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NOTES ON PROPELLER DESIGN - III.

The Aerodynamical Equations of the Propeller Blade Elements.

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Summary.

The air flow and the air force created by all elements of
the propeller blades lying in a ring located between two concen-
tric circles around the propeller axis are independent of what
happens in other rings. The aerodynamic equations of such blade
elements are established and their application is discussed.

Since the air forces of well-designed propellers are uniform-
ly distributed along the blades, the blade elements in each con-
centric ring around the propeller axis have properties practical-
ly independent of those in other rings. Hence the design or the
analysis of a propeller has to be performed by designing or ana-
lyzing consecutively the elements of a number of such rings. I
proceed immediately to the discussion of this problem.

One such short piece of the blade is generally considered
as a wing moving on a spiral path. The angle of each blade ele-
ment may always be measured from its direction of zero lift.
There are two important angles: The angle ϵ of inclination to

the propeller plane and the angle β of attack relative to the
air. The air force of the blade element has one component in the
direction of the relative motion and another one at right angle
to it. The former is due to the viscosity of the air and will be
considered later. It is small in general and does not materially
alter the character of the air flow, but it can by no means be
neglected. It may be called drag, and the other force may be call-
ed lift. By dividing the lift by the area of the blade element
and by the dynamical pressure of the relative velocity between
blade element and air, a lift coefficient C_L may be defined.
The lift coefficient thus obtained obeys the laws established in
the ordinary wing theory for infinite aspect ratio. In particu-
lar, it is never very different from $2 \pi \beta$.

The lift of the blade element, the relative velocity between
it and the air, and the absolute velocity of the blade elements
may be divided into components parallel and at right angles to
the propeller axis. The components of the relative velocity may
be written u and v, and those of the absolute velocity, U and V,
where v and V are parallel to the axis. It may be assumed that
V at the same time is the velocity of flight. The two components
of the lift may be called thrust T and tangential force P.
Let δ finally be the angle between the direction of the lift and
the axis, so that

$$T = L \cos \delta, \qquad P = L \sin \delta, \qquad v/u = \tan \delta .$$

The angle of inclination of the blade element to the plane of the

propeller is then

$$\epsilon = \delta + \beta.$$

The problem is to deduce the values of U and V if δ, C_L, u, and v are given. This is done by the application of the momentum theorem. The mass of air to which momentum is imparted is the mass of air passing the ring of radius r and breadth dr with the velocity v. Hence it is per unit of time

$$2 \pi r \, dr \, v.\rho.$$

Momentum is imparted by both the thrust and the tangential force, and it has to be assumed that half the total gain in velocity has been made when the air passes the propeller. Therefore, half this gain in velocity is

$$v \frac{i \, t}{2 \pi r} \frac{C_L}{4} . \frac{1}{\sin \delta} \qquad \text{in the tangential direction, and}$$

$$v \frac{i \, t}{2 \pi r} \frac{C_L}{4} . \frac{\cos \delta}{\sin^2 \delta} \qquad \text{in the axial direction, where i denotes the number of blades and t the breadth.}$$

Hence, writing m for

$$\frac{i \, t}{2 \pi r} \frac{C_L}{4} .$$

(1) $V = v(1 - m \cos \delta / \sin^2 \delta)$.

(2) $U = u(1 + m / \sin \delta)$.

From which it follows that

(3) $\dfrac{V}{U} = \tan \delta \, \dfrac{1 - m \cos \delta / \sin^2 \delta}{1 + m / \sin \delta}$.

The propeller coefficients, defined by

$$C_T = \frac{T}{2 \pi r \, dr \, V^2 \frac{\rho}{2}} \quad \text{and} \quad C'_Q = \frac{P}{2 \pi r \, dr \, V^2 \frac{\rho}{2}}$$

therefore is

$$(4) \qquad C_T = \frac{4 \, m \cos \delta}{(1 - m \cos \delta / \sin^2 \delta)^2} \; ; \quad C'_Q = C_T \tan \delta.$$

There remains still the drag to be considered. Its influ-
ence on the thrust and on the acceleration can be neglected.
The coefficient C'_Q is increased by

$$C''_Q = \frac{4 \, m \cos C_D/C_L}{(1 - m \cos \delta / \sin^2 \delta)^2}$$

as follows from the consideration that the tangential component
of the drag is $D \cos \delta$ and by applying the same conclusions as
before. Hence the complete coefficient of the tangential force
is

$$(5) \qquad C_Q = \frac{4 \, m \cos \delta}{(1 - m \cos \delta / \sin^2 \delta)} \; (\tan \delta \; + \; C_D/C_L).$$

The equations thus obtained are sufficient for the design
and the analysis of the blade element, but they are not yet in
such a form that they can be used conveniently. For this purpose
the equations have to be transformed. For purposes of design,
generally C_T, C_L and V/U are given, and also the number of
blades. From C_L, β and C_D can be derived directly from the
knowledge of the wing section and are to be considered as known

too. The three unknown quantities to be determined are the breadth of the blade element, t, its angle of inclination, ϵ, and C_Q, the coefficient of the tangential force. These can easily be found if the two quantities δ and m are known. They are to be determined first. From equation (3):

$$(6) \qquad m = \sin\delta\, \frac{\tan\delta - V/U}{1 + V/U} .$$

This may be substituted in equation (4)

$$C_T = \frac{4 \sin\delta \cos\delta\, \dfrac{\tan\delta - V/U}{1 + V/U}}{\left(1 - \dfrac{\tan\delta - V/U}{\tan\delta\,(1 + V/U)}\right)^2}$$

or, otherwise written

$$(7) \quad \tan\delta = V/U + \tan\delta\,(1 + V/U)\left\{1 + \frac{2\sin^2\delta}{C_T} - \sqrt{\left(1 + \frac{2\sin^2\delta}{C_T}\right)^2 - 1}\right\}.$$

The negative value of the square root has to be taken, because otherwise, if C_T is small, the second term on the right hand side becomes very great.

This equation now can be used for the determination of $\tan\delta$. It is true that the right hand side contains the unknown quantity too, but it contains it in such a way that its value is not greatly affected by a small error in $\tan\delta$. . The procedure is therefore to estimate $\tan\delta$ at first. It is slightly greater than V/U, say 10% for high velocities of flight and 20% for low speeds. Hence 1,1 to 1,2 V/U is to be substituted for $\tan\delta$

in the right hand side of the equation, thus giving a second improved value for tan δ.   Often this improved value will be close enough to the first estimation and to the final result. If not, this improved second approximation may be substituted in the right hand side of equation (7) giving a third improved approximation, and so on.

After the determination of δ, m is found by substituting δ in equation (6). Finally,

$$(8) \qquad t = \frac{4m}{C_L} \frac{2 \pi r}{i}$$

ε = β + δ and C_Q can be found by substituting m and δ in the right side of equation (5).

Thus all quantities needed for purposes of design are determined.

The analysis of a given blade element, that is, the calculation of the air force if the width of the blade element, t, its inclination, ε, and its velocity, U/V, are given, is performed similarly. The same equations are to be applied of course. It may be assumed that U/V, ε, and $\frac{i \, t}{2 \pi r}$ are given. In addition, the relation between β and C_L must be known and may be assumed to be

$$C_L = 2 \pi \beta .$$

C_T and C_Q are to be determined. In order to be able to determine them, δ and m are calculated first.

The three equations for the determination of the unknown

quantities are now

$$m = \frac{i\,t}{2\,\pi\,r}\,\frac{C_L}{4}$$

$$\delta = \epsilon - \frac{C_L}{2\pi}$$

$$V/U = \tan\delta\,\frac{1 - m\cos\delta/\sin^2\delta}{1 + m/\sin\delta}$$

with the three unknown quantities m, δ, and C_L.
The first equation can be written

$$C_L = 4\,m\,\frac{2\,\pi\,r}{i\,t}$$

and this can be substituted in the second equation, giving

$$\delta = \epsilon - \frac{2m}{\pi}\,\frac{2\,\pi\,r}{i\,t}.$$

Apply the function tan on both sides, taking into consideration
that the last term is small

$$\tan\delta = \frac{\tan\epsilon - \frac{2m}{\pi}\,\frac{2\,\pi\,r}{i\,t}}{1 + \frac{2m}{\pi}\,\frac{2\,\pi\,r}{i\,t}\,\tan\epsilon}$$

or transformed

$$m = \frac{\tan\epsilon - \tan\delta}{\frac{2}{\pi}\,\frac{2\,\pi\,r}{i\,t}\,(1 + \tan\epsilon\,\tan\delta)}$$

Now the value of m, as derived from the third equation, is given
by equation (6). The two expressions have to be equal and hence
they give an equation which contains only the one unknown quantity
$\tan\delta$. The resulting equation can be written

(9) $\tan \delta = \dfrac{\tan \epsilon \ (1 + V/U) \ \sin \delta + V/U \ \frac{4r}{i\,t} \ (\tan \epsilon \ \tan \delta \ + 1)}{(1 + V/U) \ \sin \delta + \qquad \frac{4r}{i\,t} \ (\tan \epsilon \ \tan \delta \ + 1)}$.

This equation can be used for the determination of $\tan \delta$ in the same way as was equation (6). Tan δ has to be assumed to have a value between $\tan \epsilon$ and V/U.

If $\tan \delta$ is found, C_L is quickly calculated by means of

(10) $C_L = 2 \pi \ (\epsilon - \delta)$

and

(11) $m = \dfrac{i\ t}{2\ \pi\ r} \ \dfrac{C_L}{4} = \dfrac{i\ t}{2\ \pi\ r} \ \dfrac{(\epsilon - \delta)\ \pi}{2}$.

and thus all quantities occurring in equations (4) and (5) are known and, being substituted, will give the values of C_T and C_Q.

I wish to add some remarks referring to the actual solution of equations (7) and (9). The determination of $\tan \delta$ can be made in less than one minute if the following rules are observed.

In both equations occur $\tan \delta$ and $\sin \delta$ and it would take too much time to look for $\sin \delta$ in a table after $\tan \delta$ is given or assumed. The sine can be determined from the tangent much more quickly by means of the slide rule. For

$$\sin \delta = \frac{\tan \delta}{\sqrt{1 + \tan^2 \delta}} \ ,$$

This can be solved by one move of the runner and the rule. The reader is requested to take his slide rule and to compute the following example in order to convince himself how much more

quickly he can perform the calculation than I can explain it. (See figure).

Suppose, for example, that tan δ = .40. Place the runner on the graduation .40 of the lower scale D. Then the runner indicates .16 on the upper scale A. Add 1 to this: 1 + .16 = 1.16 and move the rule so that its upper scale B shows this 1.16 under the mark of the runner. Then the left end of the runner gives sin δ = .372 on the lower scale D and $sin^2 δ$ = .138 on the upper scale A.

In the solution of equation (7), the following expressions are to be computed, in the order given, by means of the slide rule: Given V/U, C_T, and tan δ, approximately

$$(a) = \frac{2 \sin^2 \delta}{C_T}$$

$$(b) = 1 + (a)$$
$$(c) = \sqrt{(b)^2 - 1}$$
$$(d) = (b) - (c)$$

$$(e) = \tan \delta \ (1 + V/U) \cdot (d)$$

$$tg \, \delta = V/U + (e)$$

If (b) is considerably greater than 1, (d) has the value $\frac{1}{2 (b)}$.

For the solution of equation (9): Given V/U, tan ε, $\frac{4 \, r}{i \, t}$ and tan δ approximately (it lying between V/U and tan ε).

(a) $= 1 + \tan \epsilon \ \tan \delta$

(b) $= (1 + V/U) \ \sin \delta$ (c) $= \tan \epsilon \ (b)$

(d) $= \dfrac{4}{i} \dfrac{r}{t} \ (a)$ (e) $= V/U \ (d)$

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(f) $= (b) + (d)$ (g) $= (c) + (e)$

$$\tan \delta \ = \ \dfrac{(g)}{(f)} \ .$$