tag{18}$$

This latter domain is now mapped conformally into the domain $|\zeta| \geq R$ in the plane of the complex variable $\zeta = \xi + i\eta$ by means of the relation

$$G = \frac{\varphi_T}{2R}\left(\zeta + \frac{R^2}{\zeta}\right) \tag{19}$$

Equations (17) and (19) define a transformation

$$\xi = \xi(x,y), \quad \eta = \eta(x,y) \tag{20}$$

of the domain $E(P)$ into the domain $|\zeta| \geq R$. The points

$$z = Z_L, \quad z = Z_T, \quad z = \infty$$

are taken into the points

$$\zeta = -R, \quad \zeta = R, \quad \zeta = \infty$$

respectively. If R is chosen as

$$R = \frac{\varphi_T a_o}{2q_\infty} \tag{21}$$

the mapping (20) satisfies the conditions

$$\frac{\partial \xi}{\partial \eta} \longrightarrow 0, \qquad \frac{\partial \xi}{\partial x} \longrightarrow 1 \quad \text{as} \quad z \longrightarrow \infty \tag{22}$$

By virtue of the foregoing mapping there exists a one-to-one correspondence between the points of the profile P and those of the circle $|\zeta| = R$. This correspondence can be described by means of a function

$$\sigma = f(\omega)$$

such that the point $Z\,[f(\omega)S/2\pi]$ corresponds to the point $Re^{i\omega}$. Plainly $f(\omega)$ is an increasing function satisfying the conditions

$$f(0) = 0, \qquad f(\pi) = \pi, \qquad f(2\pi) = 2\pi \tag{23}$$

as well as the symmetry condition

$$f(2\pi - \omega) = 2\pi - f(\omega), \qquad 0 \leq \omega \leq \pi \tag{24}$$

In the following sections it will be shown that the knowledge of the function $f(\omega)$ implies the knowledge of the velocity distribution along P.

<u>Remark</u>: In the case of an incompressible flow the mapping just constructed is exactly the standard conformal mapping of the profile into a circle. In the case considered here the mapping (20) is conformal with respect to the Riemann metric (dS) defined in $E(P)$ by means of the formulas

$$dS^2 = g_{11}dx^2 + 2g_{12}dx\,dy + g_{22}dy^2 \tag{25}$$

where

$$g_{11} = \frac{a_o^2 + q^2 \cos^2\theta}{a_o^2 + q^2}$$

$$g_{12} = \frac{q^2 \sin\theta \cos\theta}{a_o^2 + q^2}$$

$$g_{22} = \frac{a_o^2 + q^2 \sin^2\theta}{a_o^2 + q^2}$$

The proof of this assertion follows immediately from the results of a previous report. (See reference 6.)

3. Velocity Distribution Expressed in Terms of the Function $f(\omega)$

At a point $\zeta = Re^{i\omega}$ equation (19) takes the form

$$\varphi = \varphi_T \cos\omega$$

or, by (21)

$$\varphi = 2R \frac{q_\infty}{a_o} \cos\omega \tag{26}$$

Now let $\tilde{q}(\sigma)$ denote the value of q at a point $z[\sigma S/2\pi]$ and $\tilde{\varphi}(\sigma)$ the value of φ at this point. Furthermore let

$$\omega = g(\sigma) \tag{27}$$

be the function inverse to $f(\omega)$. By (26)

$$\tilde{\varphi}(\sigma) = 2R \frac{q_\infty}{a_o} \cos g(\sigma) \tag{28}$$

On the other hand, on the profile P

$$\tilde{q}(\sigma) = \left|\frac{d\tilde{\varphi}(\sigma)}{ds}\right| = \frac{2\pi a_o}{S} \left|\tilde{\varphi}'(\sigma)\right| \tag{29}$$

so that

$$\tilde{q}(\sigma) = \frac{4\pi R}{S} q_\infty \left|\sin g(\sigma)\right| g'(\sigma) \tag{30}$$

This formula shows that the function $g(\sigma)$, and therefore also the function $f(\omega)$ determines the velocity distribution along P but for a constant factor.

A formula permitting a complete determination of the velocity distribution can be derived by introducing the so-called distorted velocity w^* defined by

$$w^* = q^* e^{-i\theta} \tag{31}$$

where q^* is the distorted speed given by

$$q^* = \frac{q}{a_o} \frac{1}{1 + \sqrt{1 + q^2/a_o^2}} \tag{32}$$

Note that q^* always satisfies the inequality $0 \leq q^* < 1$. It has been shown (see, for instance, reference 6) that the complex potential G is an analytic function of the variable w^*. Therefore w^* is an analytic function of G and hence also of the complex variable ζ. The function w^* does not vanish, except at the points $\zeta = -R$ and $\zeta = R$. The imaginary part of the logarithm of w^* is $-\theta$. Along the circle $|\zeta| = R$ the function $-\theta$ may be regarded as a function of the real variable ω, $\zeta = Re^{i\omega}$. This function possesses jumps of the magnitude α and π at $\omega = 0$ and $\omega = \pi$, respectively. It follows from known theorems of function theory that at $\zeta = -R$ the function w^* vanishes as $(\zeta + R)$ and at $\zeta = R$ as $(\zeta - R)^{\alpha/\pi}$, respectively. Furthermore

$$w^*(\infty) = q^*_\infty > 0 \tag{33}$$

Hence the function

$$\chi(\zeta) = w^*(\zeta + R)^{-1} (\zeta - R)^{-\alpha/\pi} \zeta^{1 + \alpha/\pi} \tag{34}$$

is regular for $|\zeta| > R$, continuous for $|\zeta| = R$, and nowhere equal to either zero or infinity.

Therefore $\log \chi(\zeta)$ is a one-valued analytic function which is continuous on the circle $|\zeta| = R$ and regular at $\zeta = \infty$. Set

$$\log X(Re^{i\omega}) = A(\omega) + iB(\omega) \tag{35}$$

$A(\omega)$ may be expressed in terms of $B(\omega)$ by means of the well-known formula

$$A(\omega) = -\frac{1}{2\pi}\int_0^{\pi}\left\{B(\omega+t) - B(\omega - t)\right\} \cot\frac{t}{2}\, dt + A_\infty \tag{36}$$

where A_∞ is the value of log $|X|$ at infinity. (See, for instance, reference 7, p. 243.) Now by (33) and (34)

$$A_\infty = \log q_\infty^* \tag{37}$$

$$A(\omega) = \log \tilde{q}^*[f(\omega)] - \log \left|1 + e^{i\omega}\right| \left|1 - e^{i\omega}\right|^{\frac{\alpha}{\pi}} \tag{38}$$

and

$$B(\omega) = -\tilde{\theta}[f(\omega)] - \arg(1 + e^{i\omega}) - \frac{\alpha}{\pi}\arg(e^{i\omega} - 1) + \left(1 + \frac{\alpha}{\pi}\right)\omega \tag{39}$$

Here $\tilde{\theta}(\sigma)$ and $\tilde{q}^*(\sigma)$ denote the values of θ and q^*, respectively, at the point $Z(\sigma S/2\pi)$ of P. Noting that

$$\left|1 + e^{i\omega}\right| = 2\left|\cos\frac{\omega}{2}\right|, \qquad \left|1 - e^{i\omega}\right| = 2\left|\sin\frac{\omega}{2}\right|$$

and that

$$\arg(1 + e^{i\omega}) = \begin{cases} \dfrac{\omega}{2} & \text{for } 0 < \omega < \pi \\ \dfrac{\omega}{2} + \pi & \text{for } \pi < \omega < 2\pi \end{cases}$$

$$\arg(e^{i\omega} - 1) = \frac{\omega}{2} + \frac{\pi}{2}$$

as well as that by (14)

$$\tilde{\theta}[f(\omega)] = \begin{cases} \Theta[f(\omega)] - \pi & \text{for } 0 < \omega < \pi \\ \Theta[f(\omega)] - 2\pi & \text{for } \pi < \omega < 2\pi \end{cases} \tag{40}$$

equations (38) and (39) can be written in the form

$$\log \tilde{q}^*\left[f(\omega)\right] = A(\omega) + \log\left\{2^{1+\frac{\alpha}{\pi}}\left|\cos\frac{\omega}{2}\right|\,\left|\sin\frac{\omega}{2}\right|^{\frac{\alpha}{\pi}}\right\} \tag{41}$$

$$B(\omega) = -\Theta\left[f(\omega)\right] + \frac{\alpha+\pi}{2\pi}\,\omega + \left(\pi - \frac{\alpha}{2}\right) \tag{42}$$

From (41), (42), and (36), it follows that

$$\tilde{q}^*\left[f(\omega)\right] = q_\infty^*\, 2^{1+\frac{\alpha}{\pi}}\left|\cos\frac{\omega}{2}\right|\,\left|\sin\frac{\omega}{2}\right|^{\frac{\alpha}{\pi}} e^{h(\omega)} \tag{43}$$

where

$$h(\omega) = \frac{1}{2\pi}\int_0^\pi\left\{\Theta\left[f(\omega+t)\right] - \Theta\left[f(\omega-t)\right] - \frac{\alpha+\pi}{\pi}\,t\right\}\cot\frac{t}{2}\,dt \tag{44}$$

Since by (32)

$$\frac{a_0}{q} = \frac{1}{2}\left(\frac{1}{q^*} - q^*\right) \tag{45}$$

it follows from (43) that

$$\frac{a_0}{\tilde{q}\left[f(\omega)\right]} = \frac{1}{q_\infty^*}\,\frac{1}{2^{2+\frac{\alpha}{\pi}}\left|\cos\frac{\omega}{2}\right|\,\left|\sin\frac{\omega}{2}\right|^{\frac{\alpha}{\pi}}}\left\{e^{-h(\omega)} - \lambda 2^{2\left(1+\frac{\alpha}{\pi}\right)}\cos^2\frac{\omega}{2}\left|\sin\frac{\omega}{2}\right|^{\frac{2\alpha}{\pi}} e^{h(\omega)}\right\} \tag{46}$$

where

$$\lambda = \left(q_\infty^*\right)^2 = \frac{q_\infty^2}{a_0^2}\,\frac{1}{\left[1+\sqrt{1+\frac{q_\infty^2}{a_0^2}}\right]^2} \tag{47}$$

This is the desired expression of $\tilde{q}(\sigma)$ in terms of the function $f(\omega)$. The parameter λ may be used instead of q_∞/a_0 to determine the conditions at infinity. This parameter can be easily expressed in terms of the stream Mach number (cf. reference 6):

$$\lambda = \frac{M_\infty^2}{\left[1 + \sqrt{1 - M_\infty^2}\right]^2} \tag{48}$$

The fact that the velocity distribution can be expressed in terms of the function $f(\omega)$ in two different ways permits the derivation of an integral equation for the function $f(\omega)$.

4. The Integral Equation for the Function $f(\omega)$

Equations (28) and (29) may be written in the form

$$\tilde{\varphi}\left[f(\omega)\right] = 2R\,\frac{q_\infty}{a_0}\cos\omega$$

$$\tilde{q}\left[f(\omega)\right] = \frac{2\pi a_0}{S}\left|\frac{d\tilde{\varphi}\left[f(\omega)\right]}{d\omega}\right|\frac{1}{f'(\omega)}$$

Combining these two equations yields the relation

$$f'(\omega) = \frac{2\pi}{S}\,\frac{a_0}{\tilde{q}\left[f(\omega)\right]}\,\frac{2Rq_\infty}{a_0}\left|\sin\omega\right| \tag{49}$$

Now substitute in (49) the value of a_0/q given by (46). Then

$$f'(\omega) = C\left|\sin\frac{\omega}{2}\right|^{1-\frac{\alpha}{\pi}}\left\{e^{-h(\omega)} - \lambda 2^{2\left(1+\frac{\alpha}{\pi}\right)}\left|\cos\frac{\omega}{2}\right|^2\left|\sin\frac{\omega}{2}\right|^{\frac{2\alpha}{\pi}}e^{h(\omega)}\right\} \tag{50}$$

where

$$C = 2^{1-\frac{\alpha}{\pi}}\,\frac{\pi R}{S}\,\frac{q_\infty}{a_0 q_\infty^*}$$

and $h(\omega)$ and λ are given by (44) and (47), respectively. Integrating

$$f(\omega) = C\int_0^{\omega}\left|\sin\frac{\omega'}{2}\right|^{1-\frac{\alpha}{\pi}}\left\{e^{-h(\omega')}-\lambda 2^{2\left(1+\frac{\alpha}{\pi}\right)}\left|\cos\frac{\omega'}{2}\right|^{2}\left|\sin\frac{\omega'}{2}\right|^{\frac{2\alpha}{\pi}}e^{h(\omega')}\right\}d\omega'$$

Setting $\omega = 2\pi$ here it follows from (24) that

$$2\pi = C\int_0^{2\pi}\left|\sin\frac{\omega'}{2}\right|^{1-\frac{\alpha}{\pi}}\left\{e^{-h(\omega')}-\lambda 2^{2\left(1+\frac{\alpha}{\pi}\right)}\left|\cos\frac{\omega'}{2}\right|^{2}\left|\sin\frac{\omega'}{2}\right|^{\frac{2\alpha}{\pi}}e^{h(\omega')}\right\}d\omega'$$

so that finally

$$f(\omega) = 2\pi\frac{\int_0^{\omega}\left|\sin\frac{\omega'}{2}\right|^{1-\frac{\alpha}{\pi}}\left\{e^{-h(\omega')}-\lambda 2^{2\left(1+\frac{\alpha}{\pi}\right)}\left|\cos\frac{\omega'}{2}\right|^{2}\left|\sin\frac{\omega'}{2}\right|^{\frac{2\alpha}{\pi}}e^{h(\omega')}\right\}d\omega'}{\int_0^{2\pi}\left|\sin\frac{\omega'}{2}\right|^{1-\frac{\alpha}{\pi}}\left\{e^{-h(\omega')}-\lambda 2^{2\left(1+\frac{\alpha}{\pi}\right)}\left|\cos\frac{\omega'}{2}\right|^{2}\left|\sin\frac{\omega'}{2}\right|^{\frac{2\alpha}{\pi}}e^{h(\omega')}\right\}d\omega'} \tag{51}$$

Since $h(\omega)$ is given by formula (44) this is a nonlinear integral equation for the unknown function $f(\omega)$.

5. Solution of the Integral Equation

The integral equation (51) can be written in the form

$$f(\omega) = F\left\{\omega, f(\omega')\right\}$$

where $F\left\{\omega, f(\omega')\right\}$ denotes the right-hand side of (51). The operation F is a functional transformation which takes a continuously differentiable function $f(\omega')$ satisfying the conditions

$$f(0) = 0 \qquad f(2\pi) = 2\pi$$

into a function satisfying the same end-point conditions. Therefore the solution of (51) can be attempted by the iteration method. Choose some function $f_0(\omega)$ satisfying the preceding conditions and compute successively

$$f_1(\omega) = F\left\{\omega,\ f_0(\omega')\right\}$$

$$f_2(\omega) = F\left\{\omega,\ f_1(\omega)\right\}$$

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$$f_{n+1}(\omega) = F\left\{\omega,\ f_n(\omega')\right\}$$

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If the sequence

$$f_0(\omega),\ f_1(\omega),\ \ldots\ f_n(\omega),\ \ldots$$

converges toward a function $f(\omega)$ and $\lim F(f_n) = F(f)$, this function f satisfies the integral equation.

From the purely mathematical point of view it would be necessary to supplement the preceding consideration by proving that under suitable assumption: (i) the integral equation possesses a solution, (ii) this solution can be obtained by iterations, and (iii) this solution is an increasing function. It is hoped that such proofs will be presented at some later date. At present it may suffice to state that the statements (i) to (iii) seem to be verified in the cases for which the computations have been carried out. The existence of an increasing function satisfying the integral equation seems quite obvious from physical reasons. As for the convergence of the method, reference is made to the fact that the method described here is rather similar to Theodorsen's method of conformal mapping (references 1 and 2) for which a rigorous convergence proof has been found (reference 8).

It might be noted that the desired solution $f(\omega)$ must satisfy the symmetry condition

$$f(2\pi - \omega) = 2\pi - f(\omega), \qquad f'(2\pi - \omega) = f'(\omega) \tag{52}$$

If the function $f_0(\omega)$ satisfies this condition, so will all successive approximations $f_n(\omega)$. It will therefore be sufficient to compute $f_n(\omega)$ only in the interval $0 \leq \omega \leq \pi$.

The only nontrivial step in computing the functions $f_n(\omega)$ consists in evaluating the integral

$$h_{n+1}(\omega) = \frac{1}{2\pi}\int_0^{\pi} \left\{\Lambda_{n+1}(\omega + t) - \Lambda_{n+1}(\omega - t)\right\} \cot \frac{t}{2}\, dt \tag{53}$$

where

$$\Lambda_{n+1}(\omega) = \Theta\left[f_n(\omega)\right] - \frac{\alpha + \pi}{2\pi}\,\omega \tag{54}$$

(Cf. equations (51) and (44).) It should be noted that this is a proper Riemann integral. In fact, the value of the integrand at $t = 0$ is

$$\lim_{t \to 0} \left\{\frac{\Lambda_{n+1}(\omega + t) - \Lambda_{n+1}(\omega - t)}{2t}\left(2t \cot \frac{t}{2}\right)\right\}$$

$$= 4\Lambda'_{n+1}(\omega) = 4\left\{\Theta'\left[f_n(\omega)\right] f'_n(\omega) - \frac{\alpha + \pi}{2\pi}\right\}$$

By using this information, the integral (53) can be evaluated numerically, say by the trapezoidal rule.

After $f(\omega)$ has been computed with sufficient accuracy, the velocity distribution is computed by means of formula (46).

6. Choice of the Function $f_o(\omega)$

The rapidity of convergence of the iteration method for solving equation (51) will depend upon the choice of the function $f_o(\omega)$, the 0th approximation. In order to reduce the computational work, this function should always satisfy condition (52).

A few methods of choosing the function $f_o(\omega)$ are listed, in the order of preference:

(α) Choose for $f_o(\omega)$ the solution of equation (51) for a value λ' as close as possible to the value of λ for which the equation is to be solved.

(β) Choose for $f_o(\omega)$ the solution of equation (51) for the desired value of λ or for a value λ' close to the desired value, and for a profile P' different from but close to P.

(γ) Choose for $f_o(\omega)$ the function resulting from the conformal mapping of the profile P onto a circle; that is, the solution of (51) for $\lambda = 0$.

(δ) Choose for $f_o(\omega)$ a function approximating the function resulting from the conformal mapping of the profile P onto a circle. For thin profiles such a function is given by

$$\left.\begin{aligned} f_o(\omega) &= \frac{\pi}{2}(1 - \cos\omega), \quad 0 \leq \omega \leq \pi \\ f_o(2\pi - \omega) &= 2\pi - f_o(\omega) \end{aligned}\right\} \tag{55}$$

Note that (γ) is a special case of (α) (set $\lambda' = 0$) and (δ) a special case of (β) (set $\lambda' = 0$ and choose P' as a straight segment).

7. Velocity Distribution at Points Not on the Profile

It remains to show how the knowledge of the function $f(\omega)$ permits the computation of the velocity distribution at points not on the profile. This is done by means of the following theorem which also shows that solution of the integral equation (51) actually yields a solution of the boundary value problem stated in section 1.

Note first that from the way the integral equation has been set up it follows that there exists an analytic function $w^*(\zeta)$ regular for $|\zeta| > 1$ and such that

$$w^*(e^{i\omega}) = \tilde{q}^*\left[f(\omega)\right] e^{-i\tilde{\theta}[f(\omega)]} \tag{56}$$

where $\tilde{q}^*$ is given by (43) and (44) and $\tilde{\theta}$ by (40). If $f(e^{i\omega})$ is known, $w^*(\zeta)$ can be computed, say by Cauchy's formula:

$$w^*(\zeta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{w^*(e^{i\omega})e^{i\omega}\, d\omega}{e^{i\omega} - \zeta} + q^*_\infty \qquad (57)$$

Now the following theorem holds:

Let $f(\omega)$ be an increasing function satisfying (51). Set

$$z = x + iy = C_1\left\{\int_1^\zeta \frac{1}{w^*(\zeta)}\left(1 - \frac{1}{\zeta^2}\right)d\zeta - \overline{\int_1^\zeta w^*(\zeta)\left(1 - \frac{1}{\zeta^2}\right)d\zeta}\right\} \qquad (58)$$

where C_1 is a real constant and the bar denotes the conjugate complex quantity. The transformation

$$x = x(\xi,\eta), \qquad y = y(\xi, \eta) \qquad (59)$$

of the z-plane into the ζ-plane defined by (58) (for $|\zeta| \geq 1$) is one-to-one. It takes the domain $|\zeta| > 1$ into the domain $E(P)$ exterior to the profile P. The function

$$\varphi = 2C_1 \mathrm{Re}\left(\zeta + \frac{1}{\zeta}\right) \qquad (60)$$

considered as a function of x and y is the desired potential of the compressible flow around P; that is, it satisfies the differential equation (15), the boundary condition (11) and the condition (12).

The proof of this theorem will be found in the appendix.

After φ is found, the velocity components u and v can be determined by differentiation. But it is also true that w^* considered as a function of z is the distorted velocity (cf. sec. 3) and therefore

$$u - iv = a_0 \frac{2|w^*|}{1 - |w^*|^2} \frac{w^*}{|w^*|}$$

The proof of this last statement is left to the reader.

8. Examples

As an illustration of the method, velocity distributions have been computed for a circle and for a symmetrical Joukowski profile with $\epsilon = 0.15$ (ϵ being the usual parameter determining the thickness). The following values of λ have been used

$\lambda = 0.045$ for the circle

$\lambda = 0.157$ for the Joukowski profile

These correspond to the following values of the stream Mach number

$M_\infty = 0.406$ for the circle

$M_\infty = 0.685$ for the Joukowski profile

These values of M_∞ are known to be close to the critical values. (The critical stream Mach number is the stream Mach number for which the maximum local Mach number is equal to unity.)

In the case of the circle

$$\Theta(\sigma) = \sigma + \frac{\pi}{2}, \quad \Theta'(\sigma) = 1, \ \alpha = \pi$$

It is natural to set

$$f_0(\omega) = \omega$$

This corresponds to case (γ) of section 6. The first approximation is easily computed in closed form and is equal to

$$f_1(\omega) = \omega + \frac{\lambda}{1 - 2\lambda} \sin 2\omega$$

In the case of the Joukowski profile

$$\alpha = 0$$

The function $\Theta(s)$ and $\Theta'(s)$ are given by the parametric formulas

$$s = \pi - \frac{2\epsilon_1(1+\epsilon)}{\epsilon}\left\{\epsilon_3 \tanh^{-1}(\epsilon_3 \cos t) - \epsilon_2 \tanh^{-1}(\epsilon_2 \cos t)\right\}$$

$$\Theta = \pi + t - \tan^{-1}\frac{4 \cot t\left[(\cot t)^2 + \frac{1+2\epsilon}{\epsilon^2}\right]}{\left[(\cot t)^2 + \frac{1+2\epsilon}{\epsilon^2}\right]^2 - 4(\cot t)^2}$$

$$\frac{d\Theta}{ds} = \frac{\csc t}{8\epsilon_1}\left\{\frac{(1-\epsilon)(1+3\epsilon)}{\epsilon^2}(\sin t)^4 + 6(\sin t)^2 - \frac{3\epsilon^2}{(1+\epsilon)^2}\right\}$$

where the parameter t ranges from $t = 0$ to $t = \frac{\pi}{2}$ and ϵ_1, ϵ_2, and ϵ_3 are constants determined by

$$\epsilon_1 = \frac{\pi\epsilon}{2(1+\epsilon)\left[\epsilon_3 \tanh^{-1}\epsilon_3 - \epsilon_2 \tanh^{-1}\epsilon_2\right]}$$

$$\epsilon_2 = \sqrt{1 - \epsilon^2}$$

$$\epsilon_3 = \frac{\sqrt{(1+\epsilon)(1+3\epsilon)}}{1+2\epsilon}$$

The proof of these formulas will be found in reference 9. The function s, Θ, $d\Theta/ds$ are tabulated in table I. The approximation of order 0 has again been chosen according to case (γ) of section 6. In the case of a circle the function $f(\omega)$ must satisfy the symmetry relation

$$f(\pi - \omega) = \pi - f(\omega)$$

It is therefore sufficient to consider this function in the interval $0 \leq \omega \leq \pi/2$. Accordingly the functions $f_n(\omega)$ have been computed for $\omega = 0^\circ, 10^\circ, \ldots, 90^\circ$. In the case of the Joukowski profile, the functions $f_n(\omega)$ have been computed for $\omega = 0^\circ, 10^\circ, \ldots, 180^\circ$. The convergence of the successive approximations is seen from tables II.

The resulting velocity distributions are given in tables III and plotted in figures 1 and 2. The argument δ is the argument of a point on the circle into which the profile is mapped conformally. The results obtained have been compared with those arising from the Kármán-Tsien velocity correction formula

$$\frac{q}{q_\infty} = \left(\frac{q}{q_\infty}\right)_i \frac{1 - \lambda}{1 - \lambda\left(\frac{q}{q_\infty}\right)_i} \tag{61}$$

where $\left(\frac{q}{q_\infty}\right)_i$ is the value of the ratio at local velocity to velocity at infinity for an incompressible fluid. To use this formula amounts to replacing the function $f(\omega)$ by the function arising in conformal mapping of the profile into the circle. In the case of the Joukowski profile Kaplan's results (reference 10) obtained by a modified Poggi method are also given for the sake of comparison.

It will be noticed that the present method (which consists of an actual solution of the boundary value problem for the case $\gamma = -1$) gives a greater compressibility effect than the one predicted by the approximate methods mentioned. (To evaluate this remark correctly, note that Von Kármán expressed the opinion that in the case when the assumption $\gamma = -1$ is applied to air and formula (61) is used, the error committed in using this formula seems to counteract the error committed in using the incorrect pressure-density relation.)

CONCLUDING REMARKS

It has been shown that under the assumption of the linearized pressure-volume relation and of a symmetrical flow the velocity distribution of the compressible flow past a wing section of arbitrary given shape can be determined rigorously by a method which requires not considerably more computational labor than the case of an incompressible flow. This is, of course, only the first step toward the complete solution of the velocity distribution problem. The next step should consist of extending the present method (a) to the case of the actual adiabatic pressure-density relation, (b) to the case of a circulatory flow around a not necessarily symmetrical obstacle.

Remark: After this paper was completed the author learned about a paper by Slioskin (reference 11), in which the same problem is reduced to an integro-differential equation, different from the one derived in this paper.

Brown University,
Providence, R. I., May 1945.

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APPENDIX

This appendix contains the proof of the theorem stated in section 7.

The mapping properties of the function (58) follow immediately from the following three statements.

(a) The function (58) takes the point $\zeta = \infty$ into the point $z = \infty$.

(b) The function (58) maps the circle $|\zeta| = 1$ in a one-to-one manner into the profile P.

(c) The Jacobian

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)}$$

is positive for all values of ξ and η, $\xi^2 + \eta^2 > 1$.

To verify (a), observe that as $\zeta \to \infty$, w^* approaches the value $q^* = \sqrt{\lambda} > 0$.

To verify (b), note that the integrals in (58) are independent of the path since the integrands are analytic functions of ζ. In order to obtain the image of $|\zeta| = 1$ the integration may be performed along the circle. But for $\zeta = e^{i\omega}$

$$\left(1 - \frac{1}{\zeta^2}\right) d\zeta = -2 \sin \omega \; d\omega$$

whereas $w^*(e^{i\omega})$ is given by (56). Hence the equation of the curve into which $|\zeta| = 1$ is taken may be written in the form

$$z = 2C_1 \int_0^{\omega} \left\{\frac{1}{q^*[f(\omega)]} - \tilde{q}^*[f(\omega)]\right\} \sin \omega e^{i\left\{\pi + \tilde{\theta}[f(\omega)]\right\}} d\omega$$

Set $\sigma = f(\omega)$ and note that the integral equation (51) implies that $f(2\pi) = 2\pi$. By virtue of (40), (45), and (49) the preceding equation may be written in the following forms:

$$z = C_2 \int_0^{\omega} f'(\omega) \; e^{i\Theta[f(\omega)]} \; d\omega, \qquad 0 \leq \omega < 2\pi \tag{A1}$$

$$z = C_2 \int_0^{\sigma} e^{i\Theta(\sigma)} \; d\sigma, \qquad 0 \leq \sigma < 2\pi \tag{A2}$$

where C_2 is a new positive constant. Equation (A1) shows that $|\zeta| = 1$ is taken into P (cf. equation (10) and note that P is determined but for a scale factor). Equation (A2) shows that the mapping of the circle into P is one-to-one, for by hypothesis,

$$f'(\omega) > 0 \tag{A3}$$

To verify (c), observe that it follows from (A3), (43) and (50) that $|w^*(e^{i\omega})| < 1$. Since the maximum of the modulus of an analytic function is attained on the boundary,

$$|w^*(\zeta)| < 1 \quad \text{for } |\zeta| \geq 1 \tag{A4}$$

Now the Jacobian is equal to

$$J = \operatorname{Im}\left(\frac{\partial \bar{z}}{\partial \xi}\frac{\partial z}{\partial \eta}\right) = \operatorname{Im}\left\{\overline{i\left(\frac{\partial z}{\partial \zeta} + \frac{\partial z}{\partial \bar{\zeta}}\right)}\left(\frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \bar{\zeta}}\right)\right\}$$

$$= C_1^{\,2}\operatorname{Im}\left\{i\left[\overline{\frac{1}{w^*}\left(1 - \frac{1}{\zeta^2}\right)} - w^*\left(1 - \frac{1}{\zeta^2}\right)\right]\left[\frac{1}{w^*}\left(1 - \frac{1}{\zeta^2}\right) + \overline{w^*\left(1 - \frac{1}{\zeta^2}\right)}\right]\right\}$$

$$= C_1^{\,2}\left|1 - \frac{1}{\zeta^2}\right|^2\left(\frac{1}{|w^*|^2} - |w^*|^2\right)$$

so that by (A4)

$$J > 0 \qquad \text{for} \quad |\zeta| > 1$$

Next, equations (58) and (60) may be rewritten in the form

$$x = \operatorname{Re}\chi_1(\zeta), \quad y = \operatorname{Re}\chi_2(\zeta), \quad z = \operatorname{Re}\chi_3(\zeta) \qquad \text{(A5)}$$

where

$$\chi_1(\zeta) = C_1\int_R^{\zeta}\left\{\frac{1}{w^*(\zeta)} - w^*(\zeta)\right\}\left(1 - \frac{1}{\zeta^2}\right)d\zeta$$

$$\chi_2(\zeta) = -iC_1\int_R^{\zeta}\left\{\frac{1}{w^*(\zeta)} + w^*(\zeta)\right\}\left(1 - \frac{1}{\zeta^2}\right)d\zeta$$

$$\chi_3(\zeta) = 2C_1\left(\zeta + \frac{1}{\zeta}\right)$$

Since

$$\chi_1'^2 + \chi_2'^2 + \chi_3'^2 = 0$$

(A5) is a Weierstrassian parametric representation of a minimal surface. In other words, φ considered as a function of x and y satisfies equation (15).

A simple computation shows that a line element normal to the circle $|\zeta| = 1$ is taken by the mapping (58) into a line-element normal to the profile P. Since the normal derivative of φ in the ζ-plane vanishes so does the normal derivative of φ in the z-plane. Thus φ considered as a function of x and y satisfies the boundary condition (11).

Since

$$\frac{\partial z}{\partial \xi} = \frac{\partial x}{\partial \xi} + i \frac{\partial y}{\partial \xi} = \frac{\partial z}{\partial \zeta} + \frac{\partial z}{\partial \bar{\zeta}}$$

$$\frac{\partial z}{\partial \eta} = \frac{\partial x}{\partial \eta} + i \frac{\partial y}{\partial \eta} = i \left(\frac{\partial z}{\partial \zeta} - \frac{\partial z}{\partial \bar{\zeta}}\right)$$

and $w^*(\infty) = q^*_\infty$, it follows from (58) that as $\zeta \longrightarrow \infty$

$$\frac{\partial x}{\partial \xi} \rightarrow C_1 \frac{1 - q^{*2}_\infty}{q^*_\infty}, \qquad \frac{\partial y}{\partial \xi} \rightarrow 0$$

$$\frac{\partial x}{\partial \eta} \rightarrow 0, \qquad \frac{\partial y}{\partial \eta} \rightarrow C_1 \frac{1 + q^{*2}_\infty}{q^*_\infty}$$

so that

$$\frac{\partial \xi}{\partial x} \rightarrow \frac{1}{C_1} \frac{q^*_\infty}{1 - q^{*2}_\infty}, \qquad \frac{\partial \xi}{\partial y} \rightarrow 0$$

$$\frac{\partial \eta}{\partial x} \rightarrow 0, \qquad \frac{\partial \eta}{\partial y} \rightarrow \frac{1}{C_1} \frac{q^*_\infty}{1 + q^{*2}_\infty}$$

Now, as $\zeta \rightarrow 0$

$$\frac{\partial \varphi}{\partial \xi} \rightarrow 2C_1, \qquad \frac{\partial \varphi}{\partial \eta} \rightarrow 0$$

and therefore

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \varphi}{\partial \eta}\frac{\partial \eta}{\partial x} \rightarrow \frac{2q^*_\infty}{1 - q^{*2}_\infty} = \frac{q_\infty}{a_o}$$

$$\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \varphi}{\partial \eta}\frac{\partial \eta}{\partial y} \rightarrow 0$$

as $z \rightarrow \infty$. Thus condition (12) is also verified.

Table I.

Functions entering into the computation of the velocity distribution along a symmetrical Joukowski profile with ε = .15

t	s	Ⓗ	dⒽ/ds	t	s	Ⓗ	dⒽ/ds
0.00	0.0000	3.1416		0.80	3.0267	3.8755	
.02	.0667	3.0834	-.4076	.82	3.0320	3.8980	
.04	.2548	3.0353	-.1731	.84	3.0371	3.9204	
.06	.5191	3.0034	-.0804	.86	3.0419	3.9428	
.08	.8148	2.9888	-.0220	.88	3.0465	3.9650	
.10	1.1062	2.9892	+.0238	.90	3.0510	3.9873	
.12	1.3723	3.0009	+.0647	.92	3.0551	4.0094	
.14	1.6066	3.0206	+.1040	.94	3.0591	4.0315	
.16	1.8071	3.0453	+.1437	.96	3.0629	4.0535	+5.8608
.18	1.9766	3.0730	+.1850	.98	3.0666	4.0755	+6.0844
.20	2.1193	3.1024	+.2287	1.00	3.0701	4.0975	+6.3066
.22	2.2395	3.1326	+.2757	1.02	3.0735	4.1194	+6.5277
.24	2.3407	3.1630	+.3264	1.04	3.0768	4.1412	+6.7464
.26	2.4265	3.1933	+.3813	1.06	3.0800	4.1630	+6.9630
.28	2.4995	3.2232	+.4408	1.08	3.0831	4.1848	
.30	2.5620	3.2526	+.5052	1.10	3.0861	4.2065	
.32	2.6158	3.2816	+.5750	1.12	3.0890	4.2282	
.34	2.6623	3.3101		1.14	3.0918	4.2499	
.36	2.7029	3.3381		1.16	3.0945	4.2715	
.38	2.7385	3.3655	+.8185	1.18	3.0972	4.2932	
.40	2.7697	3.3926	+.9119	1.20	3.0998	4.3148	
.42	2.7975	3.4192		1.22	3.1024	4.3363	
.44	2.8221	3.4454		1.24	3.1049	4.3579	
.46	2.8442	3.4712		1.26	3.1073	4.3794	
.48	2.8640	3.4968	+1.3500	1.28	3.1097	4.4009	
.50	2.8818	3.5220	+1.4761	1.30	3.1121	4.4224	
.52	2.8980	3.5469	+1.6089	1.32	3.1144	4.4439	
.54	2.9127	3.5715		1.34	3.1167	4.4653	
.56	2.9261	3.5959		1.36	3.1190	4.4868	
.58	2.9384	3.6201		1.38	3.1212	4.5082	
.60	2.9497	3.6441		1.40	3.1234	4.5297	
.62	2.9600	3.6679		1.42	3.1256	4.5511	
.64	2.9697	3.6915	+2.5419	1.44	3.1277	4.5725	
.66	2.9786	3.7149	+2.7190	1.46	3.1299	4.5939	
.68	2.9869	3.7382	+2.9014	1.48	3.1320	4.6153	
.70	2.9946	3.7614	+3.0895	1.50	3.1341	4.6367	
.72	3.0019	3.7844	+3.2823	1.52	3.1363	4.6581	
.74	3.0087	3.8073		1.54	3.1384	4.6794	
.76	3.0150	3.8302		1.56	3.1405	4.7008	
.78	3.0210	3.8529		1.5708	3.1416	4.7124	+10.177

Table IIa.

Successive approximations to the function $f(\omega)$ in the case of a circle (M_{∞} = .406, λ = .045)

ω deg.	ω rad.	$f_0(\omega)$	$f_1(\omega)$	$f_2(\omega)$	$f_3(\omega)$	$f_4(\omega)$	$f_5(\omega)$
0	.00000	.00000	.00000	.00000	.00000	.00000	.00000
10	.17453	.17453	.19144	.18171	.18611	.18470	.18558
20	.34907	.34907	.38086	.36239	.37082	.36808	.36976
30	.52360	.52360	.56643	.54122	.55287	.54900	.55135
40	.69813	.69813	.74683	.71769	.73138	.72673	.72951
50	.87266	.87266	.92136	.89172	.90587	.90095	.90388
60	1.04720	1.04720	1.09003	1.06355	1.07635	1.07183	1.07450
70	1.22173	1.22173	1.25352	1.23362	1.24333	1.23985	1.24191
80	1.39626	1.39626	1.41317	1.40250	1.40768	1.40586	1.40698
90	1.57030	1.57080	1.57080	1.57080	1.57079	1.57080	1.57080

Table IIb.

Successive approximations to the function $f(\omega)$ in the case of a symmetrical Joukowski profile (ϵ = .15, M_{∞} = .685, λ= .469)

ω deg.	ω rad.	$f_0(\omega)$	$f_1(\omega)$	$f_2(\omega)$	$f_3(\omega)$	$f_4(\omega)$
0	.0000	.0000	.0000	.0000	.0000	.0000
10	.1745	.0437	.0324	.0323	.0324	.0324
20	.3490	.1273	.1275	.1274	.1271	.1275
30	.5236	.2570	.2786	.2785	.2779	.2787
40	.6981	.4470	.4762	.4755	.4750	.4760
50	.8726	.6645	.7090	.7070	.7068	.7078
60	1.0472	.9078	.9649	.9609	.9615	.9620
70	1.2217	1.1682	1.2320	1.2257	1.2270	1.2272
80	1.3962	1.4332	1.4996	1.4909	1.4931	1.4928
90	1.5708	1.6956	1.7586	1.7484	1.7514	1.7508
100	1.7453	1.9502	2.0022	1.9914	1.9951	1.9943
110	1.9198	2.1857	2.2253	2.2156	2.2193	2.2182
120	2.0944	2.4016	2.4248	2.4174	2.4208	2.4199
130	2.2689	2.5922	2.5992	2.5956	2.5982	2.5972
140	2.4434	2.7555	2.7481	2.7494	2.7501	2.7495
150	2.6180	2.8902	2.8728	2.8787	2.8776	2.8778
160	2.7925	2.9964	2.9764	2.9850	2.9821	2.9831
170	2.9670	3.0773	3.0639	3.0702	3.0675	3.0686
180	3.1416	3.1416	3.1416	3.1416	3.1416	3.1416

Table IIIa

Velocity distribution about a circle (M_∞=.406)

δ \ q/q_∞	Present Method	Kármán-Tsien Method	Incompressible
0	0.000	0.000	0.000
10	.335	.333	.347
20	.675	.667	.684
30	1.014	1.000	1.000
40	1.350	1.326	1.286
50	1.671	1.636	1.532
60	1.952	1.912	1.732
70	2.185	2.134	1.879
80	2.336	2.279	1.970
90	2.389	2.329	2.000

Table IIIb

Velocity distribution about a Joukowski Profile (ε=.15, M_∞=.685)

δ \ q/q_∞	Present Method	Kármán-Tsien Method	Kaplan	Incompressible
0		.849	.839	.870
10	.835	.854	.852	.874
20	.847	.869	.867	.887
30	.873	.894	.892	.909
40	.912	.928	.927	.938
50	.957	.970	.970	.974
60	1.011	1.019	1.021	1.016
70	1.073	1.073	1.078	1.061
80	1.142	1.132	1.139	1.109
90	1.215	1.191	1.203	1.157
100	1.289	1.250	1.265	1.203
110	1.360	1.303	1.322	1.244
120	1.417	1.347	1.368	1.278
130	1.446	1.373	1.394	1.297
140	1.427	1.369	1.385	1.294
150	1.325	1.308	1.312	1.247
160	1.088	1.128	1.110	1.106
170	.645	.704	.674	.738
180	0.000	0.000	0.000	0.000

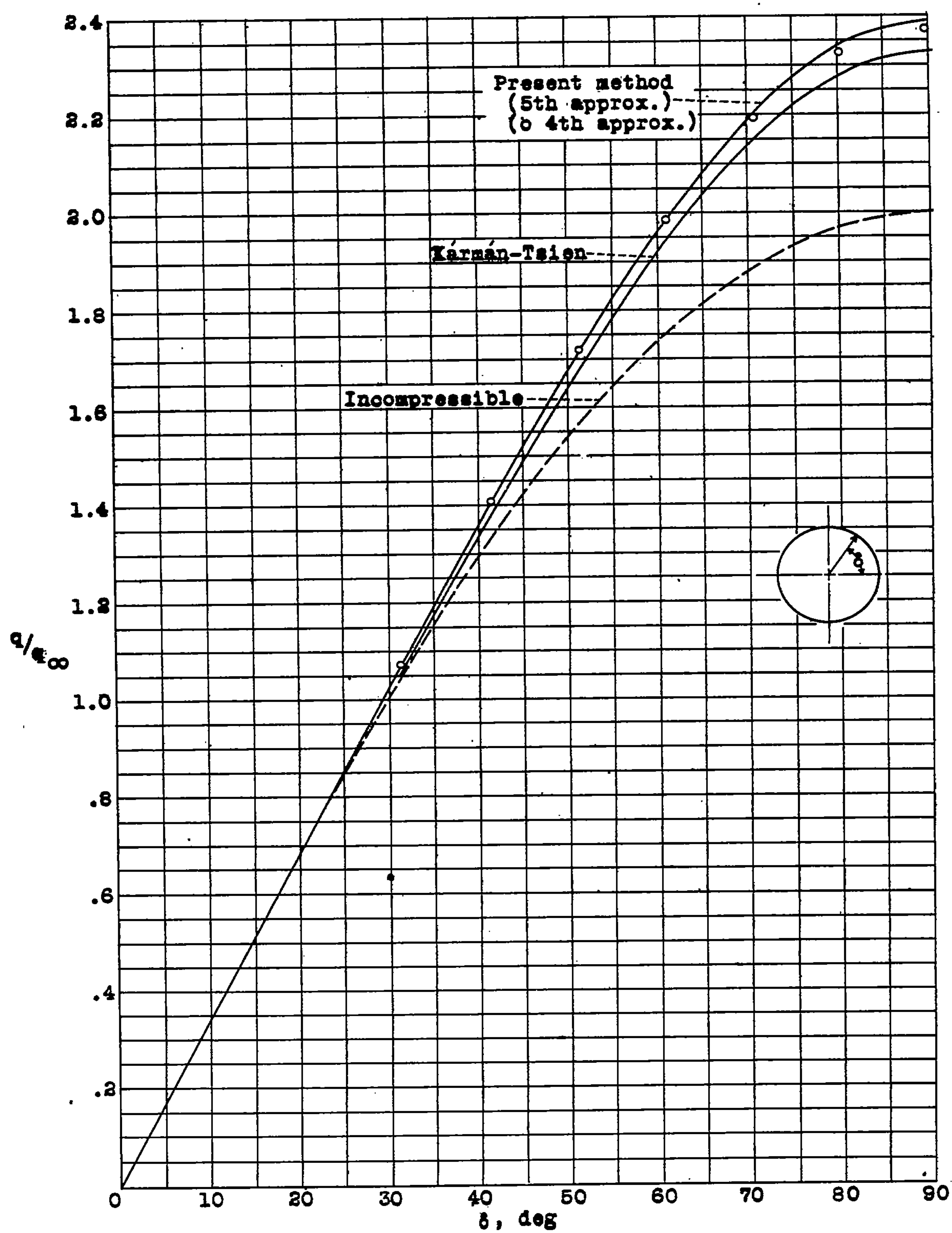

Figure 1.- Velocity distribution along a circle (M_{∞} =.406).

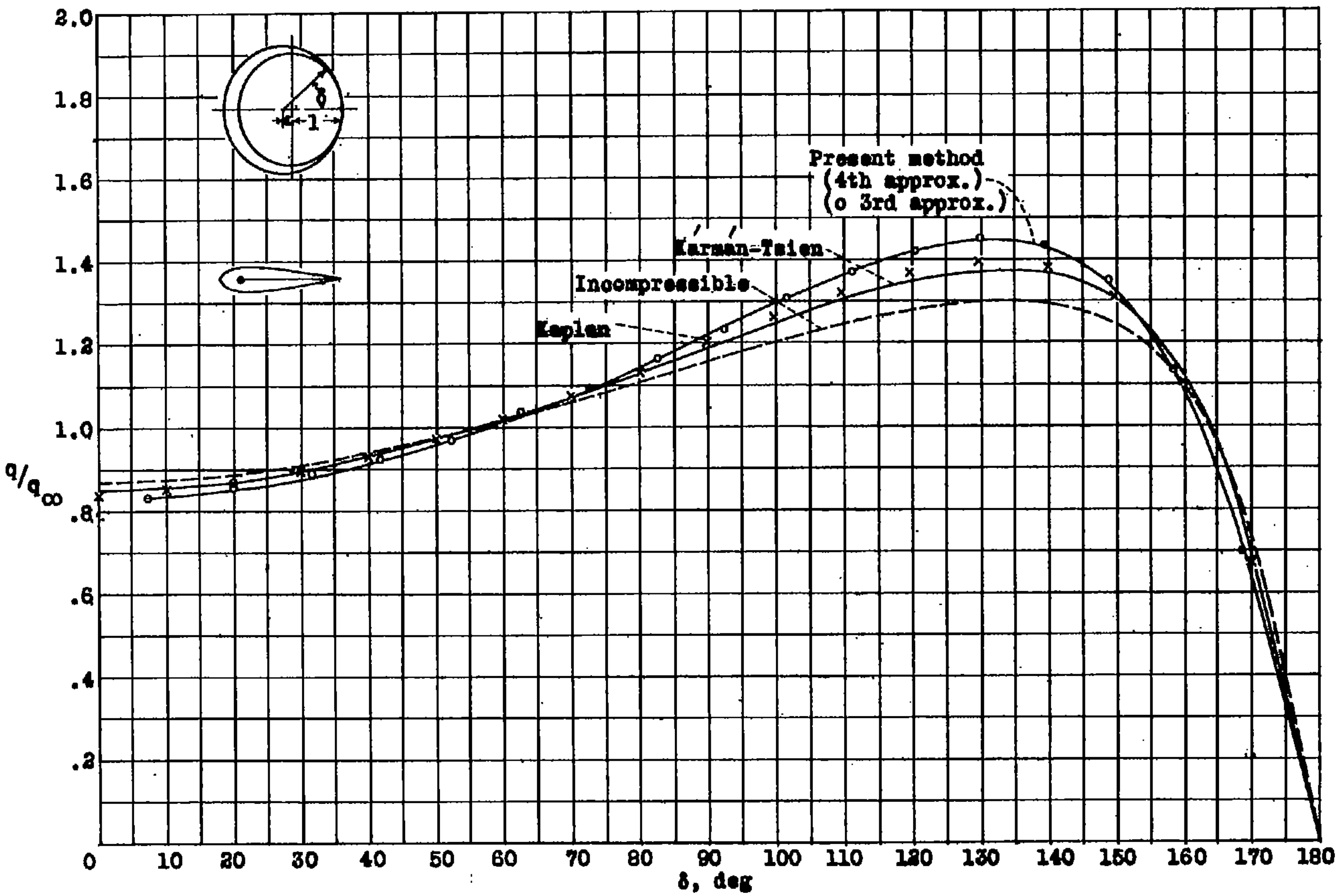

Figure 2.- Velocity distribution along a Joukowski profile (ϵ=.15, M_∞=.685).