Abstract

The paper considers the dynamics of a tethered system in the vicinity of the Earth-Moon Lagrangian points. An analysis of the coupled motion, i.e., motion of the center of mass of the tethered system and tether libration, is carried out for the case when the system is near the translunar Lagrangian point. It is noted that this motion can be stabilized by an appropriate variation of the length of the tether. It is also observed that rotation of the tether can have a stabilizing effect.

Introduction

There have been a large number of investigations of the dynamics and control of tethered satellite systems. The systems studied were usually in a low-Earth orbit. In a small number of cases, tethered satellites orbiting another planet were also considered.

There is now some interest in placing a constellation of satellites in the Sun-Earth Lagrangian points and possibly in the Earth-Moon Lagrangian points. Using a set of spacecraft as opposed to one can establish a long-baseline imaging capability. In some cases the spacecraft may be connected by tethers to ensure better relative stationkeeping. This paper examines the dynamics of a tether-connected two-spacecraft system in the vicinity of the Earth-Moon Lagrangian points. Only planar motion, i.e., motion in the plane of the motion of the primary bodies, is analyzed in this paper.

In general, the dynamics of a tethered system can be described by the motion of its center of mass and rotation of the system about this center of mass (attitude dynamics or tether libration). There have been some studies of attitude dynamics of satellites placed at the Lagrangian points. Robinson analyzed the attitude dynamics of a dumb-bell satellite at a triangular equilibrium point of the circular three-body problem and derived the stability conditions [1]. In a later study, he considered a satellite of arbitrary shape located at either a collinear or a triangular equilibrium point and determined the regions of stability as governed by the inertia ratios [2]. In these two studies, the primary bodies were assumed to be in circular orbits. Recently, Ashenberg [3] considered the nonlinear pitch dynamics of a satellite in the elliptic problem of three bodies via Poincaré maps and bifurcation diagrams. He noted that the pitch is most stable and has the smallest chaotic region at $L_2$.

The librational dynamics of a tethered system is similar to the attitude dynamics of a dumb-bell satellite. However, Robinson [1] did not consider collinear points
and neither of his studies can provide some important information such as the tether librational frequencies. This paper attempts to obtain this information.

If a mass is displaced to the right from the unstable Lagrangian point $L_2$ (Fig. 1), it has a tendency to move further to the right. Similarly, if a mass is given a perturbation to the left, it moves further in that direction. If these two masses are joined together by a tether, the destabilizing effects oppose each other. In principle, it should be possible to stabilize a tethered system near the collinear Lagrangian points by changing the tether length in a judicious manner, or probably by appropriate rotation of the tether. The paper investigates this possibility.

Very recently, it was brought to the authors’ attention that the concept of stabilizing a spacecraft system near a collinear libration point by varying the tether length had been explored by Colombo as well as by Farquhar, and the results were included in two NASA reports [4, 5]. (The journal publications will appear in the near future). However, the approach followed in this paper is quite different, and probably more convenient.

**Description of the System**

Motion of two small masses $m_1$ and $m_2$ under the gravitational influence of the Earth and the Moon is considered (Fig. 1).

![Figure 1: Geometry of the system](image)

The two masses are connected by a tether of length $\ell$, which can be changed if necessary. It is assumed that the two masses can be treated as particles and that the tether is rigid, remains straight and has negligible mass. It is assumed that the Earth (of mass $M_1$) and the Moon (of mass $M_2$) move in circular
orbits about their common center of mass $O$. The fixed distance between the Earth and the Moon is denoted by $D$, while $D_1$ and $D_2$ denote the distance of the Earth and the Moon, respectively from $O$. Clearly,

$$D_1 = \mu D, \quad D_2 = (1 - \mu) D,$$

where $\mu = M_2/(M_1 + M_2)$.

The motion is described using the conventional rotating frame $X, Y, Z$ with its origin at $O$ and the $X$-axis in the direction from $M_1$ to $M_2$. The $Y$-axis is perpendicular to the $X$-axis in the plane of the motion of the primary bodies while the $Z$-axis is along the normal to this plane. The unit vectors along the $X, Y$ and $Z$-axes are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, respectively. The axes rotate at the constant rate of $n$ about the $Z$-axis, given by

$$n = \left[ G(M_1 + M_2)/D^3 \right]^{1/2},$$

where $G$ is the universal gravitational constant.

It is convenient to use an additional set of axes $x, y, z$ parallel to the axes $X, Y, Z$ but with the origin located at the Lagrangian point of interest. The two sets of co-ordinates are related by

$$X = X_0 + x, \quad Y = Y_0 + y \quad \text{and} \quad Z = z,$$

where $(X_0, Y_0, 0)$ are the co-ordinates of the Lagrangian point under consideration.

At any arbitrary instant, let $\theta$ be the inclination of the tether to the $X$-axis, while $\mathbf{t}$ is the unit vector along the tether in the direction from $m_1$ to $m_2$. The distances $d_1$ and $d_2$ of the masses $m_1$ and $m_2$ from the tethered system center of mass $C$ are given by

$$d_1 = \beta_2 \ell \quad \text{and} \quad d_2 = \beta_1 \ell,$$

where $\beta_1 = m_1/(m_1 + m_2)$ and $\beta_2 = m_2/(m_1 + m_2)$. Furthermore, the unit vector $\mathbf{t}$ is related to $\mathbf{i}$ and $\mathbf{j}$ by

$$\mathbf{t} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}.$$

The position vector of the center of mass $C$ relative to $O$ is denoted by $\mathbf{R}$, while relative to the two primary bodies it is denoted by $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. They can be written as

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} = (X_0 + x)\mathbf{i} + (Y_0 + y)\mathbf{j},$$

$$\mathbf{R}_1 = (X + D_1)\mathbf{i} + Y\mathbf{j} = (D_1 + X_0 + x)\mathbf{i} + (Y_0 + y)\mathbf{j},$$

$$\mathbf{R}_2 = (X - D_2)\mathbf{i} + Y\mathbf{j} = (-D_2 + X_0 + x)\mathbf{i} + (Y_0 + y)\mathbf{j}.$$
The equations of motion can be obtained in various ways. Colombo [4] used Lagrange’s equations for constrained systems to obtain six equations of motion for the generalized coordinates \(x_1, y_1, z_1, x_2, y_2, z_2\) for the three-dimensional case. Since these co-ordinates are not independent, a Lagrange multiplier \(\lambda\) appears in the equations of motion. The seven unknowns can be analyzed using the six equations of motion plus the constraint equation.

Farquhar [5] obtained the equations of motion applying Newton’s second law to the two masses. The six equations of motion and the length equation are sufficient to analyze the six co-ordinates and the tether tension. Terms up to the second order in gravitational forces were retained by Farquhar.

In this paper, a Lagrangian formulation is used with three generalized co-ordinates \(x, y\) and \(\theta\). As mentioned earlier, \(x\) and \(y\) are the co-ordinates of the center of mass of the tethered system in the rotating frame located at the Lagrangian point, while \(\theta\) defines the orientation of the tether. The number of equations to be analyzed is three in the present case as opposed to five in the planar version of the approach followed by Colombo or Farquhar.

**Energy Expressions**

In order to derive the equations of motion, the kinetic and potential energy expressions must be obtained. The absolute position vectors of the two masses are given by

\[
r_1 = \mathbf{R} - d_1 t, \quad r_2 = \mathbf{R} + d_2 t. \tag{9}
\]

The absolute velocity of the two masses are then

\[
v_1 = \dot{r}_1 = \dot{\mathbf{R}} + \omega_f \times (-d_1 t) - \dot{d}_1 t, \tag{10}
\]

\[
v_2 = \dot{r}_2 = \dot{\mathbf{R}} + \omega_f \times d_2 t + \dot{d}_2 t, \tag{11}
\]

where \(\omega_f\) is the angular velocity of the rotating frame attached to the tether and is equal to \((n + \dot{\theta})\mathbf{k}\); \(n, d_1\) and \(d_2\) are given by equations (2) and (4).

The kinetic energy of the system is given by

\[
T = \frac{1}{2} m_1 v_1 \cdot v_1 + \frac{1}{2} m_2 v_2 \cdot v_2. \tag{12}
\]

Using equations (3), (4), (6), (10) and (11), equation (12) can be rewritten as

\[
T = \frac{1}{2} (m_1 + m_2) \left[ x^2 + y^2 - 2nx(Y_0 + y) + 2ny(X_0 + x) 
+ n^2 \left\{ (X_0 + x)^2 + (Y_0 + y)^2 \right\} + \beta_1 \beta_2 \left\{ (n + \dot{\theta})^2 \ell^2 + \dot{\ell}^2 \right\} \right]. \tag{13}
\]

The potential energy is given by

\[
V = -\frac{GM_1 m_1}{|\mathbf{R}_1 - d_1 t|} - \frac{GM_2 m_1}{|\mathbf{R}_2 - d_1 t|} - \frac{GM_1 m_2}{|\mathbf{R}_1 + d_2 t|} - \frac{GM_2 m_2}{|\mathbf{R}_2 + d_2 t|}; \tag{14}
\]
where \( \mathbf{R}_1 \) and \( \mathbf{R}_2 \) denote the position vector of the centre of mass of the tethered system relative to the two primary bodies and are given by equations (7) and (8), respectively. It is assumed here that the length of the tether is small compared to the distance between the two primary bodies, which implies that \( d_1 \) and \( d_2 \) are small compared to \( |\mathbf{R}_1| \) and \( |\mathbf{R}_2| \). Expanding using the binomial theorem and retaining terms up to the third order,

\[
\frac{1}{|\mathbf{R}_1 - d_1 \mathbf{t}|} = \frac{1}{R_1} \left[ 1 + \left( \frac{d_1}{R_1} \right) \mathbf{u}_1 \cdot \mathbf{t} + \left( \frac{d_1}{R_1} \right)^2 \left\{ 3 (\mathbf{u}_1 \cdot \mathbf{t})^2 - 1 \right\} \right], \tag{15}
\]

where \( R_1 = |\mathbf{R}_1| \) and \( \mathbf{u}_1 \) is the unit vector along \( \mathbf{R}_1 \). Similar expansion can be carried out for the other three terms in equation (14) and after some algebraic manipulation the potential energy can be expressed as

\[
V = -(m_1 + m_2) \left( \frac{GM_1}{R_1} \right) \left\{ 1 + \frac{1}{2} \beta_1 \beta_2 \left( \frac{\ell}{R_1} \right)^2 [3(\mathbf{u}_1 \cdot \mathbf{t})^2 - 1] \right\} + \left( \frac{GM_2}{R_2} \right) \left\{ 1 + \frac{1}{2} \beta_1 \beta_2 \left( \frac{\ell}{R_2} \right)^2 [3(\mathbf{u}_2 \cdot \mathbf{t})^2 - 1] \right\}, \tag{16}
\]

\( R_1, \mathbf{u}_1, R_2 \) and \( \mathbf{u}_2 \) can be determined from equations (7)–(8), while \( \mathbf{t} \) is given by equation (5). These can be substituted in equation (16).

**Equations Governing General Planar Motion Near \( L_1 \) or \( L_2 \)**

All three degrees of freedom, \( x, y \) and \( \theta \), are considered now. It is assumed that \( x \) and \( y \) are small compared to \( D \). The potential energy expression given by equation (16) can now be re-written in terms of \( x, y \) and \( \theta \). Retaining terms up to the third order, one obtains for the collinear libration points

\[
V = \frac{G(M_1 + M_2)}{D} (m_1 + m_2) \left[ - \left\{ I + \frac{1}{2} \beta_1 \beta_2 \left( \frac{\ell}{D} \right)^2 B (3 \cos^2 \theta - 1) \right\} \times \frac{dx}{D} + \left\{ A + \frac{3}{2} \beta_1 \beta_2 \left( \frac{\ell}{D} \right)^2 C (3 \cos^2 \theta - 1) \right\} \frac{dy}{D} + \left\{ - \beta_1 \beta_2 \left( \frac{\ell}{D} \right)^2 C (3 \sin \theta \cos \theta) \right\} \frac{d\theta}{D} \right] \times \frac{\ell}{D} + \left\{ - B \right\} \left( \frac{\ell}{D} \right)^2 \left\{ \frac{1}{2} B \right\} \left( \frac{\ell}{D} \right)^2 \left\{ C \right\} \left( \frac{\ell}{D} \right)^3 + \left\{ - \frac{3}{2} C \right\} \left( \frac{\ell}{D} \right)^2 \left( \frac{\ell}{D} \right)^2 \right], \tag{17}
\]
where

\[ I = \frac{1-\mu}{\delta_1} + \frac{\mu}{\delta_2}, \]  
(18)

\[ A = \frac{1-\mu}{\delta_1^3} \pm \frac{\mu}{\delta_2^3}, \]  
(19)

\[ B = \frac{1-\mu}{\delta_1^3} + \frac{\mu}{\delta_2^3}, \]  
(20)

\[ C = \frac{1-\mu}{\delta_1^3} \pm \frac{\mu}{\delta_2^3}, \]  
(21)

while \( \delta_1 \) and \( \delta_2 \) are \( R_{1L} \) and \( R_{2L} \), respectively. When two signs appear in an expression, the upper sign is for \( L_2 \), while the lower one is for \( L_1 \).

Using the expressions for the kinetic and potential energy given by equations (13) and (17), one can obtain the equations of motion for \( x, y \) and \( \theta \). Since it is more convenient to analyze nondimensional equations, a set of nondimensional quantities are defined as follows:

\[ \hat{x} = \frac{x}{D}, \quad \hat{y} = \frac{y}{D}, \quad \hat{\ell} = \frac{\ell}{D} \quad \text{and} \quad \tau = nt. \]  
(22)

The nondimensional equations governing the dynamics of the tethered system in the vicinity of \( L_2 \) or \( L_1 \) are then obtained as

\[ \ddot{x}' - 2\dot{y}' - (2B + 1)\dot{x}' + 3C \left( x'^2 - \frac{1}{2} y'^2 \right) + \frac{3}{4} C \beta_1 \beta_2 \ell^2 (1 + 3 \cos 2\theta) = 0 \]  
(23)

\[ \ddot{y}' + 2\dot{x}' + (B - 1)\dot{y}' - 3C \dot{x}' \dot{y}' - \frac{3}{2} C \beta_1 \beta_2 \ell^2 \sin 2\theta = 0 \]  
(24)

\[ \theta'' + 2\left( \frac{\dot{\ell}}{\dot{\ell}} \right) (\theta' + 1) + \frac{3}{2} B \sin 2\theta - \frac{3}{2} C (3 \sin 2\theta \dot{x}' + 2 \cos 2\theta \dot{y}') = 0 \]  
(25)

The right hand side of the equations is nonzero if there are external forces such as these from thruster firings.

**Numerical Results**

Equations (23)–(25) are now solved numerically using MATLAB for a variety of cases. It may be pointed out here that the results for an isolated untethered mass can be obtained from the solution of the same equations if \( \ell \) is put equal to zero; equation (25) in that case has no meaning. Any results obtained by solving equations (23)–(25) are nondimensional; but they are converted into physical units for plotting. Furthermore, the displacements of the tethered end-masses are calculated from

\[ x_1 = x - \beta_2 \ell \cos \theta, \quad y_1 = y - \beta_2 \ell \sin \theta, \]  
(26)

\[ x_2 = x + \beta_1 \ell \cos \theta, \quad y_2 = y + \beta_1 \ell \sin \theta. \]  
(27)
Figure 2 compares the displacements of two equal masses when they are free (untethered) with those when they are tethered. The length of the tether is 100 km. The displacements are measured from $L_2$. In both cases, the masses move away from $L_2$, but when they are tethered the drift is much slower. Figure 2 shows simulation for only 100 hours. However, the displacements grow exponentially and become large after 200 hours or so.

$$\ell^2 = \hat{\ell}_0^2 + \frac{1}{3c_1^2} \left[ a(\hat{x} - \hat{x}_0) + b\hat{x}' + c\theta + dy' \right].$$

(28)

with gains $a = 8$, $b = 3$, $c = 0.0002$ and $d = 1$. The value of the nominal length $\hat{\ell}_0$ is $4 \times 10^{-3}$ (i.e., $\ell_0 = 1520$ km). The initial displacements of the center of mass
are \( \dot{x}(0) = \dot{y}(0) = 3 \times 10^{-5} \) (i.e. 11.5 km). It can be seen in Figure 3 that both \( x \) and \( y \) reduce quickly; \( x \) approaches its equilibrium value of 10 km, while \( y \) tends to zero. The rotational angle of the tether is quite small. The length of the tether is large initially (about 1700 km), but soon approaches the nominal value of 1520 km. Motion of the center of mass of a rotating tether system is examined next. The rotation rate is prescribed; hence equation (25) is not used and only equations (23) and (24) are solved. A large number of computer simulations were carried out. It was found that for a given tether length, it is possible to find a rotation rate for which the motion of the center of mass seems to be bounded for several days (or even weeks). Figure 4 shows a typical case for \( \ell = 500 \) km and \( \Omega = 0.02 \) n. One mass is 9 times the other
and the co-ordinates of the bigger mass have been plotted. It appears that rotational stabilization of a tethered system has potential and needs further study.

![Figure 4: Displacements of the larger mass of a rotating tethered system; (a) $\ell = 500$ km (b) $\ell = 2472$ km, $\Omega = 0.02n$, $\beta_1 = 0.1$, $\beta_2 = 0.9$, $x(0)=y(0)= 11.5$km](image)

**Conclusion**

Connecting two masses by a tether has the effect of reducing the unstable nature of motion near the collinear Lagrangian points. In fact, by varying the tether length using the feedback of $\dot{x}$ and $\dot{x}'$, it is possible to stabilize the planar motion of the
tethered system. It was also noticed that by an appropriate choice of the rotation rate of the tethered system, the displacement of the center of mass can be maintained at low values for several days.

References


