An Analytical Solution for Relative Motion of Satellites

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Abstract

An analytical solution to the relative motion of satellites in eccentric orbits, under the effects of the nonlinearity of the gravitational field and the $J_2$-perturbation, is presented. The most notable feature of the solution is the use of the unit sphere for the description of the motion. Solutions to the relative motion coordinates and velocities are obtained by using a geometric approach, without making use of any small-angle approximations. The results obtained from a nonlinear simulation model and the analytical solutions are compared and shown to agree remarkably well with each other. Examples involving eccentric reference orbits and large relative distances are presented.

Introduction

Relative motion between satellites is a subject of interest to rendezvous and formation flying. There exists a rich body of literature on relative motion between two satellites [1]. The solutions available can be classified along the following lines:

1) The independent variable used: time, true anomaly, or eccentric anomaly.
2) Circular or eccentric orbit of the reference satellite.
3) Linearization or higher order expansion of the gravitational acceleration.
4) $J_2$-perturbation accounted for or not.

The starting point for most of the results using linearized gravitational acceleration models are the Clohessy-Wiltshire equations for circular reference orbits [2] and the Tschauner-Hempel equations for elliptic reference orbits [3]. Melton [4] presents a time-explicit solution for relative motion for elliptic orbits using a perturbation method. Vaddi et al. [5] also present a time-explicit perturbation solution to the relative motion in elliptic orbits by retaining quadratic nonlinearities of the gravitational field. Alternatively, Gim and Alfriend [6] and Garrison et al. [7] consider a geometric method for deriving the state transition matrix, utilizing small differences in orbital elements between two satellites. The results presented in [6] account for the $J_2$-perturbation. Time-explicit solutions have the advantage that Kepler’s equation need not be solved. However, geometric methods utilizing orbital elements are more accurate and a series solution for Kepler’s equation can be utilized for many practical applications.

This paper also uses the geometric method to obtain relative motion solutions for eccentric orbits under the influence of the $J_2$-perturbation. The solution, presented
in terms of differences in the orbital elements, is exact and valid for large relative
distances. However, eccentricity expansions and mean orbital elements are resorted
to, in order to express the solution in a time-explicit manner. The enabling feature
of the elegant solution is the use of the unit sphere to study the relative motion. The
actual relative motion is obtained by a scaling transformation. Several examples
are presented for eccentric orbits with and without the $J_2$-perturbation.

**Relative Motion on a Unit Sphere**

A satellite’s motion can be projected onto a unit sphere by normalizing its
Cartesian coordinates by its distance from the Earth. The point obtained thus is
called the subsatellite point. The relative motion between two subsatellite points
shown in Fig. 1, is the focus of this paper.

![Figure 1 - Projections of Two Satellites on the Unit Sphere](image)

Let $C_0$ and $C_1$, indicate the direction cosine matrices of the Local-Vertical-Local-
Horizontal (LVLH) frames of the two satellites with respect to the inertial frame.
Satellite # 0 will be designated as the reference satellite or Chief and satellite # 1
will be the Deputy. The relative motion can be expressed in the LVLH frame of the
Chief, as follows:

$$
\begin{bmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{bmatrix} = [C_0C_1^T - I] \begin{bmatrix} 1 \\
0 \\
0
\end{bmatrix}
$$

(1)

where $\Delta x$, $\Delta y$, and $\Delta z$, are respectively, the radial, along-track, and cross-track
relative positions on the unit sphere and $I$ stands for a 3X3 identity matrix. The
direction cosine matrices can be parameterized by a set of orbital elements. The
above equation can then be used to solve for the relative displacement variables
explicitly as given below:
\[ \Delta x = -1 + c^2(i_0/2)c^2(i_1/2)c(\Delta \theta + \Delta \Omega) + s^2(i_0/2)s^2(i_1/2)c(\Delta \theta - \Delta \Omega) + s^2(i_0/2)c^2(i_1/2)c(2\theta_0 + \Delta \theta + \Delta \Omega) + c^2(i_0/2)s^2(i_1/2)c(2\theta_0 + \Delta \theta - \Delta \Omega) + 1/2s(i_0)s(i_1)[c(\Delta \theta) - c(2\theta_0 + \Delta \theta)] \] 

(2)

\[ \Delta y = c^2(i_0/2)c^2(i_1/2)s(\Delta \theta + \Delta \Omega) + s^2(i_0/2)s^2(i_1/2)s(\Delta \theta - \Delta \Omega) - s^2(i_0/2)c^2(i_1/2)s(2\theta_0 + \Delta \theta + \Delta \Omega) - c^2(i_0/2)s^2(i_1/2)s(2\theta_0 + \Delta \theta - \Delta \Omega) + 1/2s(i_0)s(i_1)[s(\Delta \theta) + s(2\theta_0 + \Delta \theta)] \] 

(3)

\[ \Delta z = -s(i_0)s(\Delta \Omega)c(\theta_1) - [s(i_0)c(i_1)c(\Delta \Omega) - c(i_0)s(i_1)]s(\theta_1) \] 

(4)

where, \( s \) and \( c \), respectively, are the sine and cosine functions, \( i \) is the inclination and \( \Delta \Omega \) is the difference in the right ascensions of the two satellites. The argument of perigee of the Chief is \( \omega_0 \) and \( f_0 \) is its true anomaly. \( \theta_0 = \omega_0 + f_0 \), is the argument of latitude of the Chief. \( \Delta \theta \) is the difference between the arguments of latitude of the two satellites. The radial and along-track motions involve higher harmonics of the orbit rate of the Chief. The cross-track motion expression shows the presence of a periodic term involving the orbit rate of the Deputy. It is also clear that the fundamental frequencies of the in-plane and cross-track motions are not the same.

The above solutions are exact and valid for large angles. The effect of the \( J_2 \)-perturbation can be studied by using osculating orbital elements in the above equations. However, one can substitute mean orbital elements in order to obtain the averaged relative motion. The use of mean orbital elements simplifies the equations considerably. The mean orbital elements along with the eccentricity expansion and the well known secular drift rates for some of the elements \([8]\) are used to produce a time-explicit representation of the analytical solution. These equations are given below:

\[ \theta_j = \omega_j + M_j + 2e_j \sin(M_j) + 5/4e_j^2 \sin(2M_j) + \ldots \ , \ j = 0,1 \] 

(5)

\[ \Omega_j = \Omega_j(0) + \Omega_j \dot{t} \] 

(6)

\[ \omega_j = \omega_j(0) + \omega_j \dot{t} \] 

(7)

\[ M_j = M_j(0) + M_j \ddot{t} \] 

(8)

\[ \dot{\Omega}_j = -1.5J_2(R_e/p_j)^2 n_j \cos(i_j) \] 

(9)

\[ \dot{\omega}_j = 0.75J_2(R_e/p_j)^2 n_j \left(5\cos^2 i_j - 1\right) \] 

(10)
\[ \dot{M}_j = n_j \left[ 1 + 0.75J_2 \sqrt{1 - e_j^2} \left( R_e / p_j \right)^2 \left( 3 \cos^2 i_j - 1 \right) \right] \]  

(11)

\[ n_j = \sqrt{\frac{\mu}{a_j^3}} \]  

(12)

In the above equations, \( M_j, a_j, \) and \( e_j \) are respectively, the mean “mean” anomaly, mean semi-major axis, and mean eccentricity of the \( j^{th} \) satellite, \( R_e \) is the radius of the Earth, and \( p_j = a_j(1-e_j^2) \). The mean radius of the \( j^{th} \) satellite can be expressed as follows:

\[ r_j = a_j [1 - e_j \cos(M_j) - 1/2e_j^2(\cos(2M_j) - 1) + \ldots..] \quad j = 0,1 \]  

(13)

The eccentricity expansions in the above equations are available in Battin [8].

The actual relative motion between the two satellites can be obtained from their unit sphere counterparts by scaling, as shown below:

\[ \delta x = \eta_1 (1 + \Delta x) - \eta_0 \]  

(14)

\[ \delta y = \Delta y \eta_1 \]  

(15)

\[ \delta z = \Delta z \eta_1 \]  

(16)

where \( \delta x, \delta y, \) and \( \delta z \), are the mean radial, along-track, and cross-track relative positions, respectively.

**Results**

The first example presented considers a Chief in an orbit with mean eccentricity of 0.01, mean inclination of 70 deg, and mean semi-major axis of 6,900 km. The initial conditions of the Deputy are obtained by adding small perturbations to the mean elements of the Chief. The process for obtaining the mean element differences to shape the relative orbit is illustrated in [9]. The relative orbit selected for this example is the circular-projection orbit of radius 1 km. There is a small inclination difference between the satellites but the node difference is zero. Figures 2-4, show the differences between numerical and analytical solutions, along the three directions, for 40 orbits of the Chief. The errors are oscillatory and also contain small bias as well as secular components. The reasons for the presence of the errors can be attributed to the use of mean elements. In this example, the eccentricity expansions carried up to \( O(e^2) \). The maximum secular error in the along-track direction is very small, approximately 1 m in 40 orbits. Figure 5 shows the relative orbital motion obtained using the analytical solutions.
Figure 2 – Radial Error in km (Ex.-1)        Figure 5 – Relative Orbits (Example-1)

Figure 3 Along-Track Error in km (Ex-1) Figure 6 Along-Track Error in km (Ex 2)

Figure 4 – Cross-Track Error in km (Ex.-1) Figure 7 – Relative Orbits (Ex.-2)
Next, the same example as above is treated but the effect of $J_2$ is not included. This is accomplished by using the mean orbital elements as the osculating elements and neglecting the secular drift rates given in Eqs. (6-8). Figure 6 shows that the error in the along-track direction is of the order of 0.04 mm, with a secular growth of 0.005 mm in 40 orbits. The errors in the other directions are of the same order as that for the along-track. Figure 7 shows the relative motion between the two satellites for 40 orbits of the Chief. The effect of the $J_2$-perturbation is evident from a comparison of Figs. 5 and 7.

The third example considers a relative orbit of radius of approximately 10 km. The orbit of the Chief has mean eccentricity of 0.1, mean inclination of 45 deg, and mean semi-major axis of 7,600 km. Since the process of determining the mean element differences for the Deputy is based on small-eccentricity assumptions, the resulting orbit is not perfectly circular as can be seen from Fig. 8. This formation setup uses a node difference but no inclination difference. Eccentricity expansions for this example are carried up to $O(e^4)$. Figure 9 shows the error in the along-track direction, which grows at the rate of 12 m in 40 orbits.

![Figure 8](image1.png)  
**Figure 8** – Relative orbits with $J_2$ (Ex.-3)  
![Figure 9](image2.png)  
**Figure 9** – Along-track Error with $J_2$

Figures 10 and 11, respectively, show the relative orbits and the error in the along-track direction, when $J_2$-perturbation is neglected. These figures clearly show that the effect of eccentricity has been modelled accurately.

**Conclusions**

An analytical solution for the relative motion between satellites in elliptic orbits, perturbed by $J_2$ has been presented. Mean orbital elements and eccentricity expansions are utilised. The solution is highly accurate for the eccentricity problem and only slightly less so for the $J_2$ problem.
Figure 10 – Relative orbits without $J_2$ (Ex.-3)  Figure 11 – Along-track error

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References


