Techniques for Station Keeping Elliptically Orbiting Constellations in Along-track Formation

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Abstract

Two techniques for station keeping an orbiting constellation of satellites in an elliptical orbit after the initial deployment are developed: (1) based on an application of the linearized Tschauner-Hempel (TH) equations for the motion of a daughter satellite relative to a reference (mother) satellite together with the linear quadratic regulator (LQR) control strategy which can be used in a piecewise adaptive manner (since the TH equations for an elliptical orbit involve periodic coefficients); (2), since the mathematical model is inherently nonlinear and time varying, a control law based on a non-linear Lyapunov function is applied to daughter satellites’ osculating orbital elements.

Introduction

NASA has suggested several missions in low Earth orbit (LEO) for scientific data collection. One of these missions is the Auroral Cluster Observation System whose main objectives would be to measure the curl of the Earth’s magnetic field vector as well as detect auroral phenomenon. Of several designs proposed for this mission is an along-track formation in an elliptical orbit of up to four spacecraft with constant distances between adjacent satellites.

A novel idea for initiating such a station keeping strategy was proposed in a previous paper [1] by the authors and would involve an impulsive manoeuvre at perigee that would cause a small shift in the direction of the semi-major axis of the daughter satellite with respect to the original orbit (of the mother satellite) where the amount of the shift would depend on the perigee and apogee altitudes and the desired separation distance. It was seen that without perturbations and subsequent control effort this separation distance could be maintained with the drift from the nominal value remaining of the order of ±2%. A further improvement can be realized with a second manoeuvre involving slewing around the major axis (can further reduce the separation error to ±0.45%) [2].

In the presence of perturbations (mainly from the Earth’s oblateness) additional station keeping control would be required to counteract the resulting secular drift in the average separation distance. In this paper two techniques for subsequent station keeping
are developed and evaluated.

Application of the Linearized Tschauner-Hempel Equations with an LQR Control Strategy

The Tschaunert-Hemple equations [3], [4] describe the motion of a daughter spacecraft close to rendezvous with a reference or mother spacecraft in a nominal elliptical orbit. This equations can be linearized and recast in the familiar state-space representation as:

$$
\begin{pmatrix}
\xi' \\
\eta' \\
\zeta' \\
\bar{\xi}' \\
\eta'' \\
\bar{\zeta}'
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\frac{3\mu r}{h^2} & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\xi \\
\eta \\
\zeta \\
\bar{\xi} \\
\eta' \\
\bar{\zeta}'
\end{pmatrix}
+ 
\begin{pmatrix}
a_\xi \\
a_\eta \\
a_\zeta
\end{pmatrix}
\begin{pmatrix}
a_\xi \\
a_\eta \\
a_\zeta
\end{pmatrix}
$$

(1)

where $\xi$, $\eta$ and $\zeta$ are non-dimensionalized coordinates centered at the target or reference spacecraft. $\xi$ describes cross track (outward radial) motion, $\eta$ along track, and $\zeta$ out of the nominal orbit plane of the target spacecraft. The prime indicates differentiation with respect to the true anomaly angle, $\nu$, $r$ refers to the orbital radius, $h$ the angular momentum per unit mass, and $\mu$ is the gravitational constant (universal gravitational constant times the Earth’s mass). The non-linear term in the state matrix can be adjusted in a number of ways:

1. When it is assumed that $r$ remains constant (i.e. equal to $r(\theta)=h^2/\mu$), true for a circle and relatively short displacements, then the term becomes equal to 3.
2. If the simulation is started at perigee or apogee then evaluate $r$ at perigee or apogee, respectively, and treat as constant for sufficiently short time thereafter.
3. If several orbits are needed to correct the disturbance then use an average value for $r$ with $|\vec{h}|=|\vec{r} \times \vec{v}|=\sqrt{b^2 \mu/a}$ where $\vec{h}$ is the angular momentum, a semi-major axis and $b$ is the semi-minor axis.
4. A final consideration, is to update 1 and 2 in a piecewise adapted manner along the orbit.

The last two matrices in Eq.(1) can be identified as the control influence matrix $B$, and the control input matrix $u=[a_\xi, a_\eta, a_\zeta]^T$, in terms of the control accelerations. If the control accelerations are not provided directly along the $\xi$, $\eta$, $\zeta$ axes then the $B$ matrix (e.g. $C_i$ values, or more complicated arrangements) can be adjusted accordingly. For simplicity here, assume $u_1=c_1 a_\xi$, $u_2=c_2 a_\eta$, $u_3=c_3 a_\zeta$.

For the application of the linear quadratic regulator, assume that the state matrix is, at least, piecewise constant, and that the performance index [5]
subject to Eq.(1), where \( Q \) is a positive semi-definite state weighting matrix, and \( R \) is the control weighting matrix which is positive definite. The state vector, \( x = (\xi \eta \zeta \eta' \zeta')^T \). The result of the quadratic performance index optimization subject to the state equation yields the state feedback control law:

\[ u = -Kx \]  

where \( K \) is the optimum gain matrix given by

\[ K = R^{-1}B^TP \]  

and \( P \) is the symmetric positive definite matrix found by solving the algebraic Riccati equation:

\[ A^TP + PA + Q - PBR^{-1}B^TP = 0 \]

Assuming all states are immediately available and in the absence of noise, a parametric study was undertaken by using various weighting functions. The task is then to select the combination of values for the state and control weighting matrices which best meets criteria on convergence time, control effort (including maximum control values), and amount of transient overshoots. Only a few representative results out of many [6] are reported here.

The results presented here assume that the initial LQR correction begins near perigee at a true anomaly angle of 45°. The periodic system \( A \) matrix is evaluated at that true anomaly angle. If the responses occur in a relatively short time interval it is assumed that this value could be used throughout the manoeuvre (the case considered here). For longer time intervals this matrix would have to be re-evaluated in a piecewise adaptive manner. Figure 1 will be considered as the reference case for the others. The diagonal elements of the control \( R \) and state \( Q \) weighting matrices are given the value of ones. The off diagonal terms are assumed to be zero. The state here is defined as \( (x, y, z, \dot{x}, \dot{y}, \dot{z}) \) where \( x, y, z \) are in the radial cross-track direction in-plane, \( \dot{x}, \dot{y}, \dot{z} \) in the along-track direction, and \( \zeta \) represents out-of-plane displacement. The center of this moving coordinate system is taken at the nominal position of any daughter satellite in the orbit. Thus a displacement of 1 km along track actually represents a displacement of \( d_m + 1 \) km along track, where \( d_m \) is the desired mother-daughter separation distance for this application. Here an out-of-plane error in position of 1 km is given at the 45° true anomaly point, in addition to a 1 km error along track.

Figure 1 depicts consistence as it relates to input error. Furthermore, damped simple harmonic motion (SHM) is demonstrated as predicted by the out-of-plane Tschauner-Hempel equation. The weighting elements selected here by no means infer that these are the optimal choices or combination of choices, only to demonstrate capability and
workability.

Figure 2 shows the transient responses for the same weighting elements as in Figure 1, but with no initial out-of-plane error. As expected the over-all control requirements are much lower and overshoot amplitudes are greatly reduced.

Figure 3 shows the same case as in Figure 2, but with increased penalty on all state components. Some of the overshoots are noticeably reduced as compared with Figure 2 responses. The responses of the x component of acceleration and position are much faster. The peak control effort components are an order of magnitude larger than in Figure 2.
Figure 2 - LQR Transient Responses - Only In-Plane Initial Disturbance

Figure 3 - LQR Transient Responses - Only In-Plane Initial Disturbance
Nonlinear Control Law Based on Lyapunov Function Applied to the Osculating Orbital Elements

This is a nonlinear control law based on the non-linear mathematical model. Comparing to the control strategy described above, there is no error introduced from linearization. The variational equations of Gauss provide a convenient set of equations relating the effect of a control acceleration vector $u$ to the osculating orbital element time derivatives [7].

$$\dot{a} = (2a^2)/h \left( e \sin f u_r + (p/r) u_0 \right)$$ (a)

$$\dot{\varepsilon} = [p \sin f u_r + (p+r) \cos f + re) u_\theta]/h$$ (b)

$$i = [(r \cos \theta)/h] u_h$$ (c)

$$\dot{\Omega} = [(r \sin \theta)/(h \sin \iota)] u_h$$ (d)

$$\dot{\omega} = \frac{1}{he} [-pcosf u_r + (p+r)sinf u_\theta] - \frac{rsin \theta \cos \iota}{h \sin \iota} u_h$$ (e)

$$\dot{M} = n + [\eta/(he)] [(pcosf-2re)u_r - (p+r)sinf u_\theta]$$ (f)

where $a$ is the semi-major axis; $e$ is the eccentricity; $i$ is the inclination; $\Omega$ is the longitude of the ascending node; $\omega$ is the argument of the perigee, $M$ is the mean anomaly. We define $x = (a e i \Omega \omega M)'$ as the state variable vector and $u = (u_r u_\theta u_h)$' as the control acceleration vector, written in the Local-Vertical-Local-Horizontal frame, $u_r$ points radially away from the Earth, $u_h$ is aligned with the orbit angular momentum vector, $u_\theta$ is orthogonal to both $u_r$ and $u_h$. $f$ is the true anomaly, $r$ is the scalar orbit radius, $p=b^2/a$ is the semilatus rectum, $\theta = \omega + f$, and $h = \sqrt{p/M}$, $\eta = \sqrt{1-e^2}$, $n = \sqrt{\gamma/a^3}$

Incorporating the $J_2$ influence, Eq.(6) can be written as [7]

$$\dot{x} = B(x)u + D(x)$$ (7)

where

$$D(x) = [0, 0, 0, -3J_2(R_\text{g}/p)^2n \cos i, 3J_2(R_\text{g}/p)^2n(5 \cos^2 i - 1), n + 3J_2(R_\text{g}/p)^2 \eta n(3 \cos^2 i - 1)]^T$$ (8)

and

$$B(x) = \begin{bmatrix}
    (2a^2 \sin f)/h & (2a^2p)/(hr) & 0 \\
    (p \sin f)/h & [(p+r) \cos f + re]/h & 0 \\
    0 & 0 & (r \cos \theta)/h \\
    0 & 0 & (r \sin \theta)/(h \sin \iota) \\
    -(p \cos f)/(he) & [(p+r) \sin f]/(he) & -(r \sin \theta \cos \iota)/(h \sin \iota) \\
    \eta(p \cos f - 2re)/(he) & -\eta(p+r) \sin f \eta/(he) & 0
\end{bmatrix}$$ (9)

It is found out that the elements of $D$ are either zeros or very small, therefore, $D(x)$ is treated as a minor disturbance instead of part of the plant matrix. From Eq.(9), it is seen that the system is nonlinear and time variant, so a control law based on a Lyapunov function is applied. If the osculating orbital elements of the mother satellite
are $x_i$, the required osculating orbital elements of the first daughter satellite are $x_2$, then
\[ \Delta x = x_2 - x_1 \quad i.e. \quad x_2 = x_1 + \Delta x \tag{10} \]

Assuming that the actual osculating orbital elements of the first daughter satellite are $x_{2d}$, then
\[ \delta x = x_{2d} - x_2 \quad i.e. \quad x_{2d} = x_2 + \delta x \tag{11} \]

We define a Lyapunov function as
\[ V = \frac{1}{2} (a + be^{-\alpha t}) \cdot (a + be^{-\alpha t})^T \delta x \quad \text{where } a > 0, b > 0, \alpha > 0, \therefore \ V > 0 \tag{12} \]
then
\[ \dot{V} = -\frac{1}{2} \beta ae^{-\alpha t} \delta x^T \delta x + (a + be^{-\alpha t}) \delta x^T \dot{\delta x} \tag{13} \]
where $\delta x = x_{2d} - x_2 = x_{2d} - x_1 - \Delta x$ \text{ see (11) and (10)} \therefore \ $\dot{\delta x} = \dot{x}_{2d} - \dot{x}_1$, and $\Delta x$ does not vary with time. Substituting Eqs.(7), (8) and (9) into the above equation, and noticing that there is no control for the mother satellite, we get
\[ \dot{\delta x} = B(x)u + [D(x_{2d}) - D(x_1)] \tag{14} \]
If we ignore the effect of $D(x_{2d}) - D(x_1)$ in Eq.(14), and select
\[ u = -\beta (B^T B)^{-1} B^T \dot{\delta x} \tag{15} \]
where $\beta$ is a scalar value used to adjust the feedback gain, and substituting Eq.(15) into Eq.(14), we get
\[ \dot{\delta x} = -\beta (B^T B)^{-1} B^T \delta x \tag{16} \]
After substituting Eq.(16) into Eq.(13) there results,
\[ \dot{V} = -\frac{1}{2} \beta e^{-\alpha t} \delta x^T \delta x - \beta (a + be^{-\alpha t}) \delta x^T B (B^T B)^{-1} B^T \dot{\delta x} \tag{17} \]
\therefore \ $\Phi = B (B^T B)^{-1} B^T$ is symmetric and semi-positive definite, and $\text{Rank}(\Phi) \leq \text{Rank}(B)$, here $B$ is $6 \times 3$, \therefore \ the eigenvalues of $\Phi$ are $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0$, $0$, $0$, $0$. There must be an orthogonal matrix so that
\[ \Phi = T \Lambda T^T \quad \text{where } \Lambda = T^T \Phi T = \text{diag}(\lambda_1, \lambda_2, \lambda_3, 0, 0, 0) \tag{18} \]
Eq.(17) can be written as
Since $a>0$, $b>0$, $\alpha>0$, $\beta>0$, $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$, $: \Xi$ is positive definite, and so is $T\Xi T^T$. If $\delta x \neq 0$, then $\dot{\delta x} = - \delta x^T T \Xi T^T \delta x < 0$. That means the control law described by Eq.(15) can make the formation flight system asymptotically stable. It is obvious that the above analysis is also suitable for the control of the distance between the daughter satellites.

### Table 1 - Initial Conditions for Mother and First Daughter Satellites

<table>
<thead>
<tr>
<th>Initial Condition</th>
<th>Mother Satellite</th>
<th>Daughter Sat.1’s Required Elements</th>
<th>Daughter Sat.1’s Actual Elements</th>
<th>$\delta x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a km</td>
<td>10678.137</td>
<td>10678.137</td>
<td>10677.137</td>
<td>-1</td>
</tr>
<tr>
<td>e</td>
<td>0.34650239</td>
<td>0.34650239</td>
<td>0.346534844</td>
<td>0.000032</td>
</tr>
<tr>
<td>t deg.</td>
<td>83</td>
<td>83</td>
<td>82.9985</td>
<td>-0.0015</td>
</tr>
<tr>
<td>$\Omega$ deg.</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>$\omega$ deg.</td>
<td>10</td>
<td>11.37</td>
<td>11.37</td>
<td>0</td>
</tr>
<tr>
<td>M deg.</td>
<td>0</td>
<td>0+39.5×n</td>
<td>39.5×n</td>
<td>0</td>
</tr>
</tbody>
</table>

### Numerical Simulations

With various perturbations, including atmosphere drag, solar pressure, the Earth’s magnetic field, perturbations from the Moon and other planets, and $J_n$’s effect, where $n=2$ is the most important contribution, the osculating orbital elements do not remain constant. Reference 2 provided a program to compute the required daughter’s osculating orbital elements dynamically. The initial conditions for the mother and first daughter satellites are listed in Table 1. When $\beta=2\times10^{-4}$ (see Eq.(15)), the difference of the osculating orbital elements between the mother and the first daughter satellites as well as the control efforts of the first daughter satellite are shown in Fig.4. From Fig.4 it is seen that the oscillating orbital elements converge smoothly; the maximum control efforts (see Fig.4., g~i) are less than $10^{-4}$ m/s $^2$, e.g. if the mass of the satellite is 100 kg., the maximum control is less than 0.01 newton.
Conclusions

Simulation results for both station keeping techniques show that the response to initial errors converge smoothly with the control energy at a low level. The LQR-TH approach incorporates the robustness advantages of an LQR technique, while the Lyapunov approach is shown to result in an asymptotically stable nonlinear system.

References


