REPORT No. 19
IN TWO PARTS

PERIODIC STRESSES IN GYROSCOPIC BODIES, WITH APPLICATIONS TO AIR SCREWS*

By A. F. ZAHM

Part I.—THE GYROSCOPIC PARTICLE
Part II.—THE GYROSCOPIC THREE-DIMENSIONAL BODY

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PART 1.

THE GYROSCOPIC PARTICLE.

By A. F. Zahm.

INTRODUCTION.

The usual gyroscopic formulæ apply only to a body having kinetic symmetry about its axis of rotation, and hence offering a steady resistance to steady precession or nutation. To treat the case of unsteady gyroscopic resistance, we may find in turn the periodic stress in any precessing particle, or regular plane group of particles; then in any three-dimensional rigid body, whether or not possessing kinetic symmetry.

Particle in a rigid precessing body.—To find the rhythmic stresses in a particle of a rigid gyroscope, first assume this steadily rotating and precessing without translation, and let ω, Ω, be the angular speeds of rotation and precession. Also assume the centroid at the origin of x, y, z, as in fig. 1; and let the reference axes X, Y, Z, be respectively the rotation axis, the nutation axis, the precession axis; and call by like names the reference planes normal to these axes. Then any particle distant y from the nutation plane has, parallel to the rotation axis, the linear speed $-y\Omega = -r\Omega \cos a$, and the linear acceleration $r\omega\Omega \sin a = z\omega\Omega$, r being the distance of the particle from the rotation axis, and z its linear, a its angular distance from the precession plane.[1]

About the axes of precession and nutation, therefore, the moments of a particle of mass m are $-myz\omega\Omega$, $mz^2\omega\Omega$, and have the resultant $mr^2\omega\Omega \sin a$ about an axis perpendicular to r and the axis of rotation.

For a group of three or four particles symmetrically spaced about the axis of rotation, the resultant gyroscopic moment is easily seen, from this expression, to be constant. In general, the gyroscopic torque is constant for any particle group having kinetic symmetry about the rotation axis, or whose fundamental[2] ellipsoid is a surface of revolution about that axis. For such symmetry $\Sigma myz =$ zero, and the constant value of the torque is $\Sigma mz^2\omega\Omega = I\omega\Omega$, where $I = \Sigma mz^2$.

If, now, motion of translation be added to the above specified conditions it will not alter the values found for the gyroscopic moments, as may be inferred from the principle of the independence of

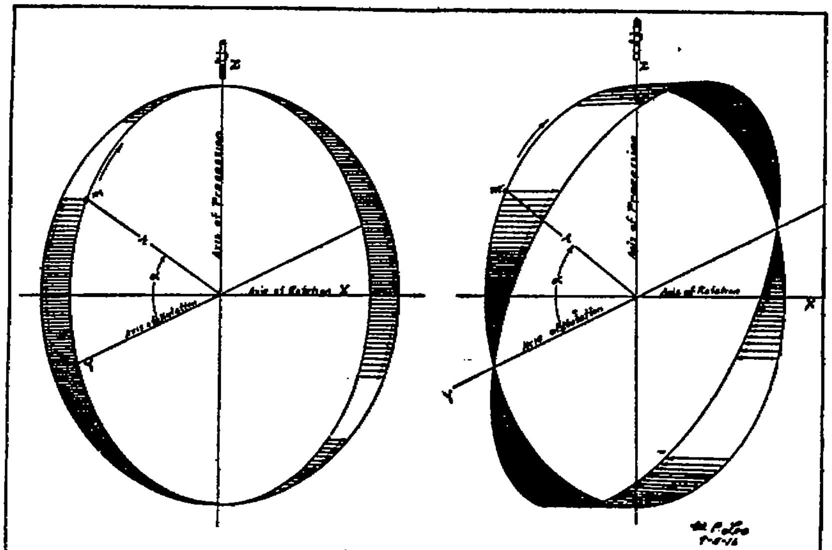
[1] These are familiar equations in elementary mechanics.

[2] The fundamental ellipsoid is the polar reciprocal of the momental ellipsoid referred to the center of mass and is a kind of space picture of the moment of inertia. In fact, the radius of inertia for any line through the center of the fundamental ellipsoid is the segment cut off this line by the perpendicular tangent plane.

the motions of translation and rotation. Also, as is well known, neither linear nor angular acceleration produces any gyroscopic effect.

If nutation, as well as precession, be assumed, the ensuing gyroscopic effects can be inferred from analogy to the case already treated.

Illustrations.—Figure 1 shows graphically for one revolution the above values of the linear velocity and acceleration of a particle in a gyroscope. The graphs, drawn upon a cylinder, are both sine curves but with a phase difference of one quadrant; that is, the acceleration of the particle is greatest when its speed along the cylinder is least, and vice versa. Hence, referring to the X direction only, it appears that every particle of a precessing gyroscope performs simple harmonic motion across its instantaneous plane of rotation.



Perspective Representation of Gyroscopic Velocity of a Particle m

Perspective Representation of Gyroscopic Acceleration of a Particle m.

Fig. 1

Figure 2 shows graphically for one revolution the foregoing value of the total gyroscopic moment of a particle, as also its rectangular components.

Figure 3 shows graphically, for several groups of particles symmetrically placed about the axis of rotation, both the component moments of each particle about the axes of precession and nutation, and the added particle moments for each group. The curves illustrate, what was seen analytically, that the component gyroscopic moments of each individual particle of the group are represented by sine square curves for the nutation axis, sine-cosine curves for the precession axis, whereas the summation of the moments about these axes, of all the particles, is zero or constant for each group of particles except the binary one, for which the summation is variable about both axes. The two-particle group, or a uniform material line joining the particles shown, has the resultant moment $\Sigma mr^2\omega\Omega \sin a$, whose extreme values are zero and $I\omega\Omega$, I being the moment of inertia of the material aggregate about its center, i. e., $I=\Sigma mr^2$.

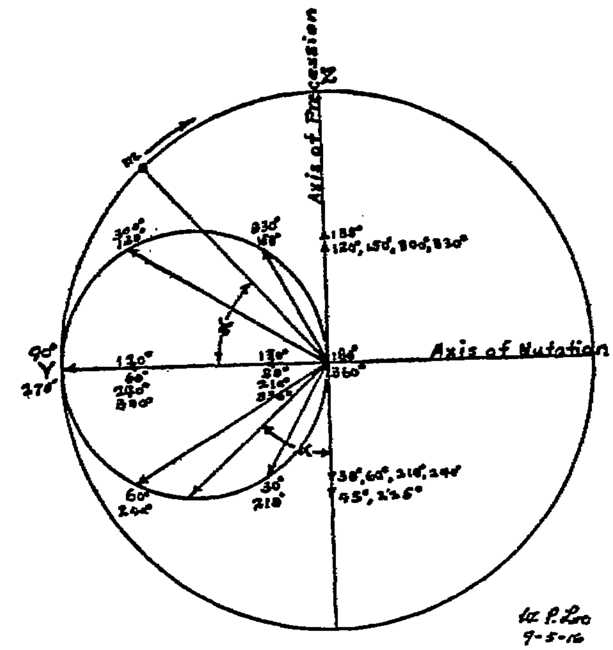


Fig 2.

Vector Diagram of Gyroscopic Torques.

Axis of rotation perpendicular to plane of paper with positive direction upward.

Component Torques Represented by Arrows on the Axes

Resultant " " " the Oblique Arrows.

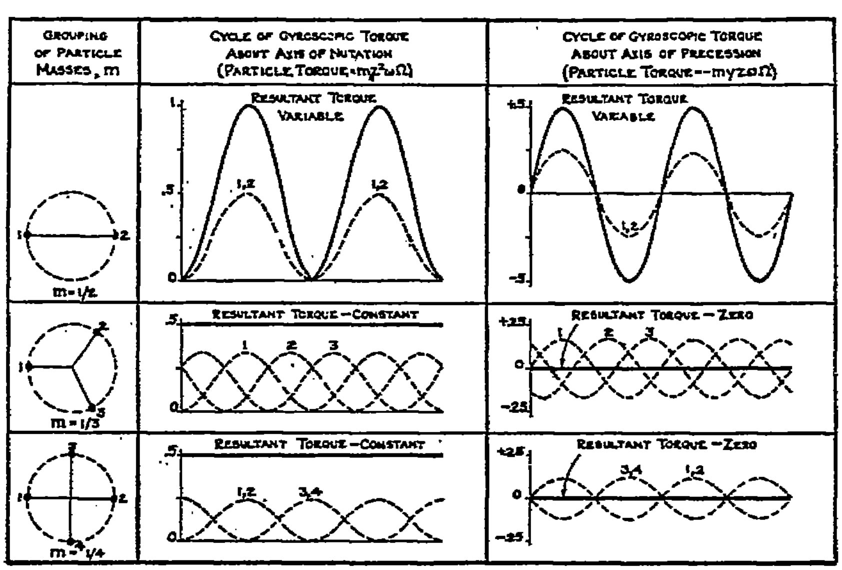


FIG. 3 —GYROSCOPIC TORQUE FOR SYMMETRIC PARTICLE GROUPS OF EQUAL TOTAL MASS

Application to propellers.—From the foregoing analysis it appears, using the same notation, that rigid and steadily precessing one-dimensional[1] propellers, and approximately also propellers with straight narrow blades, are subject to the following laws:

1. Every blade is urged to simple harmonic motion, of the same period as that of the shaft, across the instantaneous plane of rotation, and hence sustains a reversal of gyroscopic stress twice each revolution.

2. Every blade sustains a fluctuating gyroscopic moment whose magnitude at the hub is $\Sigma mr^2\omega\Omega \sin a$, which at quadrant intervals in each revolution has the successive values $I\omega\Omega$, zero, $I\omega\Omega$, zero.

3. The aggregate gyroscopic moment transmitted to the shaft by a two-blade propeller is variable and at all instants equal to twice that of one blade.

4. The aggregate gyroscopic moment transmitted to the shaft by a steady running multiblade propeller is constant and at every instant equals the geometric sum of the varying moments of the individual blades. For example, calling the maximum gyroscopic torque of one blade of a propeller unity, the constant torque of a three-blade screw is 1.5; of a four-blade screw 2.0; of an n-blade screw $n/2$.

Particle in an elastic precessing body.—All ordinary gyroscopes are practically rigid and in their ultimate parts are subject to the gyroscopic effects heretofore delineated. But propellers, more especially nonmetal ones, possess considerable flexibility. Their blades consequently yield to the gyroscopic force so as to shift the cardinal points of acceleration and velocity shown in figure 1. Also the vibrations of flexible blades are cumulative under the pulsating stresses, until the damping factors—air pressure and internal viscosity—of the blade limit its rhythmic excursions. The damping due to internal viscosity is sometimes great enough to raise the temperature of the propeller considerably, especially at or near the hub. No method of analysis is available to give an accurate estimate of the straining effects in elastic blades. But it is well enough known how fatigue induced by rapidly fluctuating and especially rapidly alternating[2] stresses shortens the life of the material.

It can be shown by elementary mechanics that the period of vibration of each particle of a rotating propeller blade, due to centrifugal force alone, is equal to the period of rotation, whatever the radial distance.[3] This property favors cumulative vibrations when the disturbing forces have the same period as the propeller. The gyroscopic force in a blade has been shown to have such a period. The varying air pressure on the blade has also that period in many instances; for example, when the air flow toward the screw is oblique to the axis, or when the air speed of approach is greater at one part of a blade revolution than another. For this reason propeller blades are sometimes designed to have under fiber stress alone a free vibrational frequency about 50 per cent greater than the frequency of rotation.

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[1] A one-dimensional propeller may be defined as a propeller composed of infinitely narrow blades symmetrically radiating from a point on the axis of rotation. The blades will here be assumed straight.

[2] Since the gyroscopic stresses alternate, the blade stresses also may alternate when the air force slackens as at low throttle.

[3] Since the radial acceleration of any particle is $r\omega^2$, the consequent frequency of vibration is—

$$N = \frac{1}{2\pi}\sqrt{\frac{r\omega^2}{r}} = \frac{\omega}{2\pi}$$

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PART 2.

THE GYROSCOPIC THREE-DIMENSIONAL BODY.

By A. F. Zahm.

Theory of the equivalent mass.—From the gyroscopic theory of a particle follows that of a body. In analyzing the motion of any given rigid body it is always possible, and sometimes more convenient, to employ in place of the actual body a kinetic equivalent; that is, a mass distribution which would, under the given system of forces, have the same translation and rotation as the actual body. For example, suppose the given body to be specified by mass M and principal moments of inertia A, B, C, at the centroid and choose for equivalent body the point-mass distribution shown in figure 4, and defined by the following equations in which a, b, c, are distances from the origin of the equal point-masses $M/6$.

$$\left.\begin{aligned} b^2+c^2&=3A/M \\ a^2+c^2&=3B/M \\ a^2+b^2&=3C/M \end{aligned}\right\} \text{-------------------- (1)}$$

Then, since the right members are given, the massless arms a, b, c, and hence the required space distributions are fully determined, providing the arms be real.

To show that a, b, c, are always real, note from equation (1) that $a^2=\frac{3}{2M}(C-A+B)$, so that a can not be imaginary unless A be greater than $C+B$. Now if m be any particle at x, y, z, of any rigid body.

$$\left.\begin{aligned} A&=\Sigma m\,(y^2+z^2) \\ B&=\Sigma m\,(x^2+z^2) \\ C&=\Sigma m\,(x^2+y^2) \end{aligned}\right\} \text{-------------------- (2)}$$

Hence, $a^2=\frac{3}{2M}(C-A+B)=\frac{3}{M}\Sigma mx^2$, which is always positive, i. e., a is always real. Similarly b and c are always real. Writing $\Sigma mx^2=Mx_1^2$ gives $a=\pm\sqrt{3}x_1$; similarly $b=\pm\sqrt{3}y_1$; $c=\pm\sqrt{3}z_1$, where x_1, y_1, z_1 are the radii of inertia referred to the principal planes.

In the most general case of rotation about three axes, each particle of the six-point equivalent mass exerts a gyroscopic torque whose magnitude and direction may be found by the method employed for a single gyroscopic particle. The component torques so found can be compounded in the usual way to obtain the resultant torque.

A six-point equivalent mass has been used here as a convenience; other distributions comprising fewer or more particles may obviously be employed.

Examples.—As illustration, suppose the model to represent a gyroscope rotating steadily about two axes only, say X and Z, its centroid being either stationary or in motion. Then the masses at a, a, taken together obviously exert no resultant moment, for they have no gyroscopic acceleration, and any acceleration they may have from their translatory motion is, in magnitude and direction, the same for both. From this it follows that such a rigid body has the same gyroscopic torque whether its mass be all in the plane of rotation $Y\ Z$ or not—that is, whether a be zero or of any finite magnitude. Hence the gyroscopic torques about Y and Z, of any rigid solid rotating about X and Z may be derived from the treatment of a point mass distribution in the $Y\ Z$ plane, say, four equal particles spaced as shown in figure 1, at b, b, c, c.

It has been proved above that a single particle of a gyroscope in steady rotation and precession has about the nutation axis a torque proportional to the sine square of its angular distance from that axis. Four equal particles symmetrically placed about the rotation axis have, therefore, about the nutation axis a resultant torque which is constant and equal to twice the maximum torque of one particle. Hence any mass having kinetic symmetry about the rotation axis, since it is the gyroscopic equivalent of a four-particle mass, has a constant torque equal to twice the maximum of one such particle. This generalization can, of course, be derived algebraically from the above-mentioned sine square law.

Summary.—The foregoing treatment of the mass equivalent of a rigid body may be summarized as follows:

Every rigid body has a mass equivalent whose motion under given forces is the same as that of the body itself. In particular, any rigid body has as mass equivalent six equal particles suitably placed on its principal centroidal axes and invariably connected by massless bonds.

A six-particle mass equivalent reduces to four particles for a plane distribution; two for a rectilinear. The arms of the six-particle mass equivalent of any rigid body equal, respectively, its radii of inertia referred to the reference planes, multiplied by $\sqrt{3}$.

The gyroscopic torque about its centroid of any rigid body is unaffected by its linear or angular acceleration or by the linear speed of its centroid.

At any instant the gyroscopic torque of a rigid body is the resultant of the torques of its equivalent mass particles.[1]

Application to an air screw.—Figure 4 shows the equivalent mass, derived from experimental data, for a standard Curtiss two-blade propeller, whose blades are notably deep and broad. From this figure we can judge the comparative gyroscopic value of the distribution of the propeller mass in each of the three axial directions, since this value varies as the square of the arm lengths, a, b, c, and as the products in pairs of the angular velocities, ω_x, ω_y, ω_z, about those arms. In fact the ratios of the three maximum gyroscopic torques of the pairs of point masses are as $c^2\omega_x\omega_z$: $b^2\omega_x\omega_y$: $a^2\omega_y\omega_z$. In practice the angular velocities may have the values 150, 0.5, 0.5 radians per

[1] As is well known, the gyroscopic torque of any particle equals its angular momentum times its deviation, or the rate of angular change of its plane of rotation.

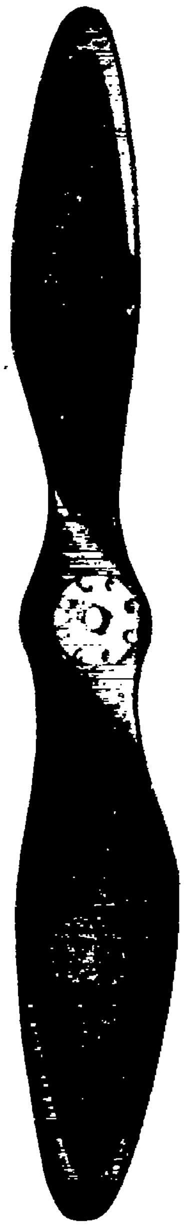
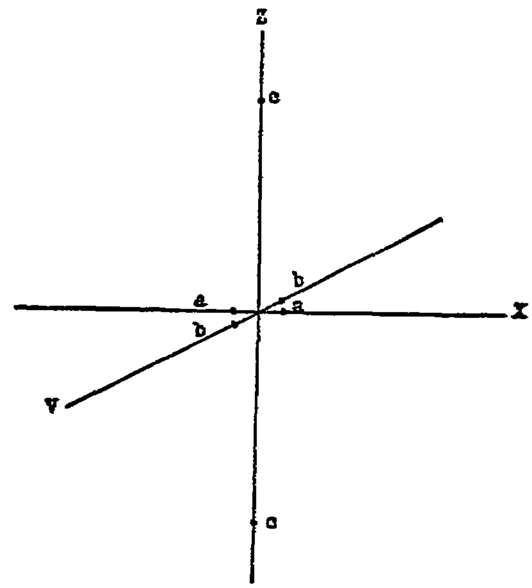


Fig. 5.

second, thus making said torque ratios $c^2\omega_x\omega_z/b^2\omega_x\omega_y=65$, and $c^2\omega_x\omega_z/a^2\omega_y\omega_z=23100$. The resultant of the three component torques is at most, therefore, only $\sqrt{1+(1/65)^2+(1/23100)^2}$ times the major one—that is, about one-eightieth of 1 per cent greater. The gyroscopic value of such a propeller, so running, may therefore with great accuracy be equated to that of a pair of simple particles.



Conclusion.—From the foregoing treatment it follows that all modern air screws obey the laws found for plane groups of particles. In particular the two-bladers exert on the shaft a rhythmic gyroscopic torque; the multibladers a steady one; both easily calculable for any given conditions of motion and mass distribution.