Attitude Error Estimator
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Abstract

This paper describes a geometrical method to predict the attitude error, that is, the *distance* between the estimated and the unknown true attitude matrices. The method, which is based on vector observations and on the sensors’ noise knowledge only, is developed according to the attitude error, whose definition and statistical parameters, are here given. The theory of the Attitude Error Estimator is developed for \( n \geq 2 \) observations. Finally, numerical tests to compare the proposed method with the standard trace of covariance matrix, are provided. Tests show that the Attitude Error Estimator describes the reliability of an attitude estimation faster and more accurately.

Introduction

Several different approaches exists to evaluate spacecraft attitude based on vector observations. During the last two decades, the improvement in this field have been achieved in term of developing faster algorithms satisfying the Wahba optimality definition. Once the attitude is evaluated, the problem to establish how much reliable the estimation is, still remains. The reliability is worldwide accepted to be represented by the trace of the covariance matrix, which is usually time consuming. Therefore, the attitude reliability evaluation usually slows down the attitude estimators. In this paper a geometrical method, which is based on the attitude error definition, is proposed as an alternative method to quantify the attitude reliability.

The idea of an “Attitude Error Estimator” (AEE) was firstly presented in a previous work (Ref. [1]), and then applied in Ref. [3] for multiple fields of view star trackers. However, the solution presented was found under linearization, and no comparisons were performed versus the trace of the covariance matrix, tr(\( P \)). This paper, presents the complete, non approximated AEE solution, and compares the results by means of numerical tests which validates the AEE as a better and faster tool to evaluate the attitude reliability with respect to the tr(\( P \)).

However, prior to enter in the mathematical details of the algorithm, the attitude error (how far an estimated attitude matrix is with respect to the unknown true attitude matrix), should be clearly defined.

The Attitude Error

The position error is easily described by the distance between two vectors, which identify, for instance, the estimated and the true positions. Unfortunately, the error
on rotation (the attitude is a rotation) cannot be so easily represented. The problem consists to define the distance between two orientations, that is, between two different attitude matrices. To this end, let \( T \) and \( A \) be the true and the estimated attitudes, respectively. Matrix \( A \) can even be randomly chosen, provided that the conditions \( A^\top A = A A^\top = I \), and \( \det(A) = +1 \), which assure \( A \) to be a rotation matrix, are satisfied. Let \( C = TA^\top \). Since \( C \) is a product of rotation matrices, \( C \) is a rotation matrix too. This matrix represents the corrective rotation that, applied to our estimated attitude, moves the estimation \( A \) to the true attitude \( T \). In fact,

\[
CA = TA^\top A = T
\]

Since \( C \) is a rotation matrix, it has its own principal axis \( e \) and principal angle \( \Phi \). Since \( e \) is the principal axis of the corrective matrix \( C \), its direction does not change during the correction and, therefore, its direction is error-less \((Ce = e)\) independently of the choice of the \( A \) matrix.

Let us consider the spherical triangle of vertex \( e \), \( b \) and \( Cb \), shown in Fig. 1, and where \( b \) is an arbitrary direction, displaced from \( e \) by the angle \( \dot{\vartheta} \). Due to the \( C \) rotation, the direction of \( b \) will be corrected to \( Cb \), which is displaced from \( e \) by the same angle \( \dot{\vartheta} \).

This implies that \( b \) is affected by the error \( \varepsilon \), where \( \cos \varepsilon = b^\top Cb \). Now, the sine law allows us to write \( \sin(\Phi/2) \sin \dot{\vartheta} = \sin(\varepsilon/2) \), that yields to the relationship

\[
\cos \varepsilon = (1 - \cos \Phi) \cos^2 \dot{\vartheta} + \cos \Phi
\]

where \( \cos \dot{\vartheta} = e^\top b = e^\top Cb = e^\top TA^\top b \), and \( \cos \varepsilon = b^\top Cb = b^\top TA^\top b \). Equation (2) provides the attitude error distribution (shown in Fig. 2), and establishes that it exists a maximum value for the error \( \varepsilon_{\text{max}} = \Phi \) associated with directions orthogonal to \( e \) (displaced from \( e \) by \( \dot{\vartheta} = \pi/2 \)), while \( \varepsilon = 0 \) for \( \Phi = 0 \), for any value of \( \dot{\vartheta} \) (all directions are error free). When \( \Phi \neq 0 \), then it results \( \varepsilon = 0 \) for all the directions aligned with \( e \), that is, for those having \( \dot{\vartheta} = 0 \) or \( \dot{\vartheta} = \pi \), regardless the value of \( A \)!

This implies that, for instance, having a spacecraft with orientation described by the matrix \( A \) (with \( A \) any rotation matrix), while \( T \) is an attitude which allows a correct pointing of an on-board telescope, then in the lucky case that the mounting angle of our telescope coincide with the principal axis of \( C \), then the telescope would point correctly!

Usually, the practical values of the maximum attitude error are so small that \( \cos \Phi \simeq 1 - \Phi^2/2 \) is a well accepted approximation. Since \( \varepsilon \leq \Phi \), then \( \cos \varepsilon \simeq 1 - \varepsilon^2/2 \) is certainly satisfied. By this substitution, Eq. (2) provides the following simple approximated expression of the error associated with a direction displaced from \( e \) by the angle \( \dot{\vartheta} \)

\[
\varepsilon(\dot{\vartheta}) \simeq \Phi \sin \dot{\vartheta} = \varepsilon_{\text{max}} \sin \dot{\vartheta}
\]
that holds for small $\Phi$. This relationship allows us to provide simple expressions for the statistical parameters of the direction error $\varepsilon$.

In fact, since the distribution of $\varepsilon$, provided by Eq. (2), is axially symmetric with respect to $e$ (see Fig. 2), then it does not depend on the azimuth $\varphi$. Therefore, the partial derivative with respect to $\varphi$ of the probability density function $P(\vartheta, \varphi)$, must be zero, that is, $\partial P(\vartheta, \varphi)/\partial \varphi = 0$, which implies $P(\vartheta, \varphi) = P(\vartheta)$. Now, any direction must have equal probability (otherwise $e$ would have preferential directions), then the infinitesimal probability $P(\vartheta) d\vartheta d\vartheta$ to have directions in the infinitesimal area $dS = (\sin \vartheta d\vartheta) d\vartheta$ must be proportional to $dS$. This implies $P(\vartheta) = k \sin \vartheta$, and the constraint $\int_0^\pi P(\vartheta) d\vartheta = 1$, provides $k = 1/2$. This allows us to evaluate the expected value

$$E\{\varepsilon\} = \bar{\varepsilon} = \int_0^\pi \varepsilon(\vartheta) P(\vartheta) d\vartheta \approx \int_0^\pi \Phi \sin \vartheta \frac{\sin \vartheta}{2} d\vartheta = \frac{\pi}{4} \Phi$$

(4)

the mean square error

$$E\{\varepsilon^2\} = \bar{\varepsilon^2} \approx \int_0^\pi \Phi^2 \sin^2 \vartheta \frac{\sin \vartheta}{2} d\vartheta = \frac{2}{3} \Phi^2$$

(5)

the variance

$$V\{\varepsilon\} = E\{(\varepsilon - \bar{\varepsilon})^2\} = \bar{\varepsilon^2} - (\bar{\varepsilon})^2 \approx \left[2/3 - (\pi/4)^2\right] \Phi^2$$

(6)

and the standard deviation

$$D\{\varepsilon\} = \sqrt{V\{\varepsilon\}} \approx \Phi \sqrt{2/3 - (\pi/4)^2}$$

(7)
These equations demonstrate that the statistical parameters of the error $\varepsilon$ (because $A$ differs from $T$), all depend on the maximum value $\varepsilon_{\text{max}} = \Phi$. Since $\varepsilon$ is constrained to an assigned shape by Eq. (2), what is characterizing it is its maximum value only (principal angle $\Phi$ of matrix $C$). The angle $\Phi$, or the expected value $E\{\varepsilon\} = \pi\Phi/4$, is the parameter identifying the distance between two orientations.

Summarizing: associated with the translation we have an isotrope distribution for the error (there is no preferential direction) which constitutes an open set (no upper limit for the error) while, associated with rotation the error distribution is anisotrope or polarized along one direction (error always zero along $e$ and maximum along orthogonal directions) which constitute a closed set (maximum error limited and always less or equal to $\pi$).

The Attitude Error Estimator

The AEE algorithm consists of estimating the attitude error directly from the knowledge of the observed directions, and the associated relative precision. The knowledge of the spacecraft attitude is not required and the attitude reliability can be quantified a priori, before the attitude estimation process.

To this end, let us consider (see Fig. 3) the case of $n = 2$ observations $b_1$ and $b_2$, which have a cone of uncertainty of $\beta_1$ and $\beta_2$, respectively. For a Gaussian error distribution we can consider $\beta_1 \simeq 3\sigma_1$ and $\beta_2 \simeq 3\sigma_2$, where $\sigma_1$ and $\sigma_2$ are the Gaussian standard deviations. Now, Ref. [2] has demonstrated that, for $n = 2$ observations Wahba’s optimality definition implies a co-planarity among the directions $b_1$, $b_2$, $Ar_1$, and $Ar_2$, where $r_1$ and $r_2$ are the reference directions associated with $b_1$ and $b_2$, and $A$ is the optimal attitude satisfying Wahba’s cost function. It is clear from Fig. 3 that the problem to find the attitude error can be seen as made of two components: 1) an in-plane error, and 2) an out-of-plane error. The in-plane error occurs when $T$ differs from $A$ in such a way that the plane identified by $b_1$ and $b_2$ coincides with the plane identified by $Tr_1$ and $Tr_2$, while the out-of-plane error increases as the inclination of these two planes increases. This means that the in-plane error cannot exceed the value of $\min(\beta_1, \beta_2)$, which is usually small, and which does not vary with the observation interstar angle $\vartheta_b$. In the contrary, the out-of-plane error is represented by the angle $\delta$, and this angle increases as the interstar angle $\vartheta_b$ decreases. In particular, when the uncertainty cones start to touch each other, then the angle $\delta$ is maximized. The angle $\delta$ is strictly related to the maximum attitude error $\Phi$. In particular, this angle represents the upper limit of all the maximum attitude errors associated to all the possible true attitude matrices $T$. Numerical tests have demonstrated that the relationship $\Phi = k\delta$, holds, where the value of $k < 1$ depends on the number of observation $n$.

Therefore, the maximum out-of-plane error ($\delta$) is achieved for $T$ such that $Tr_1$ and $Tr_2$ both lie on the cones of uncertainty with axes $b_1$ and $b_2$ and apertures $\beta_1$ and $\beta_2$, respectively. Reference [1] has provided the approximated expression $\sin^2 \delta \simeq (\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2 \cos \vartheta)/\sin^2 \vartheta$, which was based on some linearizations, and on the $\vartheta = \vartheta_r \simeq \vartheta_b$ approximation. In this paper, the complete (non approximated) analytical
solution for $\delta$, who works better than the approximated solution as the values of $\theta$ become critical, is given and compared versus $\text{tr}[P]$ to quantify the attitude estimation reliability.

Based on Fig. 3 (where $\cos \alpha = e^T Ar_1$, $\cos \theta_1 = b_1^T Ar_1$, and $\cos \theta_2 = b_2^T Ar_2$), the spherical triangles $[e, b_1, Tr_1]$ and $[e, b_2, Tr_2]$, allows us to write $[C(\bullet) = \cos(\bullet), \text{ and } S(\bullet) = \sin(\bullet)]$

$$\begin{align*}
C_\delta S_{\alpha} S_{(\theta_1 + \alpha)} + C_{\alpha} C_{(\theta_1 + \alpha)} &= C_{\beta_1} \\
C_\delta S_{(\theta_2 - \alpha)} S_{(\theta_2 + \alpha)} + C_{(\theta_2 - \alpha)} C_{(\theta_2 + \alpha)} &= C_{\beta_2}
\end{align*}$$

where $\theta_1 \simeq \beta_1 (\theta_b - \theta_r) / (\beta_1 + \beta_2)$, and $\theta_2 \simeq \beta_2 (\theta_b - \theta_r) / (\beta_1 + \beta_2)$ are demonstrated in Ref. [2], and where $\cos \theta_r = r_1^T r_2$, and $\cos \theta_b = b_1^T b_2$. Now, setting $\delta = \theta_r + \theta_b$, and rearranging Eq. (8), we obtain

$$\begin{align*}
\cos(2\alpha + \theta_1) &= \frac{2C_{\beta_1} - C_{\theta_1}(1 + C_\delta)}{(1 - C_\delta) S_\theta} \\
\sin(2\alpha + \theta_1) &= \frac{2C_{\beta_2} - C_{\theta_2}(1 + C_\delta) - C_{\theta_1}[2C_{\beta_1} - C_{\theta_1}(1 + C_\delta)]}{(1 - C_\delta) S_\theta}
\end{align*}$$

The condition $\cos^2(2\alpha + \theta_1) + \sin^2(2\alpha + \theta_1) = 1$, allows us to obtain a solving equation in term of $\cos \delta$

$$A \cos \delta - 2B \cos \delta + C = 0$$

where $A = S_\theta^2(C_{\theta_1}^2 - 1) + (C_{\theta_1} C_\theta - C_{\theta_2})^2$, $B = S_\theta^2(\xi_1 C_{\theta_1} - 1) + (\xi_1 C_\theta - \xi_2)(C_{\theta_1} C_\theta - C_{\theta_2})$, and $C = S_\theta^2(\xi_2^2 - 1) + (\xi_1 C_\theta - \xi_2)^2$, and where $\xi_1 = 2C_{\beta_1} - C_{\theta_1}$, and $\xi_2 = 2C_{\beta_2} - C_{\theta_2}$. The second order Eq. (9) provides the two solutions

$$\cos \delta = [B \pm \sqrt{B^2 - AC}] / A$$
These solutions are associated with the directions $Tr_1$ and $Tr_2$ placed on the same side or on the opposite side with respect to the $[b_1, b_2]$ plane, respectively. Thus, the searched solution is the greatest one. Now, at any possible true attitude matrix $T^{(\ell)}$, there is an associated maximum attitude error $\Phi^{(\ell)}$. Therefore, $\delta$ represents the maximum attitude error associated with the $T^{(\ell)}$ who maximizes $\Phi^{(\ell)}$. Hence, $\delta = \max[\Phi^{(\ell)}]$. Numerical tests have demonstrated that $\delta = h E \{\epsilon_{\text{max}}\}$ where $h = h(n)$ is a proportional constant.

As already described in Refs. [1, 3], when the observations are $n > 2$, then the searched solution $\delta$ is simply represented by $\delta = \min[\delta^{ij}(\beta_i, \beta_j)]$, where $\delta^{ij}(\beta_i, \beta_j)$ is associated with the $b_i$ and $b_j$ observations.

### Covariance Matrix

The covariance matrix $P$ is often derived from the Fisher information matrix and its inverse, the Cramer-Rao Lower Bound, which quantifies the error variance of the estimator. In particular, in the attitude estimation problem, let $B = \Sigma_i w_i b_i r_i^T$ be the attitude profile matrix, then $P_b = \frac{\kappa I + BB^T}{\lambda \kappa - \text{det}(B)}$, and $P_r = \frac{\kappa I + B^T B}{\lambda \kappa - \text{det}(B)}$ are the covariance matrices in the body and in the reference frames, respectively, and where $2\kappa = \lambda^2 - \text{tr}(BB^T)$. At $\lambda = \lambda_{\text{max}}$, we have $P = (\lambda_{\text{max}} I - A_{\text{opt}} B^T)^{-1}$. It is clear that $P$ provides more information than $E \{\epsilon_{\text{max}}\}$. However, in order to compare $\text{tr}[P]$ against $E \{\epsilon_{\text{max}}\}$, let us follow Ref. [4], which has shown that $A_{\text{opt}} B^T \simeq \Sigma_i w_i b_i b_i^T$, and which evaluates the covariance matrix as

$$P = (\lambda I - A_{\text{opt}} B^T)^{-1} \simeq [\Sigma_i w_i (I - b_i b_i^T)]^{-1} \quad (11)$$

### Numerical Tests

Some numerical tests have been performed to compare AEE against the $\text{tr}[P]$ provided by Eq. 11. The first plot of Fig. 4 shows the simulated errors as a function of the interstar angle $\vartheta_b$ for $n = 2$ observations together with the linear best fitting line and the value provided by AEE and $\text{tr}[P]$. In the second plot the ratio with respect the linear best fitting is given. From Fig. 4 it becomes evident that the AEE results better describe the slope of the line fitting the data. As for the speed tests (not included), AEE is much faster than $\text{tr}[P]$, especially when the approximated solution is used.

### Conclusions

This paper presents an alternative method to quantify the reliability of an attitude estimation. The proposed Attitude Error Estimator is derived from the geometrical nature of the attitude error, here shown, as well as from the spatial displacement of the observed vectors and their precision. Compared to the trace of the covariance matrix,
Figure 4: Numerical Test Results

worldwide adopted as the standard mathematical tool to quantify the attitude estimation reliability, the resulting algorithm presents two advantages: 1) the consumed time is dramatically shorter, and 2) the accuracy to describe the attitude reliability is better.

References

