POTENTIAL FLOW ABOUT ELONGATED BODIES OF REVOLUTION

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SUMMARY

This report presents a method of solving the problem of axial and transverse potential flows around arbitrary elongated bodies of revolution. The solutions of Laplace's equation for the velocity potentials of the axial and transverse flows, the system of coordinates being an elliptic one in a meridian plane, are known to be of the following form:

\[ \phi = \sum_{n=1}^{\infty} A_n Q_n(\lambda) P_n(\mu) \] (axial flow)

\[ \phi = \sum_{n=1}^{\infty} A_n Q_n(\lambda) P_n(\mu) \cos \theta \] (transverse flow)

If a power-series development of \( \lambda \) in \( \mu \) is assumed as the equation of the meridian profile in elliptic coordinates, the boundary conditions of the two types of flow yield linear equations for the determination of the coefficients \( A_n \) and \( A_m \). It is further shown that a knowledge of these coefficients leads directly to the sink-source and doublet distributions corresponding to the axial and transverse flows, respectively.

The theory is applied to a body of revolution obtained from a symmetrical Joukowsky profile, a shape resembling an airship hull. The pressure distribution and the transverse-force distribution are calculated and serve as examples of the procedure to be followed in the case of an actual airship. A section on the determination of inertia coefficients is also included in which the validity of some earlier work is questioned.

INTRODUCTION

There are two methods of handling the problem of potential flow about a body of revolution. One, the indirect method first used by Taylor (reference 1) and by G. Fuhrmann (reference 2) who computed the pressure distribution by the method of sources and sinks suggested by Rankine. Fuhrmann assumed certain sink-source distributions and calculated the pressure distribution for the streamline body resulting from the assumed sink-source system. He also constructed models of the calculated shapes and measured the pressure distributions over them when placed in a wind tunnel.

The other method, developed by von Kármán (reference 3), considered the direct problem; i.e., the calculation of the pressure distribution over a given streamline shape. He approximated the requisite sink-source distribution by a computed continuous system of sinks and sources arranged in stepwise constant intensity. The various strengths were determined from the condition that the airship hull is a streamline surface in the parallel flow and the flow induced by the sinks and sources. By satisfying this condition at an arbitrary number of points equal to the number of unknown sink and source segments, von Kármán obtained a system of linear equations for the determination of the unknown strengths of the sink-source distribution. He also treated the case of transverse flow (references 3 and 4) by the distribution of doublets along the axis of symmetry of the body of revolution and calculated the strengths of the various doublet segments in a manner similar to that used for the sink-source intensities.

The present paper is an attempt to treat the direct problem according to the methods of the potential theory. Thus, Laplace's equation for the velocity potential is set up in a system of elliptic-cylindrical coordinates \( \lambda, \mu, \theta \) and solved in conjunction with the appropriate boundary conditions for axial and transverse flows. It is then assumed that a power-series development of \( \lambda \) in \( \mu \) represents the meridian profile of the elongated body of revolution. The boundary conditions for the two types of flow may then be expressed in the form of power series in \( \mu \) valid for the entire range of \( \mu \). This method leads to two sets of linear equations, each set infinite in number of equations and each equation containing an infinite number of unknown coefficients which serve to determine the velocity potentials for the axial and transverse flows. Incidental to the major task of determining these coefficients, the sink-source and doublet distributions corresponding to the axial and transverse flows are also determined. Thus the results of this method are essentially the same as those obtained by the method of von Kármán but are obtained in a more rigorous and
direct manner. In von Kármán’s method, approximations are made prior to the analysis; whereas, in the method presented in this paper, approximations are made after the analysis has been carried through in a rigorous manner.

**FUNDAMENTAL EQUATIONS**

The fluid motion is assumed to be steady and irrotational. There then exists a velocity potential $\phi$, which is, in general, a function of the rectangular Cartesian coordinates $(x, y, z)$. In cases of rotational symmetry, however, it is appropriate to introduce the cylindrical coordinates $(\rho, \theta)$ where $\rho$ denotes the distance along the axis of symmetry, $\rho(=\sqrt{x^2+y^2})$ the perpendicular distance from this axis, and $\theta$ the angle between the $(x, \rho)$ and $(x, z)$ planes. (See fig. 1.)

![Figure 1](image)

Furthermore, since only elongated surfaces of revolution are to be considered it is natural to introduce a prolate-elliptic coordinate system in the $(x, \rho)$ plane. The equations of transformation from the coordinates $(x, \rho)$ to the prolate-elliptic coordinates $(\xi, \eta)$ are:

$$
\begin{align*}
\xi &= 2a \cosh \xi \cos \eta \\
\eta &= 2a \sinh \xi \sin \eta
\end{align*}
$$

(1)

where $0 \leq \xi \leq \infty$ and $0 \leq \eta < 2\pi$

Thus $\xi$=constant and $\eta$=constant represent confocal ellipses and hyperbolas, respectively, the distance between the foci being $4a$.

For any point in space $P(x, y, z)$ then

$$
\begin{align*}
x &= 2a(\lambda^2-1)\xi(1-\mu^2) \cos \theta \\
y &= 2a(\lambda^2-1)\xi(1-\mu^2) \sin \theta \\
z &= 2a\lambda \mu
\end{align*}
$$

(2)

where $\lambda = \cosh \theta$ and $\mu = \cos \eta$.

If, furthermore, the fluid is incompressible the velocity potential $\phi$ satisfies Laplace’s equation $\Delta \phi=0$ and since the $(\lambda, \mu, \theta)$ system of coordinates is an orthogonal one, takes the form:

$$
\begin{align*}
\frac{\partial}{\partial \lambda} \left[ (\lambda^2-1) \frac{\partial \phi}{\partial \lambda} \right] + \frac{\partial}{\partial \mu} \left[ (1-\mu^2) \frac{\partial \phi}{\partial \mu} \right] \\
+ \left( \frac{1}{\lambda^2-1} + \frac{1}{1-\mu^2} \right) \frac{\partial^2 \phi}{\partial \theta^2} &= 0
\end{align*}
$$

(3)

**FLOW PARALLEL TO THE AXIS OF SYMMETRY**

In this case the flow is the same for all meridian planes $(x, \rho)$ and therefore the velocity potential $\phi$ is a function only of $\lambda$ and $\mu$. Equation (3) then reduces to

$$
\frac{\partial}{\partial \lambda} \left[ (\lambda^2-1) \frac{\partial \phi}{\partial \lambda} \right] + \frac{\partial}{\partial \mu} \left[ (1-\mu^2) \frac{\partial \phi}{\partial \mu} \right] = 0
$$

(4)

If this equation is to be satisfied by a product

$$
\phi = \frac{1}{L(\lambda)} M(\mu)
$$

it follows that

$$
\frac{1}{L(\lambda)} \frac{d}{d\lambda} \left[ (\lambda^2-1) \frac{dL(\lambda)}{d\lambda} \right] = -\frac{1}{M(\mu)} \frac{d}{d\mu} \left[ (1-\mu^2) \frac{dM(\mu)}{d\mu} \right]
$$

which separates into two ordinary differential equations

$$
\begin{align*}
\frac{d}{d\lambda} \left[ (1-\lambda^2) \frac{dL(\lambda)}{d\lambda} \right] + cL &= 0 \\
\frac{d}{d\mu} \left[ (1-\mu^2) \frac{dM(\mu)}{d\mu} \right] + cM &= 0
\end{align*}
$$

(5)

where $c$ is an arbitrary constant.

Furthermore, if $c = -\eta(n+1)$, each of these equations is of the Legendre type and therefore the general solution of equation (4) is

$$
\phi = \sum_{n=0}^{\infty} A_n P_n(\lambda) P_n(\mu)
$$

(6)

This expression for $\phi$ has a singularity at infinity since $P_n(\lambda)$ is a polynomial of the $n$th degree in $\lambda$ and is therefore infinite for $\lambda = \infty$. Since the region outside a surface is to be considered and since it must include the region at infinity, another solution for $L(\lambda)$ is required. This solution, linearly independent of $P_n(\lambda)$, is the zonal harmonic of the second kind and is denoted by $Q_n(\lambda)$ where

$$
Q_n(\lambda) = \frac{P_n(\lambda)}{\lambda} \int_{\lambda}^{\infty} \frac{d\lambda}{(\lambda^2-1)^{n+1}}
$$

(7)

It vanishes for $\lambda = \infty$ but has a singular point for $\lambda = \pm 1$ where it is infinite like $\log(\lambda - 1)$. Thus, for example, since $P_1(\lambda) = 1$, $P_1(\lambda) = \lambda$, it is found that

$$
Q_1(\lambda) = \frac{\lambda}{\lambda^2-1} \frac{1}{2} \log \frac{\lambda+1}{\lambda-1} = \frac{1}{\lambda^2-1} \frac{1}{2} \lambda \log \frac{\lambda+1}{\lambda-1}
$$

where $|\lambda| > 1$ and

$$
Q_1(\lambda) = \lambda \int_{\lambda}^{\infty} \frac{d\lambda}{\lambda^2-1} = \frac{1}{2} \log \frac{\lambda+1}{\lambda-1} = \frac{1}{2} \lambda \log \lambda - 1
$$

It may also be shown that

$$
Q_n(\lambda) = \frac{1}{2} P_n(\lambda) \log \frac{\lambda+1}{\lambda-1} - K_n(\lambda)
$$

where $K_n(\lambda)$ is a polynomial of the $(n-1)$th degree.
Another useful expression for \( Q_\alpha(\lambda) \) is that due to F. Neumann (reference 5); namely
\[
Q_\alpha(\lambda) = \frac{1}{2} \int_{-1}^{1} \frac{P_\alpha(\lambda)}{\lambda - \lambda_1} d\lambda_1
\]  
(8)

Expanding \( \frac{1}{\lambda - \lambda_1} \) in decreasing powers of \( \lambda \),
\[
Q_\alpha(\lambda) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \int_{-1}^{1} \lambda^n P_\alpha(\lambda) d\lambda_1
\]

Expressing \( \lambda^n \) in terms of zonal harmonics (reference 5)
\[
\lambda^n = \sum_{k=0}^{\infty} \frac{1}{\lambda^{n+1}} \int_{-1}^{1} \lambda^n P_{\lambda-2n}(\lambda) d\lambda_1
\]

where the upper limit \( \left\lceil \frac{n}{2} \right\rceil \) used depends on whether \( \frac{n}{2} \) is an even and where \( (2n) = 2 \cdot 4 \cdot 6 \ldots 2n, (2n-1) = 1 \cdot 3 \cdot 5 \ldots \)
\( (2n-1); \{0\} = [1] = [1] = 1. \) Also \( (2n-1) = [2n-1] \)

Substituting this expression for \( \lambda^n \) in the foregoing equation for \( Q_\alpha(\lambda) \) it follows that
\[
Q_\alpha(\lambda) = \frac{1}{2} \sum_{n=0}^{\infty} \int_{-1}^{1} \frac{P_\alpha(\lambda) P_{\lambda-2n}(\lambda) d\lambda_1}{\lambda^{n+1}}
\]

The zonal harmonics \( P_\alpha(\lambda) \) are orthogonal functions and satisfy the following relations:
\[
\int_{-1}^{1} P_r P_s d\lambda_1 = \begin{cases} 0 & \text{if } r \neq s \\ 2 & \text{if } r = s \end{cases}
\]

Expanding the preceding expression for \( Q_\alpha(\lambda) \) with regard to \( \lambda \) and writing the terms with equal indices of \( k \) in columns and adding these columns, there is obtained, using the orthogonal property of the \( P_\alpha(\lambda) \)'s, the following expression:
\[
Q_\alpha(\lambda) = \sum_{n=0}^{\infty} \frac{(n+2k)!}{(2n+2k+1)(2k)!} \frac{1}{\lambda^{n+2k+1}} P_\alpha(\lambda) \]  
(9)

This expression is convergent for \( |\lambda| > 1 \) and divergent for \( |\lambda| \leq 1 \).

Instead of being given by equation (8) the velocity potential is now given by the following expression:
\[
\phi = \sum_{n=0}^{\infty} A_n Q_\alpha(\lambda) P_\alpha(\mu)
\]  
(10)

which gives the general solution of equation (4) for regions outside a surface of revolution and extending to infinity.

In cases of rotational symmetry where the lines of flow are in meridian planes, it is convenient to introduce Stokes' stream function \( \psi \). This function arises from the statement that the fluid is incompressible (equation of continuity) and is related to the velocity potential \( \phi \) according to the following equations:
\[
\frac{\partial \psi}{\partial \rho} = \frac{\partial \phi}{\partial z} \quad \text{and} \quad \frac{\partial \psi}{\partial z} = -\rho \frac{\partial \phi}{\partial \rho}
\]  
(11)

The lines \( \psi = \text{constant} \) represent the streamlines. It may be remarked that, unlike the two-dimensional case where both the stream function and the velocity potential satisfy Laplace's equation, Stokes' stream function does not satisfy it.

The introduction of the variables \( \lambda, \mu \) into equations (11) by means of equations (1) leads to the following relations:
\[
\frac{\partial \psi}{\partial \lambda} = 2a(1-\mu^2) \frac{\partial \phi}{\partial \mu} \quad \text{and} \quad \frac{\partial \psi}{\partial \mu} = -2a(\lambda^2-1) \frac{\partial \phi}{\partial \lambda}
\]  
(12)

If a substitution is made for \( \phi \) from equation (10) and \( P_\alpha(\mu) \) is replaced by its value obtained from Legendre's differential equation, that is:
\[
P_\alpha(\mu) = -\frac{1}{n(n+1)} \frac{d}{d\mu} \left[ (1-\mu^2) \frac{dP_\alpha(\mu)}{d\mu} \right]
\]

it is found that
\[
\psi = 2a(1-\mu^2)(\lambda^2-1) \sum_{n=0}^{\infty} \frac{A_n}{n(n+1)} \frac{dP_\alpha(\mu)}{d\mu} \frac{dQ_\alpha(\lambda)}{d\lambda} + \text{constant}
\]  
(13)

Furthermore, if the body of revolution is moving with a velocity \( U \) in the direction of the axis of symmetry \( z \), it may be conveniently supposed to be at rest and the fluid to have a translation \( -U \) superposed on its actual motion. This consideration adds a term \( 2aU \lambda \mu \) to the velocity potential and \( 2a^2U(1-\mu^2)(\lambda^2-1) \) to the stream function. Therefore
\[
\psi = 2a^2(1-\mu^2)(\lambda^2-1) \sum_{n=0}^{\infty} \frac{A_n}{n(n+1)} \frac{dP_\alpha(\mu)}{d\mu} \frac{dQ_\alpha(\lambda)}{d\lambda} + \text{constant}
\]  
(14)

At the surface of the fixed body of revolution the normal velocity of the fluid must be zero and therefore the boundary must coincide with a streamline \( \psi = \text{constant} \), say \( 0 \). Hence the boundary condition at the surface is given by
\[
\sum_{n=0}^{\infty} \frac{A_n}{n(n+1)} \frac{dP_\alpha(\mu)}{d\mu} \frac{dQ_\alpha(\lambda)}{d\lambda} + aU = 0
\]  
(15)

In order to find the velocity components \( u_\alpha, u_\beta \) in the directions of the coordinate lines \( \lambda, \mu \), respectively, it is to be noted that since the system of coordinates is an orthogonal one,
\[
\frac{\partial u_\alpha}{\partial \lambda} = -\frac{\partial \phi}{\partial \lambda} \quad \text{and} \quad \frac{\partial u_\beta}{\partial \mu} = -\frac{\partial \phi}{\partial \mu}
\]

where \( d\alpha = d\lambda^2 + d\mu^2 = d\theta^2 + d\phi^2 \).
By means of the equations of transformation (1), it is found that
\[ ds_\lambda = 2a \left( \frac{\lambda^2 - \mu^2}{\lambda^2 - 1} \right) d\lambda, \]
and
\[ ds_\mu = 2a \left( \frac{\mu^2 - 1}{1 - \mu^2} \right) d\mu. \]
Therefore:
\[ u_\lambda = -\frac{1}{2a} \left( \frac{\lambda^2 - 1}{\lambda^2 - \mu^2} \right) \frac{\partial \phi}{\partial \lambda}, \]
and
\[ u_\mu = -\frac{1}{2a} \left( \frac{1 - \mu^2}{\lambda^2 - \mu^2} \right) \frac{\partial \phi}{\partial \mu}. \]
Hence:
\[ u^2 = \frac{1}{4a^2 (\lambda^2 - \mu^2)} \left[ (\lambda^2 - 1) \frac{\partial \phi}{\partial \lambda} \right]^2 + (1 - \mu^2) \frac{\partial \phi}{\partial \mu} \right]^2. \]  
(16)

**Sink-Source Distribution**

The distribution of sinks and sources is assumed to lie along the segment of the axis of symmetry \( z = 2a \), \(-1 \leq \lambda \leq 1\), and to be of intensity \( I(\lambda) \) per unit length. At any point \( (z, \rho) \) in any meridian plane the velocity potential due to this distribution is given (reference 6, p. 60) by
\[ \phi = \frac{1}{4\pi} \int_{-2a}^{2a} \frac{I(\zeta) d\zeta}{(\zeta - z)^2 + \rho^2}. \]  
(17)

For points lying on the \( z \) axis but outside the distribution, the velocity potential is given by the simplified expression
\[ \phi = \frac{1}{4\pi} \int_{-2a}^{2a} \frac{I(\zeta) d\zeta}{\lambda - \lambda_1}. \]
Substituting for \( z \) and \( z_1 \), \( 2a \lambda \) and \( 2a \lambda_1 \), respectively, the preceding equation takes the form
\[ \phi = \frac{1}{4\pi} \int_{-1}^{1} \frac{I(2a\lambda_1) d\lambda_1}{\lambda - \lambda_1}. \]
Finally, substituting for \( \phi \) from equation (10) and noting that \( P_n(1) = 1 \) for all values of \( n \),
\[ \sum_{n=0}^{\infty} A_n Q_n(\lambda) = \frac{1}{4\pi} \int_{-1}^{1} \frac{I(2a\lambda_1) d\lambda_1}{\lambda - \lambda_1}. \]  
(18)

This is an integral equation for the unknown function \( I(2a\lambda) \). It may be solved in the following manner:

From F. Neumann's expression for \( Q_n(\lambda) \) given by equation (8) the following development is suggested for the distribution function:
\[ I(2a\lambda) = \sum_{n=0}^{\infty} a_n P_n(\lambda), \]
where \(-1 \leq \lambda \leq 1\).
It then follows directly from equation (8) that
\[ \sum_{n=0}^{\infty} \left( A_n - \frac{1}{2} a_n \right) Q_n(\lambda) = 0, \]
for all values of \( \lambda \).
Hence
\[ a_n = 2\pi A_n, \]
and
\[ I(2a\lambda) = 2\pi \sum_{n=0}^{\infty} A_n P_n(\lambda). \]  
(19)

Thus, given the potential function \( \phi \), that is the \( A_n \)'s, this expression determines the equivalent sink-source distribution.

**Flow Normal to the Axis of Symmetry**

The differential equation for the velocity potential in the case of transverse flow is given by equation (3). Recalling that this expression is Laplace's equation in the coordinates \( \lambda, \mu, \nu \), it may be solved by supposing \( \phi \) to be a product \( N(\lambda, \mu) R(\theta) \). Replacing \( \phi \) in equation (3) by this product, the following pair of differential equations is obtained:
\[ \begin{align*}
\frac{d^2R}{d\theta^2} + k^2 R &= 0, \\
\frac{\partial}{\partial \lambda} \left[ (\lambda^2 - 1) \frac{\partial N}{\partial \lambda} \right] + \frac{\partial}{\partial \mu} \left[ (1 - \mu^2) \frac{\partial N}{\partial \mu} \right] &= 0.
\end{align*} \]  
(20)

The general solution of the first equation is given by
\[ R = A \cos k \theta + B \sin k \theta, \]
where \( A \) and \( B \) are arbitrary constants.
Putting \( N(\lambda, \mu) = L(\lambda) M(\mu) \) in the second equation leads to the following pair of ordinary differential equations:
\[ \begin{align*}
\frac{d}{d\lambda} \left[ (1 - \lambda^2) \frac{dL}{d\lambda} \right] + \left( c - \frac{k^2}{L - \lambda} \right) L &= 0, \\
\frac{d}{d\mu} \left[ (1 - \mu^2) \frac{dM}{d\mu} \right] + \left( c - \frac{k^2}{\mu^2} \right) M &= 0.
\end{align*} \]  
(21)

where \( c \) is an arbitrary constant.

Both of the latter equations are of the form of the differential equation for the associated Legendre functions provided that \( c = n(n + 1) \). Accordingly,
\[ M(\mu) = P_n^k(\mu) \]
and \( L(\lambda) = P_n^k(\lambda) \)
where, for example,
\[ P_n^k(\mu) = (1 - \mu^2)^{\frac{k}{2}} \frac{d^k P_n(\mu)}{d\mu^k}. \]

The general solution of equation (3) may then be written as
\[ \phi = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_n^k(\mu) Q_n^k(\lambda) A_{nk} \cos k\theta + B_{nk} \sin k\theta. \]
This expression, however, has a singularity at infinity and since only the region outside a given surface of revolution is of interest, the infinite region, or the neighborhood of \( \lambda = \infty \), must be considered. Therefore \( P_n^k(\lambda) \) is replaced by the associated Legendre function of the second kind \( Q_n^k(\lambda) \), where by definition,
\[ Q_n^k(\lambda) = (\lambda^2 - 1)^{\frac{k}{2}} \frac{d^k Q_n(\lambda)}{d\lambda^k}. \]
Then
\[ \phi = \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_n^k(\mu) Q_n^k(\lambda) A_{nk} \cos k\theta + B_{nk} \sin k\theta. \]  
(22)

If the body of revolution moves with a uniform velocity \( V \) in the direction of the \( z \) axis, it may be
supposed to be at rest and the fluid to have a translation \( \mathbf{V} \) superposed on its actual motion. Then
\[
\phi = \phi_1 + xV
\]  
(23)

Consider the body profile in any one of the meridian planes \( \theta \). At any arbitrary point of it the normal derivative of \( \phi \) is given by
\[
-\frac{\partial \phi}{\partial n} ds = \frac{\partial \phi}{\partial s_\mu} ds_\mu - \frac{\partial \phi}{\partial s_\lambda} ds_\lambda
\]  
(24)

Since the normal velocity along the meridian curve is zero, it follows from equations (23) and (24) that
\[
\left( \frac{\partial \phi_1}{\partial s_\lambda} + V \frac{\partial x}{\partial s_\lambda} \right) ds_\mu - \left( \frac{\partial \phi_1}{\partial s_\mu} + V \frac{\partial x}{\partial s_\mu} \right) ds_\lambda = 0
\]  
Also, \( x = \rho \cos \theta \), so that
\[
\frac{\partial x}{\partial s_\lambda} = \cos \theta \frac{\partial \rho}{\partial s_\lambda} \quad \text{and} \quad \frac{\partial x}{\partial s_\mu} = \cos \theta \frac{\partial \rho}{\partial s_\mu}
\]

Therefore
\[
\left( \frac{\partial \phi_1}{\partial s_\lambda} + V \cos \theta \frac{\partial \rho}{\partial s_\lambda} \right) ds_\mu - \left( \frac{\partial \phi_1}{\partial s_\mu} + V \cos \theta \frac{\partial \rho}{\partial s_\mu} \right) ds_\lambda = 0
\]  
(25)

In order that the condition of no flow normal to the body of revolution be valid for all values of \( \theta \), there must be chosen from among all the solutions given by equation (22) that one which has \( \cos \theta \) as a factor; namely:
\[
\phi_1 = \sum_{n=1}^{\infty} A_n P_n^1(\mu) Q_n(\lambda) \cos \theta
\]
or
\[
\phi_1 = 2aV \cos \theta \left( \lambda^2 - 1 \right)^{1/2} \left( 1 - \mu^2 \right)^{1/2} \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda}
\]  
(26)

where \( C_n = \frac{A_n}{2aV} \)

Furthermore
\[
\frac{d\lambda}{\lambda^2 - 1} = 2a \left( \frac{\lambda^2 - \mu^2}{\lambda^2} \right)^{1/2} d\lambda \quad \text{and} \quad \frac{d\mu}{1 - \mu^2} = 2a \left( \frac{\lambda^2 - \mu^2}{\lambda^2} \right)^{1/2} d\mu
\]

so that equation (25) becomes
\[
\frac{\partial}{\partial \lambda} \left( \phi_1 + \rho V \cos \theta \right) = \frac{1 - \mu^2}{\lambda^2 - 1} \frac{d\lambda}{d\mu}
\]

Finally, by means of equation (26) and the differential equation for the Legendre polynomials the foregoing boundary condition takes the following form:
\[
\sum_{n=1}^{\infty} C_n \left[ \frac{d(\lambda^2)}{d\mu} \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} - n(n+1) \frac{d}{d\mu} (P_n Q_n) \right] = \frac{d(\lambda^2)}{d\mu}
\]  
(27)

**DISTRIBUTION OF DOUBLETs**

The doublets are assumed to have their axes in the \( x \) direction and to lie along the segment \(-2a \leq z \leq 2a\) of the axis of symmetry \( z \). The velocity potential at any point \((x, \rho)\) of some meridian plane \( \theta \) then takes the form (see reference 6):
\[
\phi_i = \frac{\rho \cos \theta}{4\pi} \int_{-2a}^{2a} \frac{J(z_1)}{\sqrt{(x - z_1)^2 + \rho^2}} dz_1
\]

where \( J(z_1) \) is the intensity of the doublets per unit length.

Substituting for \( \phi_i \) from equation (26) it follows that
\[
V \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} = \frac{1}{4\pi \rho^2} \int_{-2a}^{2a} \frac{J(z_1)}{\sqrt{x^2 + (z - z_1)^2 + \rho^2}} dz_1
\]

For points lying on the \( z \) axis but outside the distribution this equation takes the following simplified form:
\[
V \sum_{n=1}^{\infty} \frac{n(n+1)}{2} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} = \frac{1}{16\pi \rho^2} \int_{-2a}^{2a} \frac{J(2a\lambda_1) d\lambda_1}{\lambda_1^3 - 1} \quad \text{where} \quad \lambda_1 \text{ is replaced by } 2a\lambda_1, \quad z \text{ by } 2a\lambda_1, \quad \text{and} \quad \left( \frac{dP_n}{d\mu} \right)_{\mu=1} \text{ by } \frac{n(n+1)}{2}.
\]

This is an integral equation for the unknown function \( J(2a\lambda_1) \). In the solution of this integral equation it is necessary that a development of \( \frac{1}{\lambda^3 - 1} \) as a series of Legendre polynomials in \( \lambda \) be obtained. The form of this development is suggested by Neumann’s equation (8). Thus assume that
\[
\frac{1}{\lambda^3 - 1} = \sum_{n=1}^{\infty} b_n P_n(\lambda) Q_n(\lambda)
\]

Then substituting this expression for \( \frac{1}{\lambda^3 - 1} \) in Neumann’s equation and making use of the orthogonality relations satisfied by the Legendre polynomials, it is found that
\[
b_n = 2n + 1
\]

Therefore
\[
\frac{1}{\lambda^3 - 1} = \sum_{n=1}^{\infty} \left( 2n + 1 \right) P_n(\lambda) Q_n(\lambda)
\]

Differentiating this last expression once with regard to \( \lambda \) and once with regard to \( \lambda_1 \), it follows that
\[
\frac{1}{\lambda^3 - 1} = \sum_{n=1}^{\infty} \frac{2n+1}{2} \frac{dP_n(\lambda)}{d\lambda_1} \frac{dQ_n(\lambda)}{d\lambda}
\]

Equation (28) then becomes
\[
V \sum_{n=1}^{\infty} C_n n(n+1) \frac{dQ_n}{d\lambda} = \sum_{n=1}^{\infty} \frac{2n+1}{32\pi \rho^2} \int_{-2a}^{2a} \frac{J(2a\lambda_1) d\lambda_1}{\lambda_1^3 - 1} \]

It is now obvious that the following assumption must be made:
\[
J(2a\lambda_1) = -8\pi \rho^2 V(1 - \lambda^2) \sum_{n=1}^{\infty} \frac{dP_n}{d\lambda_1}
\]
and substituting this expression in equation (29) it follows that
\[ \sum_{n=1}^{\infty} (C_n - c_n) \frac{dQ_n}{d\lambda} = 0 \]
In order that this equation be valid for arbitrary values of \( \lambda \),
\[ c_n = C_n \]
and therefore
\[ J(2\lambda_n) = -8\pi a^2 V (1 - \lambda_n^2) \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\lambda} \]  
(30)

Thus, given the velocity potential \( \phi_1 \), that is, the \( C_n \)'s, this expression determines the equivalent doublet distribution.

**DETERMINATION OF THE COEFFICIENTS \( A_\lambda \) AND \( C_\lambda \)**

Any symmetrical profile may be represented by a power series in \( \mu (= \cos \eta) \). That is
\[ \lambda = \sum_{n=0}^{\infty} a_n \mu^n \]  
(31)
The rapidity of convergence of this series depends, however, on the choice of origin with respect to the profile. Since \( \lambda = a_0 \) defines an ellipse, the rapidity of convergence of the foregoing series may be looked upon as a measure of the resemblance of the profile to the ellipse \( \lambda = a_0 \). The proper choice of origin may be attained in the following manner. The radius of curvature \( R \) of an ellipse at the end of its major axis is given by
\[ R = \frac{B^2}{A} \]
where \( A \) and \( B \) are its semimajor and semiminor axes, respectively.

Eliminating \( \eta \) from equations (1), the following equation of a system of confocal ellipses results:
\[ \frac{a^2}{(2a \cosh \xi)^2} + \frac{b^2}{(2a \sinh \xi)^2} = 1. \]  
(The distance between foci is \( 4a \).)

In terms of elliptic coordinates then
\[ R = 2a \frac{\sinh \xi \cdot \xi}{\cosh \xi} \]
Furthermore, for an elongated ellipse the semimajor axis \( 2a \cosh \xi \) is large compared to the semiminor axis \( 2a \sinh \xi \). This limitation means that \( \xi \) is small. Neglecting powers of \( \xi \) higher than the second it follows that (see reference 7)
\[ R \approx 2a \xi^2 \]  
(approximately)
The ends of the ellipse are at
\[ \pm 2a \cosh \xi = \pm 2a \left( 1 + \frac{\xi^2}{2} + \cdots \right) = \pm \left( 2a + \frac{R}{2} \right) \]  
(approximately)
and therefore the focus of an elongated ellipse very nearly bisects the line joining the end of the semimajor axis and the center of curvature. Thus the proper choice of origin is the point bisecting the line of length \( 4a \) extending from the point midway between the leading edge and the center of curvature of that edge to a point midway between the trailing edge and the center of curvature of that edge. Having thus chosen a reference frame (\( z, \rho \)) in which to present the profile, the next step is to obtain the series equation (31). This equation may be obtained with the help of the following expressions. From equation (1) it can be found that
\[ \lambda = \frac{1}{2} \sqrt{\frac{(z - \lambda)^2}{2a - 1}} + \frac{\rho}{2a} \]
\[ \mu = \frac{1}{2} \sqrt{\frac{(z - \lambda)^2}{2a - 1}} - \frac{\rho}{2a} \]  
where \(-1 \leq \mu \leq 1 \).

A series of corresponding values of \( \lambda \) and \( \mu \) are thus obtained. In order to express \( \lambda \) as a polynomial in \( \mu \) of, say, degree \( n \), it is most convenient to employ the method of least squares for determining the \((n+1)\) constants \( a_n \) (reference 8).

**FLOW PARALLEL TO THE AXIS OF SYMMETRY**

The boundary condition for this type of flow is given by equation (15). In that expression functions of the type \( \frac{dQ_n}{d\lambda} \) appear and these are to be expressed as power series in \( \mu \).

Suppose the meridian profile to be given by the following analytic expression:
\[ \lambda = a_0 + \mu \sum_{n=0}^{\infty} a_{1,n} \mu^n \]  
(33)
Then on the profile, \( \frac{dQ_n}{d\lambda} \) may be looked upon as a function of \( \mu \) and can be developed in a Taylor series in the neighborhood of \( \mu = 0 \) or \( \lambda = a_0 \). That is,
\[ \frac{dQ_n}{d\lambda} = \sum_{p=0}^{\infty} \left( (\lambda - a_0)^p \frac{d^{p+1}Q_n}{d\lambda^{p+1}} \right) \]
where \( \frac{d^{p+1}Q_n}{d\lambda^{p+1}} = \left( \frac{d^{p+1}Q_n}{d\lambda^{p+1}} \right) \lambda = a_0 \)
Substituting for \( \lambda = a_0 \) from equation (33), it follows that
\[ \frac{dQ_n}{d\lambda} = \sum_{p=0}^{\infty} \mu^p \left( \sum_{n=0}^{\infty} a_{1,n} \mu^n \right)^p \frac{d^{p+1}Q_n}{dx_0^{p+1}} \]
In the following the expansion of \( S^p \) in powers of \( \mu \) is to be determined (reference 9, p. 122), \( p \) being any positive integer and where
\[ S = \sum_{\xi=0}^{\infty} a_{1,\xi} \mu^\xi \]
Thus
\[ S^2 = \sum_{q=0}^{\infty} a_{1,q} \mu^q \]
or
\[ S^2 = \sum_{r=0}^{\infty} \sum_{p=-r}^{\infty} a_{1,p-r} \mu^p \]
Expanding \( S^2 \) with respect to \( r \) and writing the terms with equal indices of \( p \) in columns and adding these columns,
\[ S^2 = \sum_{p=0}^{\infty} \mu^p \sum_{r=0}^{\infty} a_{1,r} a_{1,p-r} = \sum_{p=0}^{\infty} a_{2,p} \mu^p \]
where
\[ a_{2,p} = \sum_{r=0}^{p} a_{1,p-r} a_{1,r} \]
In a similar manner,
\[ S^2 = \sum_{p=0}^{\infty} a_{3,p} \mu^p \]
where
\[ a_{3,p} = \sum_{r=0}^{p} a_{1,p-r} a_{1,r} \]
where
\[ a_{p,0} = a_{p,0} \]
\[ a_{p,1} = p a_{p,0} a_{1,1} \]
\[ a_{p,2} = p a_{p,1} a_{1,2} - \frac{p(p-1)}{2} a_{p,2} a_{1,1} \]
\[ a_{p,3} = p a_{p,2} a_{1,3} + p(p-1) a_{p,3} a_{1,1} - \frac{p(p-1)(p-2)}{3} a_{p,2} a_{1,1} \]
and so on.

The boundary condition also contains terms of the type \( \frac{dP_\alpha}{d\mu} \) where \( P_\alpha(\mu) \) is the Legendre polynomial in \( \mu \) of degree \( n \) and is given by
\[ P_\alpha(\mu) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} (-1)^j (2n-2j-1)! \mu^{n-2j-1} \]
(36)

where the upper limit for \( j \) is \( \left\lfloor \frac{n}{2} \right\rfloor \) according as \( n \) is \( \text{even} \) or \( \text{odd} \).

Then
\[ \frac{dP}{d\mu} = \sum_{j=0}^{n-2} \frac{n-2-j}{2} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!/[2j]} \mu^{n-2j-1} \]
(37)

Substituting for \( \frac{dQ_\alpha}{d\lambda} \) and \( \frac{dP}{d\mu} \) the expressions given by equations (34) and (37) into the boundary condition equation (15),
\[ \sum_{n=0}^{\infty} \frac{A_n}{n(n+1)} \sum_{j=0}^{\infty} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!/[2j]} \mu^{n-2j-1} \sum_{p=0}^{\infty} \mu^p \sum_{r=0}^{\infty} a_{1,p-r} \mu^r + aU = 0 \]
Expanding according to powers of \( n \) and writing the terms with equal indices of \( j \) in columns and then adding these columns,
\[ \sum_{n=0}^{\infty} \sum_{j=0}^{n(n+1)} (-1)^j \frac{[2n-2j-1]}{(n-2j-1)!/[2j]} \mu^{n-2j-1} \sum_{p=0}^{\infty} \mu^p \sum_{r=0}^{\infty} a_{p-r} \mu^r + aU = 0 \]
Putting $n-2j-1+m=q=m$ this becomes

$$\sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \mu^m (-1)^j \frac{A_{2j+1+m}^{m+q}}{(2j+1+m-q)(2j+2+m-q)} \frac{[2j+1+2m-2q]}{m!(2j)!} \sum_{p=0}^{q} \frac{a_{p,3-q}}{p!} \frac{d^{n+1}Q_{m+1+q}}{da_0^n} + aU = 0$$

Expanding with respect to $q$ and writing the terms with equal indices of $m$ in columns and adding these columns,

$$\sum_{m=0}^{\infty} \mu^m \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \frac{A_{2j+1+m}^{m+q}}{(2j+1+m-q)(2j+2+m-q)} \frac{[2j+1+2m-2q]}{m!(2j)!} \sum_{p=0}^{q} \frac{a_{p,3-q}}{p!} \frac{d^{n+1}Q_{m+1+q}}{da_0^n} + aU = 0 \quad (38)$$

If this expression is to be valid everywhere on the boundary surface, it must hold for the entire range of $\mu$. It follows that the coefficients of the various powers of $\mu$ are identically zero. Finally, the introduction of $k$ and $n$ by means of the substitutions $q=m-k$ and $p=m-n$, respectively, leads to the following expression of the boundary condition:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_{n-k,3-k}}{k!(m-n)!} \sum_{j=0}^{\infty} (-1)^j \frac{[2j+1+2k]A_{2j+1+k}^{m+n+k}}{(2j+1+k)(2j+2+k)(2j)!} \frac{d^{n+1}Q_{m+n+k}}{da_0^n} = -aU\delta_n$$

where

$$\delta_n = 1 \quad \text{if} \quad m=0$$

$$\delta_n = 0 \quad \text{if} \quad m \neq 0 \quad \text{and} \quad m=0, 1, 2, \ldots \infty.$$

Equation (39) represents a set of linear equations infinite in number and containing an infinite number of unknowns $A_\lambda$. It provides a formal and rigorous solution of the problem of potential flow about a body of revolution, parallel to its axis of symmetry.

In the foregoing equations the only unknowns are the $A_\lambda$'s. The $a_{p,3-q}$'s are related to the coefficients of the power series of $\lambda$ in $\mu$ (giving the meridian profile equation (33)) and are evaluated by means of equations (35). Finally, the $Q_\lambda$'s and their derivatives are well-defined Legendre functions.

For example, if the meridian profile is an ellipse $\lambda=a_0$, then equation (39) becomes

$$\sum_{j=0}^{\infty} (-1)^j \frac{A_{2j+1+m}^{m+q}}{(2j+1+m)(2j+2+m)} \frac{[2j+2m+1]}{m!(2j)!} \frac{dQ_{2j+1+m}}{da_0} = -aU$$

For $m=1, 2, 3, \ldots$ this is an infinite set of linear homogeneous equations for the unknowns $A_2, A_3, \ldots$ and, since the determinant of the coefficients is different from zero, the only solution is that $A_2, A_3, \ldots$ are zero. From the first equation, i. e., $m=0$, it is then found that

$$A_1 = \frac{2aU}{dQ_1} = -\frac{2aU}{\frac{1}{2} \log \frac{a_0+1}{a_0-1} - \frac{a_0}{a_0^2-1}}$$

Hence

$$\phi = \frac{2aU}{\frac{1}{2} \log \frac{a_0+1}{a_0-1} - \frac{a_0}{a_0^2-1}} \left( \frac{1}{2} \lambda \log \frac{\lambda+1}{\lambda-1} \right)^\mu$$

If $A$ and $B$ are the semimajor and semiminor axes of the meridian ellipse and $\epsilon$ its eccentricity, then

$$2a = Ae, \quad a_0 = \frac{1}{\epsilon}, \quad 2a(a^2-1)^{1/2} = B$$

so that

$$\phi = \frac{UA}{\frac{1}{2} \log \frac{1+\epsilon}{1-\epsilon} \frac{1}{1-\epsilon^2}} \left( \frac{1}{2} \lambda \log \frac{\lambda+1}{\lambda-1} \right)^\mu$$

This result agrees with the well-known expression for the velocity potential of a prolate ellipsoid of revolution (reference 10, p. 132).

**FLOW NORMAL TO THE AXIS OF SYMMETRY**

The case of flow normal to the axis of symmetry will now be treated in a manner similar to the case of parallel flow. The boundary condition is given by equation (27):

$$\sum_{n=1}^{\infty} \left[ \frac{d(Q_\mu)}{d\mu} \frac{dP_\mu}{d\mu} dQ_\mu - \frac{1}{n(n+1)} \frac{d(Q_\mu)}{d\mu} (P_\mu Q_\mu) \right] = \frac{d(Q_\mu)}{d\mu}$$

From equation (33),

$$\frac{d(Q_\mu)}{d\mu} = a_0 + \sum_{n=0}^{\infty} (n+2)a_{3,n} \lambda^{n+1}$$

(40)
Referring to equation (38) it follows that
\[
\sum_{n=1}^{\infty} C_n \frac{d}{d\mu} \frac{dP_n}{d\mu} = a_0 \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{j+1+2m-q} \left[ \frac{(2j+m-q)(2j+1+2m-q)}{(m-q)!} \right] C_{2j+1+m-q} \sum_{p=0}^{q} \frac{d^p Q_n}{d\alpha^p} \frac{d^{p+1} Q_{n+m-q}}{d\alpha^{p+1}}
\]

or
\[
\sum_{n=1}^{\infty} C_n \frac{d}{d\mu} \frac{dP_n}{d\mu} = a_0 \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{j+1+2m-q} \left[ \frac{(2j+m-q)(2j+1+2m-q)}{(m-q)!} \right] C_{2j+1+m-q} \sum_{p=0}^{q} \frac{d^p Q_n}{d\alpha^p} \frac{d^{p+1} Q_{n+m-q}}{d\alpha^{p+1}}
\]

or
\[
\sum_{n=1}^{\infty} C_n \frac{d}{d\mu} \frac{dP_n}{d\mu} = a_0 \sum_{m=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{j+1+2m-q} \left[ \frac{(2j+m-q)(2j+1+2m-q)}{(m-q)!} \right] C_{2j+1+m-q} \sum_{p=0}^{q} \frac{d^p Q_n}{d\alpha^p} \frac{d^{p+1} Q_{n+m-q}}{d\alpha^{p+1}}
\]

(41)

Analogous to equation (34)
\[
Q_n = \sum_{q=0}^{\infty} \sum_{p=0}^{q} \frac{d^p Q_n}{d\alpha^p}
\]

Hence, in a manner similar to the derivation of equation (38)
\[
\sum_{n=1}^{\infty} n(n+1) C_n \frac{d}{d\mu} (P_n Q_n) =
\]
\[
\sum_{m=1}^{\infty} m \sum_{q=0}^{m-1} \sum_{j=0}^{\infty} (-1)^{j+m-q} \left[ \frac{(2j+m-q)(2j+1+m-q)}{(m-q)!} \right] C_{2j+1+m-q} \sum_{p=0}^{q} \frac{d^p Q_n}{d\alpha^p} \frac{d^{p+1} Q_{n+m-q}}{d\alpha^{p+1}}
\]

(42)

Substituting equations (40), (41), and (42) in the boundary condition and equating the coefficients of the various powers of \( \mu \) equal to zero, the following set of equations is obtained:
\[
\sum_{n=0}^{\infty} (-1)^{n+1} \left[ \frac{2n+1}{[2n]} \right] C_{2n+1} \left[ a_0 \frac{dQ_{2n+1}}{da_0} - (2n+1)(2n+2) Q_{2n+1} \right] - 2na_0 C_{2n} \frac{dQ_{2n}}{da_0} = a_0 \text{ for } h=0
\]

and (after rearranging as for equation (39))
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{h-1} \sum_{m=0}^{k} \sum_{q=0}^{m} \sum_{j=0}^{\infty} (-1)^{j+1+2m-q} \left[ \frac{(2j+1+2m)}{[2j]} \right] C_{2j+1+m} \frac{d^{n+1} Q_{2j+1+m}}{d\alpha^{n+1}}
\]
\[
+ a_0 \sum_{n=0}^{\infty} \sum_{k=0}^{h-1} \sum_{m=0}^{k} \sum_{q=0}^{m} \sum_{j=0}^{\infty} (-1)^{j+1+2m} \left[ \frac{(2j+1+2m)}{[2j]} \right] C_{2j+1+m} \frac{d^{n+1} Q_{2j+1+m}}{d\alpha^{n+1}}
\]
\[
- (h+1) \sum_{n=0}^{\infty} \sum_{k=0}^{h-1} \sum_{m=0}^{k} \sum_{q=0}^{m} \sum_{j=0}^{\infty} (-1)^{j+1} \frac{(2j+m)(2j+1+m)}{[2j]} C_{2j+1+m} \frac{d^{n+1} Q_{2j+1+m}}{d\alpha^{n+1}} = (h+1)a_{1,1}-
\]

where \( h=1, 2, 3, \ldots \). \( \infty \).

This set of equations represents a formal and rigorous solution of the problem of potential flow about a body of revolution with flow normal to the axis of symmetry. The only unknown quantities are the infinite number of \( C_n \)'s. The other quantities appearing in the equations are determined as in equations (39).

If the meridian profile is an ellipse \( \lambda=a_0 \), the \( a_{1,n} \)'s are all zero and, from the second expression of equation (43),
\[
\sum_{n=0}^{\infty} (-1)^{n+1} \frac{[2n+2h+1]}{h!} \left[ a_0 \frac{dQ_{2n+1+h}}{da_0} - (2n+2) Q_{2n+1+h} \right] C_{2n+1+h} = 0
\]

where \( h=1, 2, \ldots \). \( \infty \).
This expression represents an infinite set of linear homogenous equations in the unknowns \( C_0, C_1, \ldots \) and, since the determinant of the coefficients is different from zero, it immediately follows that the only solution is \( C_2 = C_3 = C_4 = \ldots = 0 \). From the first expression of equation (43), it follows that

\[
C_1 = \frac{a_0}{dQ_4 da_0 - 2Q_4(a_0)}
\]

This result could have been easily obtained from the general expression for the velocity potential given by equation (26). Thus, assume that

\[
\frac{d^2 \phi}{dn^2} = 2aV \cos \theta \left( \frac{1 - \mu^2}{a_0^2 - 1} \right) \left[ C_1(a_0^2 - 1) \frac{d^2 Q_4}{da_0^2} + C_1 \frac{dQ_4}{da_0} + a_0 \right] \frac{d\lambda}{dn} = 0
\]

From Legendre's equation

\[
(a_0^2 - 1) \frac{d^2 Q_4}{da_0^2} = -2a_0 \frac{dQ_4}{da_0} + 2Q_4(a_0)
\]

Hence (see reference 10, p. 133)

\[
C_1 = -\frac{a_0}{a_0^2 - 2Q_4(a_0)} = -\frac{a_0^2}{2} \log \frac{a_0 + 1}{a_0 - 1} + \frac{a_0^2 - 1}{a_0 - 1}
\]

In the appendix an application of the boundary condition (equations (39) and (43)) for axial and transverse flows, respectively, is made to a body of revolution obtained from a symmetrical Joukowsky profile.

INERTIA COEFFICIENTS OF BODIES OF REVOLUTION

It is of some interest to obtain the coefficients of inertia for axial and transverse flows and also to compare them with those of an ellipsoid of revolution of equal fineness ratio (references 11 and 12).

When a body moves in a fluid at rest at infinity the total kinetic energy of the fluid is given by

\[
2T = -\sigma \int \int \phi \frac{\partial \phi}{\partial n} dS
\]

where \( \phi \) is the velocity potential of the fluid motion, \( \frac{\partial \phi}{\partial n} \) the normal derivative of \( \phi \) where the positive direction of the normal to the surface of the body is into the fluid and the integration is performed over the surface of the body; \( \sigma \) denotes the density of the fluid.

\[
2T = -8\pi a^3 \sigma U \int_1^\infty \left( 1 - \mu^2 \right)^{\frac{1}{2}} \frac{d\lambda}{d\mu} - (\lambda^2 - 1) \mu \sum_{n=1}^\infty A_n P_n(\mu) \Phi_n(\lambda) d\mu
\]

If \( M \) is the mass of fluid displaced by the body, then the coefficient of inertia \( k_1 \) is the quantity multiplying \( MU^2 \) in the expression for \( 2T \).

If the body is a prolate spheroid \( \lambda = a_0 \), the foregoing expression for \( 2T \) becomes:

\[
2T = \frac{4}{3} \pi a(2a)^3 U^2 (a_0^2 - 1) \frac{1}{2} a_0 \log \frac{a_0 + 1}{a_0 - 1} - \frac{a_0}{a_0^2 - 1} \frac{1}{2} a_0 \log \frac{a_0 + 1}{a_0 - 1}
\]

\[
\phi = 2aV \cos \theta \left( \lambda^2 - 1 \right)^{\frac{1}{2}} (1 - \mu^2)^{\frac{1}{2}} C_1 \frac{dP_1}{d\lambda}
\]

If the body moves in the positive direction of the \( y \) axis with constant velocity \( V \), it may be supposed to be at rest and the fluid to have a translation \(- V \) superposed on its actual motion. Accordingly

\[
\phi = 2aV \cos \theta \left( \lambda^2 - 1 \right)^{\frac{1}{2}} (1 - \mu^2)^{\frac{1}{2}} \left( C_1 \frac{dQ_4}{d\lambda} + 1 \right)
\]

At the surface of the ellipsoid of revolution generated by the ellipse \( \lambda = a_0 \), the normal velocity of the fluid must be zero. Therefore

\[
\text{FLOW PARALLEL TO THE AXIS OF SYMMETRY}
\]

Since the velocity potential of this type of flow is independent of the angular coordinate \( \theta \), the following equation may be written for the element of surface:

\[
dS = 2\pi nds
\]

where \( ds \) denotes the element of length along a meridian profile. Hence

\[
2T = -2\pi \sigma \int_0^\infty \rho \phi \frac{\partial \phi}{\partial n} ds
\]

If the body moves in the direction of its axis of symmetry with a uniform velocity \( U \) the boundary condition is

\[
\frac{\partial \phi}{\partial n} ds = -U \frac{\partial \phi}{\partial n} ds
\]

Also, according to equation (24)

\[
-\frac{\partial \phi}{\partial n} ds = \left[ \frac{1 - \mu^2}{\lambda^2 - 1} \right]^{\frac{1}{2}} \frac{\partial \phi}{\partial \mu} d\lambda - \left( \lambda^2 - 1 \right) \mu \frac{\partial \phi}{\partial \mu} d\mu
\]

Therefore

\[
2T = -8\pi a^2 \sigma U \int_1^\infty \phi \left( 1 - \mu^2 \right) \frac{d\lambda}{d\mu} - (\lambda^2 - 1) \mu \sum_{n=1}^\infty A_n P_n(\mu) \Phi_n(\lambda) d\mu
\]

In general then

\[
2T = -8\pi a^2 \sigma U \int_1^\infty \left( 1 - \mu^2 \right) \frac{d\lambda}{d\mu} - (\lambda^2 - 1) \mu \sum_{n=1}^\infty A_n P_n(\mu) \Phi_n(\lambda) d\mu
\]

But 2a = \( A_e \), \( a_0 = \frac{1}{e} \) and 2a(\( a_0^2 - 1 \)) = B where \( A, B \) are the semimajor and semiminor axes, respectively, and \( e \) the eccentricity of the elliptical meridional section. Therefore

\[
2T = \frac{1}{2\pi} \int_{1-\frac{e}{1-e}}^{1+\frac{e}{1-e}} \frac{1}{1-e^2} \frac{1}{2\pi} \frac{1}{1-e} \frac{1}{1+e} \frac{4}{3} \pi AB^2 \sigma U^2 = k_1 MU^2
\]
The coefficient of inertia for a prolate ellipsoid in uniform axial motion is then given by

\[
\kappa_a = \frac{\frac{1}{2e} \log \frac{1+e}{1-e} - 1}{\frac{1}{2e^2} - \frac{1}{2e} \log \frac{1+e}{1-e}}
\]

(47)

(See reference 10, p. 144)

Equation (46) is now evaluated for the case of a body of revolution obtained from the Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \). (See appendix.) The volume of this body is found to be

\[
Q = \frac{4}{3} \pi (2a)^3 \times 0.05342
\]

so that the expression for \( 2T \) may be written:

\[
2T = \frac{2\pi}{3} (2a)^3 \times 0.003139 \sigma Q U^2
\]

or \( \kappa_a = 0.0881 \). (See table I.) Compare this value of \( \kappa_a \) with that of a prolate spheroid whose fineness ratio is the same as for the above-mentioned body of revolution. The fineness ratio \( f \) is defined as the ratio of the length to the maximum diameter of the body. The maximum diameter is obtained from equation (53) by means of the condition \( \frac{dp}{d\theta} = 0 \) and the length of the body is given by

\[
l = 2a (\lambda_{n+1} + \lambda_{n-1})
\]

By means of these expressions it is found that \( f = 4.208 \). The fineness ratio for an ellipse is given by

\[
f = \frac{A}{B} = \frac{1}{\sqrt{1-e^2}}
\]

or

\[
e = \sqrt{1 - \frac{1}{f^2}} = 0.971
\]

where \( e \) is the eccentricity of the ellipse.

Substituting this value of \( e \) into equation (47), the following value of \( \kappa_a \) is obtained,

\[
\kappa_a = 0.0757
\]

A theorem enunciated by Munk (reference 13) states that when the disturbance caused by a body moving in an infinite fluid is replaced by fictitious sinks and sources, the total mass is the sum of the products obtained by multiplying the intensity of each source or sink by the potential of the parallel flow. This theorem will now be shown to be only a first approximation and to hold exactly only for ellipsoids of revolution. Thus from equation (19),

\[
I(z_1) = 2\pi \sum_{n=1}^{\infty} A_n P_n(\lambda_1)
\]

where \( z_1 = 2a \lambda_1 \)

The strength per length \( dz \) is then

\[
4\pi \sigma \sum_{n=1}^{\infty} A_n P_n(\lambda_1) d\lambda_1
\]

The velocity potential at \( (z_1, 0) \) of the parallel flow is given by

\[
\phi = 2a U \lambda_1
\]

Hence, according to Munk's theorem

\[
2T_{total} = 8\pi \sigma a^2 U \sum_{n=1}^{\infty} A_n \int_{-1}^{1} \lambda_1 P_n(\lambda_1) d\lambda_1
\]

or

\[
2T_{total} = \frac{4}{3} \pi (2a)^3 \sigma Q U^2
\]

Therefore

\[
2T_{knife} = \frac{4}{3} \pi (2a)^3 \frac{A_1}{Q} \left( \frac{1}{2aU} - 1 \right) MU^2
\]

where \( Q \) is the volume of the body and \( M \) is the mass of the displaced fluid.

The coefficient of inertia for axial flow is therefore

\[
\kappa_a = \frac{4}{3} \pi (2a)^3 \frac{A_1}{Q} \frac{1}{2aU}
\]

This expression for \( \kappa_a \) is valid for a prolate ellipsoid but is not valid for a more general shape.

It is obvious from this expression that Munk's theorem applies exactly only to ellipsoids of revolution since only the coefficient \( A_1 \) appears.\(^1\) In order to provide a numerical comparison between Munk's theorem and the exact method, the foregoing equation is evaluated for the body of revolution whose meridian curve is the Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \). It yields a value \( \kappa_a = 0.0717 \) as compared with the more exact value \( \kappa_a = 0.0867 \) already obtained by means of the fundamental equation (46).

FLOW NORMAL TO THE AXIS OF SYMMETRY

For flow normal to the axis of symmetry the velocity potential depends not only on the elliptic coordinates \( \lambda, \mu \), but also on the cylindrical coordinate \( \theta \). Hence, the equation for the element of surface \( dS \) is

\[
dS = \rho d\theta ds
\]

and equation (44) becomes

\[
2T = -\sigma \int \int \rho \frac{\partial \phi}{\partial n} dS d\theta
\]

If the body moves in the direction of the transverse axis \( O\xi \) with a constant velocity \( V \) the boundary condition is

\[
\frac{\partial \phi}{\partial n} dS = -V \cos \theta \frac{\partial \rho}{\partial n} ds
\]

\(^1\) The above criticism of Munk's theorem has been found to be incorrect. (It is to be noted that the volume of the body contains all of the coefficients \( A_n \) implicitly.) This theorem may be readily verified by applying Green's second theorem to the space internal to the shape and enforcing the appropriate distribution of sinks and sources \( \gamma \). Then if \( \phi_1 = U \xi \) and \( \phi_2 = \phi \) (where \( \Delta \xi = \gamma = 0 \)), Green's second theorem together with equation (44) immediately yields the following expression on Munk's theorem:

\[
\frac{2T_{knife}}{\sigma} + U \xi (\text{volume of body}) = \int \int U \xi d\phi = \int \int U \xi d\gamma
\]
Also, in general
\[ \phi = 2aV \cos \theta (\lambda^2 - 1)^{1/2} (1 - \mu^2)^{1/2} \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} \]
Hence
\[ 2T = (2a)^3 \pi \sigma V^2 \int_{-1}^{1} (1 - \mu^2) (\lambda^2 - 1) \left( \frac{d\lambda}{d\mu} + \lambda \right) \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} \frac{dQ_n}{d\lambda} \, d\mu \]
(48)

For a prolate spheroid, \( \lambda = a_0 \) and
\[ \phi = 2aV \cos \theta (\lambda^2 - 1)^{1/2} (1 - \mu^2)^{1/2} \frac{C_n dQ_n}{d\lambda} \]
where
\[ C_n = \frac{a_0}{a_0 - 2Q_1(a_0)} \]
Therefore
\[ 2T = \frac{a_0}{a_0^2 - 1} \frac{a_0 + 1}{2} \log \frac{a_0 + 1}{a_0 - 1} \frac{4}{3} \pi A B^2 \sigma V^2 = k_T MV^2 \]
or
\[ k_T = - \frac{1 - e^2}{2e^2} \log \frac{1 + e}{1 - e} \]
(49)

(See reference 10, p. 145.)
For the body of revolution whose meridian curve is the Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \) (see appendix) it is seen from equation (48) that
\[ 2T = \frac{\pi \sigma V^2 (2a)^3}{3} \times 0.059587 = 0.8366 \times MV^2 \]
Therefore \( k_T = 0.8366 \). (See table II.)
According to equation (49) for the prolate ellipsoid of equal fineness ratio \( f = 4.208 \) and \( k_T = 0.8689 \).
According to Munk’s theorem the inertia coefficient \( k_T \) of transverse flow may be obtained from the doublet distribution along the axis of symmetry. Again, as in the case of axial flow, this theorem is a first approximation and holds exactly only for ellipsoids of revolution since an expression for \( k_T \) is obtained that contains only the coefficient \( C_1 \). Thus from equation (30) it follows that

From equation (45), it follows that
\[ \frac{dP_n}{dn} ds = -8a^2 (1 - \mu^2)^{1/2} (\lambda^2 - 1)^{1/2} (\mu d\lambda + \lambda d\mu) \]

Then according to Munk
\[ 2T_{total} = \sigma V \int_{-1}^{1} J(2a\lambda_1) 2ad\lambda_1 \]
or
\[ 2T_{total} = -16 \pi a^2 V^2 \sum_{n=1}^{\infty} C_n \int_{-1}^{1} (1 - \lambda_1^2) \frac{dP_n}{d\lambda_1} \frac{dP_1}{d\lambda_1} \, d\lambda_1 \]
since
\[ \frac{dP_1}{d\lambda_1} = 1 \]
Hence
\[ 2T_{total} = - \frac{8}{3} \pi (2a)^3 V^2 C_1 \]
and
\[ 2T_{total} = - \frac{8}{3} \pi (2a)^3 C_1 + \frac{Q}{Q} MV^2 \]
or
\[ k_T = - \frac{8}{3} \pi (2a)^3 C_1 + \frac{Q}{Q} \]

In order to give a numerical comparison between Munk’s theorem and the exact method, the foregoing equation is evaluated for the body of revolution whose meridian curve is the symmetrical Joukowsky profile \( \epsilon_1 = 0.15, \epsilon_2 = 0.10 \). It yields a value \( k_T = 0.8210 \) as compared with the more exact value \( k_T = 0.8366 \) obtained from the fundamental equation (48).

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, Va., November 12, 1934.
APPENDIX

APPLICATION OF THE ANALYSIS TO SURFACES OF REVOLUTION OBTAINED FROM SYMMETRICAL JOUKOWSKY PROFILES

By means of the mapping function

\[ t = t' + \frac{a^2}{t} \]  

(50)

the circle \( k_1 \) of radius \( a \) in the \( t' \) plane is transformed into the line segment \((-2a, 0; 2a, 0)\) in the \( t \) plane and the circle \( k_2 \) of radius \((1+e_1+e_2)a\) with center at \((e_1, a, O)\) is transformed into a symmetrical Joukowski profile \( J \) in the \( t \) plane. (See fig. 2.)

![Diagram showing mapping of circle into Joukowski profile](image)

**Figure 2.**—Mapping of a circle into a symmetrical Joukowski profile.

If in the \( t' \) plane \( PQ = a e^{t_0}, PO = a e^t, \) angle \( POQ = \eta, \) and angle \( PQz' = \phi \) then, according to the law of cosines,

\[ e^{t_0 + t} = 1 + 2\delta \cos \phi + \delta^2 \]  

(51)

where \( \delta = \frac{e_1}{1+e_1+e_2} \).

Again, by the law of sines

\[ \tan \eta = \frac{\sin \phi}{\delta + \cos \phi} \]  

(52)

Putting \( t' = a e^{t+it} \) into equation (48),

\[ t = 2a \cosh (\xi + i\eta) \]

or \( z = 2a \cosh \xi \cos \eta, \rho = 2a \sinh \xi \sin \eta \).

The latter two equations are, in fact, the equations of transformation from the rectangular coordinates \((z, \rho)\) to the elliptic coordinates \((\xi, \eta)\). Since \((z, \rho)\) refer to points of the Joukowski profile \( J \), using equations (51) and (52), the following parametric equations of the system of symmetrical Joukowski profiles may be obtained

\[
\begin{align*}
\lambda &= \frac{\xi \left(1 + 2\delta + \delta^2\right)^4}{2\delta + \delta^2} + \frac{\delta}{2\delta \left(1 + 2\delta + \delta^2\right)^4} \\
\mu &= \frac{\delta + \nu}{(1 + 2\delta + \delta^2)^4}
\end{align*}
\]  

(53)

where \( \lambda = \cosh \xi, \mu = \cos \eta, \) and \( \nu = \cos \phi \) (the independent parameter).

From these equations \( \lambda \) can be expressed as a power series in \( \mu \) by means of a Maclaurin's expansion in the neighborhood of \( \mu = 0 \) (i.e. \( \nu = -\delta \)). Thus,

\[ \lambda = a_0 \left(1 + \alpha \gamma + \frac{1}{2} \gamma^2 - \frac{1}{8} \gamma^3 + \frac{1}{16} \gamma^4 \ldots\right) \]  

(54)

where

\[ a_0 = \frac{e_1^2 \left(1 - \delta^2\right) + \delta^2}{2\delta \left(1 + \delta^2\right)^4}, \quad \alpha = \frac{e_1^2 \left(1 - \delta^2\right) + \delta^2}{e_1^2 \left(1 - \delta^2\right) + \delta^2} \quad \text{and} \quad \gamma = \frac{\delta}{\sqrt{1 - \delta^2} \mu} \]

FLOW PARALLEL TO THE AXIS OF SYMMETRY

Equation (39) is a set of linear equations for the infinite number of unknown coefficients \( A_n \) and provides a solution of the problem of axial potential flow about a body of revolution. In practice it is necessary to evaluate only the first few coefficients \( A_n \). From equation (39), neglecting the \( A_n \)'s after \( A_6 \), the following equations are obtained:

\[ \lambda = a_0 \left(\alpha \gamma + \frac{1}{2} \gamma^2 \right) \]

In fact, if the expression for \( \mu \) from equation (53) is herein substituted the equation for \( \lambda \) is obtained.
\[ -5 \frac{dQ_1}{da_0} A_1 + 2 \frac{dQ_3}{da_0} A_3 - \frac{dQ_4}{da_0} A_4 = 16a U \]

\[ 8a_{1,0} \frac{d^2 Q_3}{da_0^2} A_1 + 8 \frac{dQ_3}{da_0} A_2 = 2a_{1,0} \frac{d^2 Q_3}{da_0^2} A_3 - \frac{dQ_4}{da_0} A_4 \]

\[ 8 \left( 2a_{1,1} \frac{d^2 Q_3}{da_0^2} + a_{1,0} \frac{d^2 Q_4}{da_0^2} \right) A_4 + 16a_{1,0} \frac{d^2 Q_3}{da_0^2} A_3 + 2 \left( \frac{10}{da_0} - \frac{2a_{1,1}}{da_0} \right) \frac{dQ_3}{da_0} A_4 = 0 \]

\[ \frac{dQ_3}{da_0} - \frac{a_{1,0}}{da_0} \frac{dQ_3}{da_0^2} A_3 \]

\[ 8 \left( 6a_{1,0} \frac{d^2 Q_3}{da_0^2} + 6a_{1,0} \frac{d^2 Q_4}{da_0^2} + a_{1,0} \frac{d^2 Q_5}{da_0^2} \right) A_1 + 24 \left( 2a_{1,1} \frac{d^2 Q_3}{da_0^2} + a_{1,0} \frac{d^2 Q_4}{da_0^2} \right) A_2 \]

\[ + 2 \left[ 6(5a_{1,1} - a_{1,0}) \frac{dQ_3}{da_0} - 6a_{1,0} a_{1,0} \frac{dQ_3}{da_0^2} - a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_3 \]

\[ + \left[ \frac{14}{da_0} - 6a_{1,0} \frac{d^2 Q_3}{da_0^2} - 3a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_4 \]

\[ + 6 \left( \frac{14}{da_0} - 6a_{1,0} \frac{d^2 Q_3}{da_0^2} - 3a_{1,0} \frac{d^2 Q_4}{da_0^2} \right) A_4 \]

\[ + \left[ \frac{6}{da_0} - 14a_{1,0} \frac{d^2 Q_3}{da_0^2} + 6a_{1,0} a_{1,0} \frac{dQ_3}{da_0} + a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_5 = 0 \]

\[ 8 \left[ 24a_{1,0} \frac{d^2 Q_3}{da_0^2} + 12(2a_{1,0} a_{1,1} + a_{1,0}) \frac{dQ_3}{da_0} + 12a_{1,0} a_{1,0} \frac{dQ_3}{da_0} + a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_3 \]

\[ + 32 \left( 6a_{1,0} \frac{d^2 Q_3}{da_0^2} + 6a_{1,0} a_{1,0} \frac{dQ_3}{da_0} + a_{1,0} \frac{d^2 Q_4}{da_0^2} \right) A_3 \]

\[ + 2 \left[ 24(5a_{1,1} - a_{1,0}) \frac{dQ_3}{da_0} + 12(5a_{1,0} a_{1,0} - a_{1,1}) \frac{dQ_3}{da_0} - 12a_{1,0} a_{1,0} \frac{d^2 Q_3}{da_0^2} - a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_3 \]

\[ + 24 \left( 2(5a_{1,1} - a_{1,0}) \frac{dQ_3}{da_0} + 6a_{1,0} a_{1,0} \frac{dQ_3}{da_0} - a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_4 \]

\[ + 504 \frac{dQ_3}{da_0} - 24(14a_{1,1} - a_{1,0}) \frac{dQ_3}{da_0} + 12(2a_{1,0} a_{1,1} + a_{1,0} - 14a_{1,0}) \frac{dQ_3}{da_0} + 12a_{1,0} a_{1,0} \frac{d^2 Q_3}{da_0^2} + a_{1,0} \frac{d^2 Q_4}{da_0^2} \right] A_5 = 0 \]

The coefficients of the unknown \( A_i \)'s can be calculated simply by a knowledge of power-series development of \( \lambda \) in \( \mu \) (i.e., the quantities \( a_0, a_{1,0}, a_{1,1}, \ldots \) are obtained from equation (33)). The zonal harmonics \( Q_n(a_0) \) are given by means of the recursion formula:

\[(n+1)Q_{n+1}(a_0) - (2n+1)a_n Q_n(a_0) + nQ_{n-1}(a_0) = 0\]

In the application of this recursion formula it is necessary to calculate \( Q_0(a_0) \) and \( Q_1(a_0) \) independently where

\[ Q_0(a_0) = \frac{1}{2} \log \frac{a_0 + 1}{a_0 - 1} \]

and \( Q_1(a_0) = a_0 Q_0(a_0) - 1 \)

The first derivatives \( \frac{dQ_1}{da_0} \) are then obtained from the following relation:

\[ \frac{dQ_{n+1}}{da_0} + \frac{dQ_n}{da_0} = (1 + a_0) \frac{dQ_0}{da_0} + Q_0(a_0) + 3Q_1(a_0) + 5Q_2(a_0) + \ldots + (2n+1)Q_n(a_0) \]

where it is necessary to determine \( \frac{dQ_0}{da_0} \) independently.

In order to calculate the higher derivatives the preceding recursion formula may be repeatedly differentiated with regard to \( a_0 \). The higher derivatives of \( Q_3 \) and \( Q_4 \) are obtained independently by means of the (\( r \)-1)th derivative of Legendre's differential equation:

\[(a_o^2 - 1) \frac{d^{r+1}Q_3}{da_0^{r+1}} + 2ra_3 \frac{d^rQ_3}{da_0^r} - (n-r+1) \frac{d^{r-1}Q_3}{da_0^{r-1}} = 0\]

If the constants \( a_0, a_{1,1}, a_{1,2}, \ldots \) and the various derivatives of \( Q_i \) are known, the coefficients of the unknowns \( A_i \) in equation (54) are easily calculated. The resulting system of linear equations can then be solved for the \( A_i \)'s which in turn determine the potential function \( \phi \) given by equation (10). A knowledge of the potential function yields directly the velocity \( u \) given by equation (16). Finally, according to Bernoulli's equation, the pressure \( p \) on the surface is given by

\[ p + \frac{1}{2} \rho u^2 = p_0 + \frac{1}{2} \rho U^2 \]

or

\[ C_p = \frac{p - p_0}{p_0} = 1 - \left( \frac{u}{U} \right)^2 \]

where \( p_0 = \frac{1}{2} \rho U^2 \)

The numerical work is straightforward but somewhat tedious owing largely to a lack of tables of the zonal harmonics of the second kind.

As an illustration of the procedure here outlined, consider the body of revolution whose meridian curve is
the Joukowsky profile (reference 14) defined by 

\[ \lambda = 0.10 \] 

and \( \nu = 0.10 \). (See fig. 2.) From equation (53) there can be written then for the power-series development of \( \lambda \):

\[ \lambda = 1.02340 + 0.02630 \mu + 0.007476 \mu^2 - 0.0000279 \mu^4 + \ldots, \]  

(56)

a very rapidly convergent series.

<table>
<thead>
<tr>
<th>19.385 ( A_1 )</th>
<th>(-3.780 \ A_3)</th>
<th>(+1.400 \ A_5=2aU )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.372 ( A_1 )</td>
<td>(-7.635 \ A_3)</td>
<td>(+4.333 \ A_4)</td>
</tr>
<tr>
<td>(-1.152 \ A_1)</td>
<td>(+1.294 \ A_2)</td>
<td>(-0.810 \ A_3)</td>
</tr>
<tr>
<td>(-0.500 \ A_1)</td>
<td>(+0.685 \ A_2)</td>
<td>(-0.763 \ A_3)</td>
</tr>
<tr>
<td>(-27.957 \ A_1)</td>
<td>(+39.785 \ A_2)</td>
<td>(-57.432 \ A_4)</td>
</tr>
</tbody>
</table>

The solution is given by \(^2\)

\[ A_1=0.0573\times2aU, \ A_2=0.0726\times2aU, \ A_3=0.0296\times2aU, \ A_4=0.0065\times2aU, \ A_5=0.0015\times2aU \]

The sink-source distribution obtained from equation (19) is:

\[ I(2a\lambda) = 4\pi a U \left( -0.0339 + 0.0157 \lambda + 0.0846 \lambda^2 + 0.0606 \lambda^3 + 0.0283 \lambda^4 + 0.0119 \lambda^5 \right) \]  

(58)

Figure 3 shows a graph of this function with \( I(2a\lambda)/4\pi a U \) as ordinate and \( \lambda \) as abscissa.

![Figure 3. Sink-source and doublet distributions.](image)

In order to obtain the pressure distribution, the following expressions can be evaluated at a sufficient number of points of the boundary:

\[ \left( \frac{\mu}{\nu} \right)^2 \times \frac{1}{\lambda^2 - \mu^2} \left( \frac{1}{2aU} \sum_{n=1}^{\infty} A_n \frac{dQ_n}{d\lambda} P_n(\mu) + \mu \right) + (1 - \mu^2) \left( \frac{1}{2aU} \sum_{n=1}^{\infty} A_n Q_n(\lambda) \frac{dP_n}{d\mu} + \lambda \right) \]  

(59)

Note here that the velocity potential

\[ \phi = 2aU \left[ 0.0573 \ P_1(\mu) Q_\mu(\mu) + 0.0726 \ P_4(\mu) Q_\mu(\mu) + \ldots + 0.0015 \ P_6(\mu) Q_\mu(\mu) \right] \]

is exact for the body of revolution obtained by superposing a uniform velocity \( U \) on the flow from the sink-source distribution given by equation (58). This body is a very good approximation to the actual body obtained by revolving the Joukowsky profile about the axis of symmetry, so that in calculating the pressure distribution it is permissible to use the \( \lambda, \mu \) values as given by equation (55).

Table IV shows the sequence of operations to be followed in obtaining the pressure distribution and figure 4 presents graphically the pressure distribution.
FLOW NORMAL TO THE AXIS OF SYMMETRY

From equations (43), for the first five coefficients $A_1, A_2, A_3, A_4, A_5$ the following set of linear equations can be obtained:

$$A_1 = a_0 \frac{dQ_1}{da_0} - 2Q_1$$
$$A_2 = 3a_{1,0} \frac{dQ_2}{da_0}$$
$$A_3 = -3 \left( \frac{1}{2} a_0 + \frac{dQ_3}{da_0} - 6Q_3 \right)$$
$$A_4 = -15 a_{1,0} \frac{dQ_4}{da_0}$$
$$A_5 = \frac{15}{4} \left( \frac{1}{2} a_0 + \frac{dQ_5}{da_0} - 15Q_5 \right)$$

$$A_1 = a_{1,0} \left( \frac{a_0 \frac{dQ_1}{da_0} - 2Q_1}{a_0} \right)$$
$$A_2 = 3 \left[ a_{1,0} \frac{dQ_2}{da_0} + a_{2,0} (a_{1,1} + a_0) \frac{dQ_2}{da_0} - 3a_{1,1} \frac{dQ_1}{da_0} \right]$$
$$A_3 = -3 \left[ a_{1,0} \frac{dQ_3}{da_0} + a_{2,0} (a_{1,1} + a_0) \frac{dQ_3}{da_0} + (3a_{1,2} - 7a_{1,0}) \frac{dQ_3}{da_0} \right]$$
$$A_4 = -15 \left[ a_{1,0} \frac{dQ_4}{da_0} + a_{2,0} (a_{1,1} + a_0) \frac{dQ_4}{da_0} + (3a_{1,2} - 28a_{1,0}) \frac{dQ_4}{da_0} - 20Q_4 \right]$$
$$A_5 = \frac{15}{4} a_{1,0} \left( \frac{a_0 \frac{dQ_5}{da_0} - 29Q_5}{a_0} \right)$$

$$A_1 = a_{2,0} a_{1,0} \frac{dQ_1}{da_0} + a_{2,0} (a_{1,1} - a_{1,0}) \frac{dQ_1}{da_0} - 3a_{1,1} \frac{dQ_1}{da_0}$$
$$A_2 = \frac{3}{2} \left[ a_{1,0} \frac{dQ_2}{da_0} + a_{2,0} (3a_{1,1} - a_0) \frac{dQ_2}{da_0} + (3a_{1,2} - 7a_{1,0}) \frac{dQ_2}{da_0} \right]$$
$$A_3 = -\frac{3}{4} \left[ a_{1,0} \frac{dQ_3}{da_0} + a_{2,0} (a_{1,1} - 16a_{1,0}) \frac{dQ_3}{da_0} + (5a_0 + 33a_{1,0}) \frac{dQ_3}{da_0} + 30Q_3 \right]$$
$$A_4 = -15 \left[ a_{1,0} \frac{dQ_4}{da_0} + a_{2,0} (3a_{1,1} - a_0) \frac{dQ_4}{da_0} + (3a_{1,2} - 28a_{1,0}) \frac{dQ_4}{da_0} \right]$$
$$A_5 = \frac{15}{4} a_{1,0} \left( a_{2,0} \frac{dQ_5}{da_0} - 29Q_5 \right)$$

$$A_1 = a_{2,0} a_{1,0} \frac{dQ_1}{da_0} + a_{2,0} (a_{1,1} - a_{1,0}) \frac{dQ_1}{da_0} - 3a_{1,1} \frac{dQ_1}{da_0}$$
$$A_2 = \frac{3}{2} \left[ a_{1,0} \frac{dQ_2}{da_0} + a_{2,0} (3a_{1,1} - a_0) \frac{dQ_2}{da_0} + (3a_{1,2} - 7a_{1,0}) \frac{dQ_2}{da_0} \right]$$
$$A_3 = -\frac{3}{4} \left[ a_{1,0} \frac{dQ_3}{da_0} + a_{2,0} (a_{1,1} - 16a_{1,0}) \frac{dQ_3}{da_0} + (5a_0 + 33a_{1,0}) \frac{dQ_3}{da_0} + 30Q_3 \right]$$
$$A_4 = -15 \left[ a_{1,0} \frac{dQ_4}{da_0} + a_{2,0} (3a_{1,1} - a_0) \frac{dQ_4}{da_0} + (3a_{1,2} - 28a_{1,0}) \frac{dQ_4}{da_0} \right]$$
$$A_5 = \frac{15}{4} a_{1,0} \left( a_{2,0} \frac{dQ_5}{da_0} - 29Q_5 \right)$$
\[ A_i^2 = 3 \left[ a_{i,2} \frac{d^2 Q_i}{da_0^2} + a_{i,3} \frac{5 a_{i,1} + a_0 \frac{d^2 Q_i}{da_0^2} + a_{i,1} \frac{5 4 a_{2,1,2} + a_{i,1}^2}{2 f} + a_0 a_{i,1} - \frac{3}{2} a_{i,2} \frac{d^2 Q_i}{da_0^2} + \left( 5 a_{1,3} a_{i,2} + 5 a_{i,2,1,1} + 3 a_{i,1} - 10 a_{i,2,1,1} \right) \frac{d^2 Q_i}{da_0^2} + \left( 5 a_{1,4} - 11 a_{i,4} \right) \frac{d^2 Q_i}{da_0^2} \right] \]

\[ \frac{d^2 Q_i}{da_0^2} = -\frac{1}{2} \left[ a_{i,2} \frac{d^2 Q_i}{da_0^2} + a_{i,3} \frac{5 a_{i,1} + a_0 \frac{d^2 Q_i}{da_0^2} + a_{i,1} \frac{5 4 a_{2,1,2} + a_{i,1}^2}{2 f} + a_0 a_{i,1} - \frac{22}{3} a_{i,2} \frac{d^2 Q_i}{da_0^2} \right] \]

\[ \frac{d^2 Q_i}{da_0^2} = -\frac{1}{2} \left[ a_{i,2} \frac{d^2 Q_i}{da_0^2} + a_{i,3} \frac{5 a_{i,1} + a_0 \frac{d^2 Q_i}{da_0^2} + a_{i,1} \frac{5 4 a_{2,1,2} + a_{i,1}^2}{2 f} + a_0 a_{i,1} - \frac{22}{3} a_{i,2} \frac{d^2 Q_i}{da_0^2} \right] + 15 (a_{1,4} - a_{i,4}) \frac{d^2 Q_i}{da_0^2} + \left( 15 a_{1,4} - 138 a_{i,4} + 161 a_{i,1} \right) \frac{d^2 Q_i}{da_0^2} \]

Again, substituting the numerical values given by equation (53) and table III into the foregoing expressions, there result the following equations:

\[-22.402 \quad C_1 = -1.363 \quad C_2 = 34.018 \quad C_3 = 2.578 \quad C_4 = -39.602 \quad C_5 = 1.02340 \]

\[ 25.036 \quad C_1 = -67.392 \quad C_2 = 45.658 \quad C_3 = 163.240 \quad C_4 = 66.762 \quad C_5 = 0.05260 \]

\[-20.665 \quad C_1 = 78.034 \quad C_2 = -132.860 \quad C_3 = -221.940 \quad C_4 = 497.330 \quad C_5 = 0.02243 \]

\[-15.235 \quad C_1 = 64.050 \quad C_2 = -178.260 \quad C_3 = -202.310 \quad C_4 = 391.330 \quad C_5 = 0 \]

\[ 10.437 \quad C_1 = 46.495 \quad C_2 = 150.380 \quad C_3 = -361.150 \quad C_4 = 197.750 \quad C_5 = 0.00014 \]

The solution is given by

\[ C_1 = -0.0486 \]
\[ C_2 = -0.0178 \]
\[ C_3 = -0.0027 \]
\[ C_4 = -0.0028 \]
\[ C_5 = -0.00005 \]

The doublet distribution function \( J(2a\lambda_i) \) then becomes:

\[ J(2a\lambda_i) = 8\pi a^2 \left( 1 - \lambda_i^2 \right) \left( 0.0447 + 0.0512\lambda_i + 0.0187\lambda_i^2 + 0.0049\lambda_i^3 + 0.0022\lambda_i^4 \right) \]

The graph of this function with \( J/8\pi a^2 \) as ordinate and \( \lambda_i \) as abscissa is shown in figure 3.

**DETERMINATION OF THE TRANSVERSE-FORCE DISTRIBUTION**

When the axial flow is combined with the transverse flow some information regarding the distribution of the force over the surface of the body can be obtained by introducing the notion of the transverse-force coefficient. For the pressure difference at the surface, according to Bernoulli's equation:

\[ p - p_0 = \frac{\sigma}{2} (U^2 + V^2 - q^2) \]

where \( p_0 \) is the pressure at an infinite distance from the body and \( q \) is the velocity of the fluid at the surface, supposing the body to be at rest and the fluid to strike it at an angle \( \alpha \) where \( \tan \alpha = \frac{U}{V} \).

Now \( q \) has three components—in the directions \( d\lambda_i, ds_i, \) and \( d\theta \). They may be denoted by \( g_\lambda, g_s, \) and \( g_\theta \), respectively. Also denote by \( u_\lambda \) and \( u_s \) the velocity components of the axial flow and by \( v_\lambda \) and \( v_s \) the velocity components of the transverse flow taken in the plane \( \theta = 0 \). Then,

\[ g_\lambda = u_\lambda + v_\lambda \cos \theta \]
\[ g_s = u_s + v_s \cos \theta \]
\[ g_\theta = v_s \sin \theta \]

Introducing these values into Bernoulli's equation,
It is only the term
\[ \sigma u_0 v_0 + u_0 v_0 \cos \theta = \sigma (\bar{u} \bar{v}) \cos \theta \]
that need be considered, for the other terms have equal values for \( \theta \) and \( \pi - \theta \) and accordingly vanish when integrated over the entire cross section. The resulting transverse force, relative to an annular element of width unity, is therefore

\[ \frac{dQ}{dz} = \sigma \int_0^{2\pi} \cos \theta \rho \, d\theta = \sigma \pi \rho (\bar{u} \bar{v}) \]

that need be considered, for the other terms have equal values for \( \theta \) and \( \pi - \theta \) and accordingly vanish when integrated over the entire cross section. The resulting transverse force, relative to an annular element of width unity, is therefore

\[ \frac{\beta}{2a_2 \sin 2\alpha} = \frac{(\lambda^2 - 1)(1 - \mu^2)}{\lambda^2 - \mu^2} \left[ \left( \sum_{n=1}^{\infty} A_n \frac{dP_n}{d\mu} \right) \frac{dQ_n}{d\lambda} + \mu \right] + \left( \sum_{n=1}^{\infty} C_n \frac{dP_n}{d\mu} Q_n(\lambda) + \lambda \right) \left( \sum_{n=1}^{\infty} C_n \frac{dQ_n}{d\lambda} \right) \left[ n(n+1)Q_n(\lambda) - \lambda \frac{dQ_n}{d\lambda} \right] \]

Tables IV and V give the numerical data for the evaluation of the right-hand side of equation (61) and figure 5 represents graphically these numerical results.

FIGURE 5.—Transverse-force distribution.

with \[ \frac{\beta}{2a_2 \sin 2\alpha} \] as ordinate and \( \frac{\bar{u}}{a_2} (= \lambda \mu) \) as abscissa.

According to theory the positive and negative areas included by the \( \beta \) curve and the \( x \) axis are equal; that is, there is no resultant lift force but only a simple couple.

REFERENCES


2. Klemperer defines the transverse-force coefficient by

\[ \beta = \frac{dQ}{dz} = 2\pi \rho \left( \frac{\bar{u} \bar{v}}{U^2 + V^2} \right) \]

or

\[ \beta = \pi \rho \left( \frac{\bar{u} \bar{v}}{U^2 + V^2} \right) \sin 2\alpha \] (63)

By means of the velocity potentials of the axial and transverse flows this last expression takes the following form.


Mr. Upson has kindly pointed out to the author the very close check of his approximation formula (equation (10), N. A. C. A. Report 403) for the transverse force with the more exact equation (63).
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According to the empirical rule
\[ f(x) = A \cdot e^{-|x|} + B \cdot e^{x} \]

TABLE II

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According to the empirical rule
\[ f(x) = A \cdot x^2 + B \cdot x + C \]

TABLE III

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According to the empirical rule
\[ f(x) = A \cdot e^{-x} + B \cdot e^{x} \]

FIGURE 2

Potential flow about elongated bodies of revolution

\[ \int_{0}^{L} p \cdot n \cdot dA = \sum_{i=1}^{n} \left( p_{i} \cdot n_{i} \cdot dA_{i} \right) \]

\[ \int_{0}^{L} \rho \cdot v \cdot dA = \sum_{i=1}^{n} \left( \rho_{i} \cdot v_{i} \cdot dA_{i} \right) \]
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